A method for constructing coreflections for nearness frames

John Frith and Anneliese Schauerte

Abstract

We provide a fairly general method, which is straightforward and widely applicable, for constructing some coreflections in the category of nearness frames. The method captures all coreflective subcategories with $1-1$ coreflection maps; this includes the well-known uniform, totally bounded and separable coreflections.

The primary application of this method answers in the affirmative the question of Dube and Mugochi ([15]) as to whether strong nearness frames are coreflective in nearness frames. We show that the strong coreflection can change the underlying frame, in contrast to Dube and Mugochi's almost uniform coreflection in the category of interpolating nearness frames.

The method also finds application in categories other than nearness frames, for instance, prenearness frames and nearness σ-frames. We conclude with an application to the unstructured setting where we recover the regular and completely regular coreflections in frames.

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1 Introduction

Reflections in topology and, latterly, coreflections in pointfree topology, provide fundamental insights into the categories of topological spaces and frames.

Examples range from the relatively easy, for instance the totally bounded coreflection of nearness frames or quasi-nearness biframes ([14] and [17]) to the fairly intricate, for instance the paracompact coreflection of frames ([11] and [13]). The coreflection to the compact objects in completely regular frames is given by the Stone-Čech compactification ([7], [8], [9]); in uniform frames by the Samuel compactification ([10]). Analogues of these in σ-frames ([5]) and in the asymmetric setting of quasi-uniform biframes ([16]) also exist.

This paper provides a fairly general method, which is straightforward and widely applicable, for constructing some coreflections in the category of nearness frames. In fact we capture all coreflective subcategories with $1-1$ coreflection maps; this includes the well-known uniform, totally bounded and separable coreflections.

Our primary application of this method answers the question of Dube and Mugochi ([15]) as to whether strong nearness frames are coreflective in nearness frames; the answer is in the affirmative. We show that the strong coreflection can change the underlying frame, in contrast to Dube and Mugochi's almost uniform coreflection in the category of interpolating nearness frames.

Our general method also finds application in categories other than nearness frames, for instance, prenearness frames and nearness σ-frames. We conclude with an application to the unstructured setting where we recover the regular and completely regular coreflections in frames. In a subsequent paper, we use this method in the asymmetric setting to discuss coreflective subcategories of quasi-nearness and quasi-uniform biframes.

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2 Background

See [28], [20], [31], [29], [3], [12], [18], [30] and [6] for background information on frames, uniform frames and nearness frames. For further varied uses of coreflections in related categories, see [27], [32], [23], [21] and [33] (but note the latter's non-standard terminology). See [22] and [1] for category theory. In this paper, we take all subcategories to be full and isomorphism-closed.

Definition 2.1 1. A *frame* L is a complete lattice in which the distributive law

$$x \wedge \bigvee\{y : y \in Y\} = \bigvee\{x \wedge y : y \in Y\}$$

holds for all $x \in L, Y \subseteq L$. A *frame map* is a set function between frames which preserves finite meets and arbitrary joins, and thus also the top (denoted 1) and the bottom (denoted 0) of the frame. For any $x \in L$, $x^* = \bigvee\{t \in L : t \wedge x = 0\}$ is the *pseudocomplement* of x.

2. For a frame L, $C \subseteq L$ is a *cover of* L if $\bigvee C = 1$.
For covers C and D of L, $C \wedge D = \{c \wedge d : c \in C, d \in D\}$ is again a cover of L.
We say that C *refines* D if for any $c \in C$ there exists $d \in D$ with $c \leqslant d$; we then write $C \leqslant D$.
For $a, b \in L$ and C a cover of L we write $a \lhd_C b$ if $Ca = \bigvee\{c \in C : c \wedge a \neq 0\} \leqslant b$. If $CC = \{Cc : c \in C\} \leqslant D$, we write $C <^* D$.

3. A non-empty collection of covers, $\mathcal{N}L$, of L is a *prenearness* on L if it is filtered by meet and refinement. $\mathcal{A} \subseteq \mathcal{N}L$ is a *base* for $\mathcal{N}L$ if every member of $\mathcal{N}L$ is refined by a member of $\mathcal{A}$. The members of $\mathcal{N}L$ are called *uniform* covers. The pair $(L, \mathcal{N}L)$ is a *prenearness frame*.
A prenearness $\mathcal{N}L$ on L that satisfies the property that for each $D \in \mathcal{N}L$ there exists $C \in \mathcal{N}L$ with $C <^* D$ is a *preuniformity* on L.
If a prenearness $\mathcal{N}L$ satisfies that, for any $x \in L$, $x = \bigvee\{t \lhd_C x : C \in \mathcal{N}L\}$, it is a *nearness* on L. (We refer to this latter condition as the *compatibility condition*.) The pair $(L, \mathcal{N}L)$ is a *nearness frame*.
A preuniformity $\mathcal{N}L$ on L that satisfies the compatibility condition is called a *uniformity* on L. The pair $(L, \mathcal{N}L)$ is a *uniform frame*.

4. For a prenearness $\mathcal{N}L$ on L we write $a \lhd b$ *in* $(L, \mathcal{N}L)$ (or $a \lhd_{\mathcal{N}L} b$) if there is $C \in \mathcal{N}L$ such that $a \lhd_C b$.

5. For (pre)nearness frames $(L, \mathcal{N}L)$ and $(M, \mathcal{N}M)$, a frame map f from L to M is a *uniform* map if for every $C \in \mathcal{N}L$, $f[C] = \{f(c) : c \in C\} \in \mathcal{N}M$. The category of nearness frames and uniform maps is denoted by **NearFrm**.

6. We note that for any nearness frame $(L, \mathcal{N}L)$, the underlying frame L is regular. This means that, for any $x \in L$, $x = \bigvee\{t : t \prec x\}$ where $t \prec x$ iff $t^* \vee x = 1$. Further the underlying frame of a uniform frame is

completely regular. (For the definition of complete regularity, see p90 in [28].)

Definition 2.2 Let $(L, \mathcal{N}L)$ be a nearness frame. $(L, \mathcal{N}L)$ is said to be

- *totally bounded* if each uniform cover is refined by a finite uniform cover.
- *separable* if each uniform cover is refined by a countable uniform cover.
- *strong* if whenever C is a uniform cover, so is $\check{C} = \{t \in L : t \lhd c$ in $(L, \mathcal{N}L)$ for some $c \in C\}$.
- *interpolating* if whenever $a \lhd b$ there exists t such that $a \lhd t \lhd b$.
- *almost uniform* if it is both strong and interpolating.
- *fine* if $\mathcal{N}L = \mathrm{Cov}L$, that is, if all covers of L are uniform.
- *finitely fine* if the finite covers of L form a base for $\mathcal{N}L$.

3 The construction

We introduce the notion of a sub nearness frame and show that the sub nearness frames of a given nearness frame form a complete lattice. This is an important ingredient in our method for establishing certain coreflections in the category of nearness frames.

Definition 3.1 Let $(L, \mathcal{N}L)$ and $(M, \mathcal{N}M)$ be nearness frames. We call $(L, \mathcal{N}L)$ a *sub nearness frame* of $(M, \mathcal{N}M)$ if L is a subframe of M and $\mathcal{N}L \subseteq \mathcal{N}M$. We note that this is equivalent to the identical embedding from $(L, \mathcal{N}L)$ to $(M, \mathcal{N}M)$ being a uniform map. We then write $(L, \mathcal{N}L) \leqslant (M, \mathcal{N}M)$.

Proposition 3.2 Let $(L, \mathcal{N}L)$ be a nearness frame. The collection of all sub nearness frames of $(L, \mathcal{N}L)$ forms a complete lattice.

PROOF. The relation $\leqslant$ given in Definition 3.1 is indeed a partial order. The bottom element is clearly the two element frame with its unique nearness (except in the case where L is degenerate, in which case it is L itself).
Let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ be a non-empty collection of sub nearness frames of $(L, \mathcal{N}L)$.

• Let $\widetilde{L}$ be the subframe of L generated by $\bigcup\limits_{\alpha \in I} L_\alpha$.

• Define $\mathcal{N}\widetilde{L}$ as follows: $C \in \mathcal{N}\widetilde{L}$ iff $C \subseteq \widetilde{L}$ and there exists a natural number n and $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$ for $j = 1, \ldots, n$ such that $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$.

We now show that $(\widetilde{L}, \mathcal{N}\widetilde{L})$ is the join of $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$, by noting the following points:

1. For each $\alpha \in I$, $\mathcal{N}L_\alpha \subseteq \mathcal{N}\widetilde{L}$.

2. For $a, b \in L_\alpha$, $a \lhd b$ in $(L_\alpha, \mathcal{N}L_\alpha)$ implies that $a \lhd b$ in $(\widetilde{L}, \mathcal{N}\widetilde{L})$, since $Ca \leqslant b$ for some $C \in \mathcal{N}L_\alpha$ gives $Ca \leqslant b$ for that same $C \in \mathcal{N}\widetilde{L}$.

3. $\mathcal{N}\widetilde{L}$ is closed under finite meets.

4. If $C \in \mathcal{N}\widetilde{L}$, $D \subseteq \widetilde{L}$ and $C \leqslant D$, then $D \in \mathcal{N}\widetilde{L}$.

5. For any $a \in \widetilde{L}$, $a = \bigvee\{b \in \widetilde{L} : b \lhd a \text{ in } (\widetilde{L}, \mathcal{N}\widetilde{L})\}$. To see this, take $t \in L_\alpha$, for some $\alpha \in I$. Then
   $t = \bigvee\{s \in L_\alpha : s \lhd t \text{ in } (L_\alpha, \mathcal{N}L_\alpha)\}$ since $(L_\alpha, \mathcal{N}L_\alpha)$ is a nearness frame.
   So $t = \bigvee\{s \in L_\alpha : s \lhd t \text{ in } (\widetilde{L}, \mathcal{N}\widetilde{L})\}$ by 2 above.
   So $t = \bigvee\{u \in \widetilde{L} : u \lhd t \text{ in } (\widetilde{L}, \mathcal{N}\widetilde{L})\}$.
   Now an arbitrary member of $\widetilde{L}$ is an arbitrary join of finite meets of such t's, and so can be expressed in the desired form.

6. $(\widetilde{L}, \mathcal{N}\widetilde{L})$ is a nearness frame from 3, 4 and 5 above.

7. $(\widetilde{L}, \mathcal{N}\widetilde{L})$ is a sub nearness frame of $(L, \mathcal{N}L)$, since $\widetilde{L}$ is a subframe of L and $\mathcal{N}\widetilde{L} \subseteq \mathcal{N}L$. The latter follows since $\mathcal{N}L_\alpha \subseteq \mathcal{N}L$ and if $C \in \mathcal{N}\widetilde{L}$ with $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$, for some $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$, then $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \in \mathcal{N}L$ and so $C \in \mathcal{N}L$.

8. If $(M, \mathcal{N}M)$ is a sub nearness frame of $(L, \mathcal{N}L)$ such that $(L_\alpha, \mathcal{N}L_\alpha) \leqslant (M, \mathcal{N}M)$ for all $\alpha \in I$, then L_α is a subframe of M and $\mathcal{N}L_\alpha \subseteq \mathcal{N}M$

for all $\alpha \in I$. So $\widetilde{L}$ is a subframe of M and $\mathcal{N}\widetilde{L} \subseteq \mathcal{N}M$. We see that $(\widetilde{L}, \mathcal{N}\widetilde{L})$ is indeed the join of $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$, as required.

∎

Throughout this paper, we will use P to denote an arbitrary property that a nearness frame might have. We introduce the idea of a P-approximation of a nearness frame and use it to construct a functor from **NearFrm** to itself, in the case that the property P is preserved by uniform images.

Definition 3.3 Let $(L, \mathcal{N}L)$ be a nearness frame.

1. We call those sub nearness frames of $(L, \mathcal{N}L)$ that have property P, the *P-approximations* of $(L, \mathcal{N}L)$.
2. Define $\Gamma_P(L, \mathcal{N}L)$ to be the join of all the P-approximations of $(L, \mathcal{N}L)$ (as provided in Proposition 3.2 of course).

By Proposition 3.2 $\Gamma_P(L, \mathcal{N}L)$ is a nearness frame. We make no claim that $\Gamma_P(L, \mathcal{N}L)$ necessarily satisfies property P, but will, of course, be most interested in those properties P where it does.

We note that $\Gamma_P(L, \mathcal{N}L)$ is defined in the case where a given nearness frame has no P-approximations. It is the empty join.

Definition 3.4

1. Let $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ be a uniform map between nearness frames.
   We define $h(L, \mathcal{N}L) = (h[L], h[\mathcal{N}L])$ where $h[L] = \{h(x) : x \in L\}$ and $h[\mathcal{N}L] = \{h[C] : C \in \mathcal{N}L\}$.
   It is straightforward to check that $h(L, \mathcal{N}L)$ is a sub nearness frame of $(M, \mathcal{N}M)$ and that $h : (L, \mathcal{N}L) \to h(L, \mathcal{N}L)$ is a uniform map.
2. If a property P satisfies the condition that, whenever a nearness frame $(L, \mathcal{N}L)$ has property P, then $h(L, \mathcal{N}L)$ has property P for any uniform h, we say that *P is preserved by uniform images.*

Proposition 3.5 Let P be a property that is preserved by uniform images. Then $\Gamma_P : \mathbf{NearFrm} \to \mathbf{NearFrm}$ is a functor.

PROOF. Γ_P was defined on objects in Definition 3.3. We define Γ_P on morphisms as follows. Let $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ be a uniform map between nearness frames. We show below that $\Gamma_P h : \Gamma_P(L, \mathcal{N}L) \to \Gamma_P(M, \mathcal{N}M)$ given by restricting the domain and codomain of h is again a uniform map. For brevity we write $\Gamma_P(L, \mathcal{N}L) = (\widetilde{L}, \mathcal{N}\widetilde{L})$ and $\Gamma_P(M, \mathcal{N}M) = (\widetilde{M}, \mathcal{N}\widetilde{M})$.

Let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ be the collection of all P-approximations of $(L, \mathcal{N}L)$. For any $\alpha \in I$, by assumption, $h(L_\alpha, \mathcal{N}L_\alpha)$ is a P-approximation of $(M, \mathcal{N}M)$. So $h(L_\alpha, \mathcal{N}L_\alpha) \leqslant (\widetilde{M}, \mathcal{N}\widetilde{M})$. Then $h[L_\alpha] \subseteq \widetilde{M}$ for all $\alpha \in I$, giving $h[\widetilde{L}] \subseteq \widetilde{M}$. Further, $h[\mathcal{N}L_\alpha] \subseteq \mathcal{N}\widetilde{M}$ for all $\alpha \in I$, so $h[\mathcal{N}\widetilde{L}] \subseteq \mathcal{N}\widetilde{M}$.

That Γ_P preserves identities and composition is clear. ∎

We are now in a position to provide the promised construction of certain P-coreflections.

Theorem 3.6 Let P be a property satisfying the conditions:

1. P is preserved by uniform images, and
2. for any nearness frame $(L, \mathcal{N}L)$, the join $\Gamma_P(L, \mathcal{N}L)$ of all P-approximations of $(L, \mathcal{N}L)$ has property P.

Then the nearness frames with property P form a full monocoreflective subcategory of all nearness frames.

PROOF. Let P be a property described as above and let $(L, \mathcal{N}L)$ be a nearness frame. We show that the identical embedding $\eta_P : \Gamma_P(L, \mathcal{N}L) \to (L, \mathcal{N}L)$ is the desired coreflection map.
Let $(M, \mathcal{N}M)$ be a nearness frame with property P and $f : (M, \mathcal{N}M) \to (L, \mathcal{N}L)$ a uniform map. Now $f(M, \mathcal{N}M)$ is a sub nearness frame of $(L, \mathcal{N}L)$ and has property P, so is a P-approximation of $(L, \mathcal{N}L)$. This makes the identical embedding $i : f(M, \mathcal{N}M) \to \Gamma_P(L, \mathcal{N}L)$ a uniform map and we have the following obvious commuting diagram:

$$\begin{tikzcd}
\Gamma_P(L,\mathcal{N}L) \arrow[r, "\eta_P"] & (L,\mathcal{N}L) \\
f(M,\mathcal{N}M) \arrow[u, "i"] & (M,\mathcal{N}M) \arrow[l, "f"'] \arrow[u, "f"']
\end{tikzcd}$$

The factorization of f is unique, because η_P is $1-1$, and hence a monomorphism. ∎

Definition 3.7 We call the coreflection constructed in Theorem 3.6 the *P-coreflection of nearness frames.*

We note that a morphism $h : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ in **NearFrm** is an isomorphism iff $h : L \to M$ is a frame isomorphism and $h[\mathcal{N}L] = \mathcal{N}M$. Further, if $f : (L, \mathcal{N}L) \to (M, \mathcal{N}M)$ is a morphism in **NearFrm** and f is $1-1$, then $(L, \mathcal{N}L)$ is isomorphic to $f(L, \mathcal{N}L)$ which is a sub nearness frame of $(M, \mathcal{N}M)$.

In the next result, we show that any full, isomorphism-closed coreflective subcategory, **K**, of **NearFrm** for which the coreflection maps are all $1-1$, can in fact be obtained by the construction of Theorem 3.6. We simply define $(L, \mathcal{N}L)$ to have property P whenever $(L, \mathcal{N}L)$ is an object of **K**. The details follow:

Proposition 3.8 Let **K** be a full, isomorphism-closed coreflective subcategory of **NearFrm** for which the **K**-coreflection maps are all $1-1$. Define P by stating that a nearness frame $(L, \mathcal{N}L)$ satisfies P iff $(L, \mathcal{N}L)$ is an object of **K**. Then the P-coreflection and the **K**-coreflection of any nearness frame are isomorphic.

PROOF. Let $(L, \mathcal{N}L)$ be a nearness frame and denote its **K**-coreflection map by $\eta : K(L, \mathcal{N}L) \to (L, \mathcal{N}L)$. Since η is $1-1$, $\eta K(L, \mathcal{N}L)$ is a sub nearness frame of $(L, \mathcal{N}L)$; in fact it is a P-approximation of $(L, \mathcal{N}L)$. So $\eta K(L, \mathcal{N}L) \leqslant \Gamma_P(L, \mathcal{N}L)$.
Let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ denote the set of all P-approximations of $(L, \mathcal{N}L)$.

Let $\alpha \in I$. Then the identical embedding $i : (L_\alpha, \mathcal{N}L_\alpha) \to (L, \mathcal{N}L)$ is a uniform map. Since $(L_\alpha, \mathcal{N}L_\alpha) \in \mathbf{K}$, i factors through η; that is, there exists a unique uniform map $h : (L_\alpha, \mathcal{N}L_\alpha) \to K(L, \mathcal{N}L)$ such that $\eta h = i$. Since i is $1-1$, h is also $1-1$. Then $h(L_\alpha, \mathcal{N}L_\alpha)$ is a sub nearness frame of $K(L, \mathcal{N}L)$, and so $\eta h(L_\alpha, \mathcal{N}L_\alpha)$ is a sub nearness frame of $\eta K(L, \mathcal{N}L)$. This makes $(L_\alpha, \mathcal{N}L_\alpha)$ a sub nearness frame of $\eta K(L, \mathcal{N}L)$, for all $\alpha \in I$. Then since $\Gamma_P(L, \mathcal{N}L)$ is the join of all such $(L_\alpha, \mathcal{N}L_\alpha)$, we get $\Gamma_P(L, \mathcal{N}L) \leqslant \eta K(L, \mathcal{N}L)$. So finally, $\Gamma_P(L, \mathcal{N}L) = \eta K(L, \mathcal{N}L)$ as desired. ∎

4 The strong coreflection

Our first application of the coreflection method provided in the last section concerns the subcategory of strong nearness frames. Dube and Mugochi [15] asked whether this forms a coreflective subcategory of nearness frames; we now answer this question in the affirmative. We then investigate aspects of this coreflection. We compare it to Dube and Mugochi's almost uniform coreflection in [15], which is taken in the category of interpolating nearness frames. We conclude by looking at the question as to whether the strong coreflection always preserves the underlying frame.

Proposition 4.1 The strong nearness frames form a coreflective subcategory of all nearness frames.

PROOF. We apply Theorem 3.6. The fact that a uniform image of a strong nearness frame is again strong is familiar. (See [6] p 21.)

For the second condition, let $(L, \mathcal{N}L)$ be a nearness frame and let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ be the set of strong approximations of $(L, \mathcal{N}L)$, that is, those sub nearness frames of $(L, \mathcal{N}L)$ that are strong when regarded as nearness frames in their own right. Let $(\widetilde{L}, \mathcal{N}\widetilde{L})$ be the join of $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$. (So this is $\Gamma_P(L, \mathcal{N}L)$ where P is the property of being strong.) We show that $(\widetilde{L}, \mathcal{N}\widetilde{L})$ is strong.

Let $C \in \mathcal{N}\widetilde{L}$. Let $\check{C} = \{x \in \widetilde{L} : x \lhd c \text{ in } (\widetilde{L}, \mathcal{N}\widetilde{L})\,,\ \text{for some } c \in C\}$; we show that $\check{C} \in \mathcal{N}\widetilde{L}$. By the construction of $\mathcal{N}\widetilde{L}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ and $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$ such that $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$. Now $\check{D}_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$ where

$\breve{D}_{\alpha_j} = \{t \in L_{\alpha_j} : t \lhd d \text{ in } (L_{\alpha_j}, \mathcal{N}L_{\alpha_j}), \text{ some } d \in D_{\alpha_j}\}$, since all the $(L_\alpha, \mathcal{N}L_\alpha)$ are strong. This gives $\breve{D}_{\alpha_1} \wedge \ldots \wedge \breve{D}_{\alpha_n} \in \mathcal{N}\widetilde{L}$. To conclude the proof, we show that $\breve{D}_{\alpha_1} \wedge \ldots \wedge \breve{D}_{\alpha_n} \leqslant \breve{C}$. (In fact we show $\breve{D}_{\alpha_1} \wedge \ldots \wedge \breve{D}_{\alpha_n} \subseteq \breve{C}$.)

Take $x_j \in \breve{D}_{\alpha_j}$, $j = 1, \ldots, n$. Then $x_j \lhd d_j$ in $(\widetilde{L}, \mathcal{N}\widetilde{L})$ for some $d_j \in D_{\alpha_j}$ (as already noted in the proof of Proposition 3.2). So $x_1 \wedge \ldots \wedge x_n \lhd d_1 \wedge \ldots \wedge d_n$ in $(\widetilde{L}, \mathcal{N}\widetilde{L})$. Since $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$, this means that $x_1 \wedge \ldots \wedge x_n \in \breve{C}$. ▮

In [15], the almost uniform nearness frames are shown to be coreflective in the interpolating nearness frames, with a coreflection that leaves the underlying frame unchanged. The construction given there is as follows. Let $(L, \mathcal{N}L)$ be an interpolating nearness frame. For $C, D \in \mathcal{N}L$, write $D \lhd C$ if, for each $d \in D$, there exists $c_d \in C$ such that $d \lhd c_d$ in $(L, \mathcal{N}L)$. Define $\mathcal{AU}L = \{C \in \mathcal{N}L : D \lhd C \text{ for some } D \in \mathcal{N}L\}$. Then $(L, \mathcal{AU}L)$ is the desired almost uniform coreflection of $(L, \mathcal{N}L)$. (See Lemma 2.2 of [15].) In the next result, we show that, for an interpolating nearness frame, Dube and Mugochi's almost uniform coreflection and our strong coreflection coincide.

Proposition 4.2 For any interpolating nearness frame, its strong coreflection taken in nearness frames and its almost uniform coreflection taken in interpolating nearness frames, are the same.

PROOF. Let $(L, \mathcal{N}L)$ be an interpolating nearness frame. Let $(\widetilde{L}, \mathcal{N}\widetilde{L})$ be its strong coreflection (as constructed in Proposition 4.1) and let $(L, \mathcal{AU}L)$ be its almost uniform coreflection (as described above). We show that $(\widetilde{L}, \mathcal{N}\widetilde{L}) = (L, \mathcal{AU}L)$.

First note that $(L, \mathcal{AU}L)$ is a strong approximation of $(L, \mathcal{N}L)$, so $(L, \mathcal{AU}L) \leqslant (\widetilde{L}, \mathcal{N}\widetilde{L})$. For the reverse inequality, let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ be the set of all strong approximations of $(L, \mathcal{N}L)$. It suffices to show that $(L_\alpha, \mathcal{N}L_\alpha) \leqslant (L, \mathcal{AU}L)$ for all $\alpha \in I$, since then $(\widetilde{L}, \mathcal{N}\widetilde{L}) \leqslant (L, \mathcal{AU}L)$. To this end, fix $\alpha \in I$. Certainly L_α is a subframe of L; it remains to show that $\mathcal{N}L_\alpha \subseteq \mathcal{AU}L$. Take $C \in \mathcal{N}L_\alpha$. Then $\breve{C} \in \mathcal{N}L_\alpha$ where $\breve{C} = \{x \in L_\alpha : x \lhd c \text{ in } (L_\alpha, \mathcal{N}L_\alpha), \text{ some } c \in C\}$. Then $\breve{C} \lhd C$ in $(L, \mathcal{N}L)$. Further, since $\breve{C} \in \mathcal{N}L_\alpha$, it follows that $\breve{C} \in \mathcal{N}L$. This shows that $C \in \mathcal{AU}L$ as desired. ▮

We now mention some consequences of the result above which follow directly from the explicit description of the almost uniform coreflection given by Dube and Mugochi.

Remark 4.3

1. The strong coreflection of an interpolating nearness frame is interpolating.
2. The strong coreflection of an interpolating nearness frame does not change the underlying frame.
3. We note that the construction of $(L, \mathcal{AU}L)$ in [15] (see their Lemma 3.2) uses the Axiom of Choice.

The second remark above points to the need for an example in which the underlying frame of the strong coreflection changes. For this we need the lemma below, concerning total boundedness.

We note that a strong nearness frame is totally bounded if and only if each uniform cover has a finite subcover (which need not be uniform). (See Remark 2, p43 of [4].) It is this criterion we use below.

Lemma 4.4 The strong coreflection of a totally bounded nearness frame is totally bounded.

PROOF. Let $(L, \mathcal{N}L)$ be a totally bounded nearness frame and let $(\widetilde{L}, \mathcal{N}\widetilde{L})$ be its strong coreflection. Let $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ be the set of all strong approximations of $(L, \mathcal{N}L)$. For $C \in \mathcal{N}\widetilde{L}$, there exist $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$, $j = 1, \ldots, n$ with $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$. For $j = 1, \ldots, n$, $D_{\alpha_j} \in \mathcal{N}L$, so there exists a finite $E_j \in \mathcal{N}L$ with $E_j \leqslant D_{\alpha_j}$. Let $E = E_1 \wedge \ldots \wedge E_n$. Then E is a finite member of $\mathcal{N}L$. Since $E \leqslant C$, for each $e \in E$, there exists $c_e \in C$ such that $e \leqslant c_e$. Then $\{c_e : e \in E\}$ is a finite subcover of C, as desired. ∎

Example 4.5 This is an example of a nearness frame whose strong coreflection changes the underlying frame.
Let L be a regular frame that is not completely regular. Let $\mathcal{N}L$ be the finitely fine nearness on L. Let $(\widetilde{L}, \mathcal{N}\widetilde{L})$ be the strong coreflection of $(L, \mathcal{N}L)$. Since $(L, \mathcal{N}L)$ is obviously totally bounded, so is $(\widetilde{L}, \mathcal{N}\widetilde{L})$, by Lemma 4.4. This makes $(\widetilde{L}, \mathcal{N}\widetilde{L})$ uniform (see [14]) and hence $\widetilde{L}$ completely regular. So $\widetilde{L}$ is not the same as L. ∎

5 Familiar coreflections revisited

We now look briefly at three familiar coreflections of nearness frames: the totally bounded, the separable and the uniform. Of course, if one already knows the existence and description of these (as in [14] and [3], [25], [2]), Proposition 3.8 guarantees that they can be obtained by our general construction. The point here, however, is that these coreflections can be independently and easily obtained using our method.

Example 5.1 The totally bounded coreflection of nearness frames:
It is clear that total boundedness is preserved by uniform images. Let $(L, \mathcal{N}L)$ be a nearness frame, $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ the set of all its totally bounded approximations, and $(\widetilde{L}, \mathcal{N}\widetilde{L})$ the join of these. For $C \in \mathcal{N}\widetilde{L}$ there exist $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}, j = 1, \ldots, n$ with $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$. Since all $(L_\alpha, \mathcal{N}L_\alpha)$ are totally bounded, there exist finite $E_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$ with $E_{\alpha_j} \leqslant D_{\alpha_j}$ for each j. Then $E = E_{\alpha_1} \wedge \ldots \wedge E_{\alpha_n}$ is finite, $E \in \mathcal{N}\widetilde{L}$ and $E \leqslant C$. Theorem 3.6 gives the totally bounded coreflection. ∎

Example 5.2 The separable coreflection of nearness frames:
The argument is the same as in Example 5.1 above, with "finite" replaced by "countable" in the appropriate places. ∎

Example 5.3 The uniform coreflection of nearness frames:
It is familiar that a uniform image of a uniform frame is uniform. (See, for instance, [4], Lemma 2.2, p 25.) Let $(L, \mathcal{N}L)$ be a nearness frame, $\{(L_\alpha, \mathcal{N}L_\alpha) : \alpha \in I\}$ the set of all its uniform approximations and $(\widetilde{L}, \mathcal{N}\widetilde{L})$ the join of these. For $C \in \mathcal{N}\widetilde{L}$ there exist $D_{\alpha_j} \in \mathcal{N}L_{\alpha_j}, j = 1, \ldots, n$ with $D_{\alpha_1} \wedge \ldots \wedge D_{\alpha_n} \leqslant C$. Since all the $(L_\alpha, \mathcal{N}L_\alpha)$ are uniform, there exist $E_{\alpha_j} \in \mathcal{N}L_{\alpha_j}$ with $E_{\alpha_j} <^* D_{\alpha_j}$ for $j = 1, \ldots, n$. Then $E = E_{\alpha_1} \wedge \ldots \wedge E_{\alpha_n} \in \mathcal{N}\widetilde{L}$ and $E <^* C$. Theorem 3.6 again gives the uniform coreflection. ∎

Baboolal and Ori's description ([2]) of the uniform coreflection of nearness frames is as follows. Let $(L, \mathcal{N}L)$ be a nearness frame. Call $C \in \mathcal{N}L$ a *normal* *cover* with respect to $\mathcal{N}L$ if $C = C_1$ in some sequence, (C_n), of members of $\mathcal{N}L$ with $C_{n+1} <^* C_n$ for all $n = 1, 2, \ldots$. Denote the set of normal covers with respect to $\mathcal{N}L$ by $\mathcal{M}$. Define $k : L \to L$ by $k(x) = \bigvee\{y \in L : y \lhd_{\mathcal{M}} x\}$ where $y \lhd_{\mathcal{M}} x$ means that there exists $C \in \mathcal{M}$ with $Cy \leqslant x$. Let $M = \mathrm{Fix}(k)$ and $\mathcal{N}M = \{k[C] : C \in \mathcal{M}\}$. Then $(M, \mathcal{N}M)$ is the uniform coreflection of $(L, \mathcal{N}L)$.

We have already looked at situations where the strong coreflection of nearness frames does or does not change the underlying frame. We continue this theme in the lemma below.

Lemma 5.4 The uniform coreflection of an interpolating nearness frame does not change the underlying frame.

PROOF. Let $(L, \mathcal{N}L)$ be an interpolating nearness frame. We show that, for $x, y \in L$, $x \lhd_{\mathcal{N}L} y$ iff $x \lhd_{\mathcal{M}} y$ (using the notation of the paragraph above). Then $k(x) = x$ for all $x \in L$, making $\mathrm{Fix}(k) = L$.

Trivially $x \lhd_{\mathcal{M}} y$ implies that $x \lhd_{\mathcal{N}L} y$. Suppose then that $x \lhd_{\mathcal{N}L} y$. Then $\{x^*, y\} \in \mathcal{N}L$; we show that $\{x^*, y\} \in \mathcal{M}$. Take $a, b \in L$ with $x \lhd_{\mathcal{N}L} a \lhd_{\mathcal{N}L} b \lhd_{\mathcal{N}L} y$, and let $C = \{x^*, a\} \wedge \{a^*, b\} \wedge \{b^*, y\}$. Then $C \in \mathcal{N}L$ and a straightforward calculation shows that $C <^* \{x^*, y\}$. This procedure can clearly be repeated, making $\{x^*, y\}$ a normal cover. ∎

We briefly continue this exploration of familiar coreflections with a mention of some properties which involve only the underlying frame.

Example 5.5 The nearness frames with underlying frames that are completely regular form a coreflective subcategory of nearness frames. This follows because an image of a completely regular frame is completely regular and a subframe generated by completely regular subframes is again completely regular. A similar statement holds with "zero-dimensional" replacing "completely regular". (A frame is *zero-dimensional* if every element is a join of complemented elements.) ∎

We conclude this section by noting that the order in which coreflections in the category of nearness frames are taken is significant:

Remark 5.6 It is known that the totally bounded coreflection (in nearness frames) of a uniform frame is uniform. (See, for example, [10], p72.) Also, the uniform coreflection of a totally bounded nearness frame is totally bounded: this is clear from the description given by [2] or by an argument like that of Lemma 4.4 above. So the totally bounded and the uniform coreflections commute.
By contrast, we note that the totally bounded and the strong coreflections (in nearness frames) do not commute. This is not surprising, given that Dube and Mugochi ([15]) showed that, for interpolating nearness frames, their almost uniform coreflection does not commute with the totally bounded one. Indeed, their example ([15] Example 3.9) applies exactly to our context too.

6 Different home categories

In this section of the paper, we consider some other categories in which our general technique applies. For ease of reference we call such categories "home" categories.

Prenearness frames

Here we use prenearness and preuniform frames as home categories.

Proposition 6.1 Nearness frames form a coreflective subcategory of prenearness frames.

PROOF. The home category is prenearness frames and property P is the compatibility condition for nearness frames. (See Definition 2.1.) The entire discussion from Definition 3.1 through to Theorem 3.6 can be applied to prenearness frames instead of nearness frames, by simply omitting references to compatibility. It remains to show that using compatibility as property P complies with the conditions of Theorem 3.6: preservation by uniform images is straightforward, and the join of all P-approximations has property P by an argument identical to that in Proposition 3.2 (5). ∎

Proposition 6.2 (a) Uniform frames form a coreflective subcategory of preuniform frames.
(b) Uniform frames form a coreflective subcategory of prenearness frames.

PROOF. We omit the proofs since they proceed along lines now very familiar. ∎

The result of Proposition 6.2 (a) is not new, and can be obtained by a different argument. The lemma below appears in [11], where it is stated that the result is essentially an omitted exercise in [19]:

Lemma 6.3 (Lemma 2 of [11]) Any preuniformity $\mathcal{M}$ on L determines an interior operator k on L such that $M = \mathrm{Fix}(k)$ is a subframe of L and $\mathcal{U} = \{k[A] : A \in \mathcal{M}\}$ is a uniformity on M that generates $\mathcal{M}$. The map $k : L \to L$ is given by $k(a) = \bigvee\{x \in L : x \lhd_{\mathcal{M}} a\}$. Although the authors do not mention it, coreflectivity is easily seen from their result.

A similar argument to the above shows:

Proposition 6.4 The interpolating nearness frames form a coreflective subcategory of the interpolating prenearness frames.

We note that interpolation is used to show that the map k in question is idempotent, but omit the details of the proof.

As far as we can see, the method of proof of Lemma 6.3, using an interior operator, cannot be used to prove Proposition 6.1.

Nearness σ-frames

We give here just some of the details of nearness σ-frames; more details can be found in [26]. A σ-frame L is a lattice with top and bottom element in which all countable joins exist and where finite meets distribute over countable joins. A cover is defined to be a *countable* set whose join is the top element. A σ-frame map preserves top, bottom, finite meets and countable joins. A nearness on a σ-frame L is a non-empty collection, $\mathcal{N}L$, of (necessarily countable) covers of L filtered by finite meet and refinement such that, for any $x \in L$, x is a countable join of elements uniformly below it (compatibility). Members of $\mathcal{N}L$ are called uniform covers. Uniform maps preserve uniform covers. This gives the category **NearσFrm**. Sub nearness σ-frames, the partial order on sub nearness σ-frames and P-approximations are defined in the obvious way. We mention only that in constructing a join of sub nearness σ-frames, we consider only elements that can be written as *countable* joins of finite meets of elements from the given sub nearness σ-frames. This then allows one to consider all the constructions mooted in Section 3 of this paper; in particular, we can construct the sub nearness σ-frame $\Gamma_P(L, \mathcal{N}L)$ for any property P and any nearness σ-frame $(L, \mathcal{N}L)$.

The proposition below indicates that nearness σ-frames form a suitable home category.

Proposition 6.5 (a) The collection of all sub nearness σ-frames of a nearness σ-frame forms a complete lattice.
(b) If P is a property preserved by uniform images, then Γ_P : **NearσFrm** $\to$ **NearσFrm** is a functor.
(c) Let P be a property preserved by uniform images and such that the join of all P-approximations has property P. Then the nearness σ-frames with property P form a full (mono)coreflective subcategory of all nearness σ-frames.

PROOF. Omitted. ∎

Applying the general method as outlined before allows us to conclude the following. We note that (a) is new; (b) and (c) appear in [24].

Corollary 6.6 (a) The strong nearness σ-frames form a coreflective subcategory of **NearσFrm**.
(b) The uniform σ-frames form a coreflective subcategory of **NearσFrm**.
(c) The totally bounded nearness σ-frames form a coreflective subcategory of **NearσFrm**.

PROOF. Apply Proposition 6.5 with arguments similar to those of Proposition 4.1, Example 5.3 and Example 5.1. ∎

Subcategories of nearness frames as home categories

We note that it is possible to use some subcategories of nearness frames as a suitable home category and then apply our general method to identify some of their coreflective subcategories.

Proposition 6.7 (a) The strong totally bounded nearness frames form a coreflective subcategory of all totally bounded nearness frames.
(b) The strong totally bounded nearness frames form a coreflective subcategory of all strong nearness frames.

PROOF. (a) Use totally bounded nearness frames as home category; this involves checking that the join of totally bounded sub nearness frames is again totally bounded, of course. For property P use the strong property.
(b) Use strong nearness frames as home category and for property P use the totally bounded property. ∎

Applications to unstructured frames

We conclude with two corollaries that link this work with familiar coreflections in the unstructured setting. For this, we note that, for any frame L (not necessarily regular), $(L, \mathrm{Cov}L)$ is a prenearness frame. Here $\mathrm{Cov}L$ denotes all covers of L.

Corollary 6.8 (a) For any frame L, the nearness coreflection of the prenearness frame $(L, \mathrm{Cov}L)$ provides the regular coreflection of L as its underlying frame.
(b) For any frame L, the uniform coreflection of the prenearness frame $(L, \mathrm{Cov}L)$ provides the completely regular coreflection of L as the underlying frame.

PROOF. (a) Denote by $i : (M, \mathcal{N}M) \to (L, \mathrm{Cov}L)$ the nearness coreflection of $(L, \mathrm{Cov}L)$. We show that $i : M \to L$ is the regular coreflection of L. The frame M is obviously regular, since $(M, \mathcal{N}M)$ is a nearness frame. Let $f : N \to L$ be a frame map from a regular frame N into L. Let $\mathcal{N}N$ be any

nearness structure compatible with N (for example, Cov$N$ would do). Then $f : (N, \mathcal{N}N) \to (L, \mathrm{Cov}L)$ is a uniform map, and so there exists a unique uniform map $h : (N, \mathcal{N}N) \to (M, \mathcal{N}M)$ such that $ih = f$. Then $h : N \to M$ provides the desired factorization, which is unique since i is $1 - 1$.
(b) A similar argument to (a) above applies with uniformities replacing nearnesses. ∎

We remark that the completely regular coreflection of a frame can also be obtained by applying Proposition 6.2 (a) (instead of Proposition 6.2 (b)) and using normal covers of L instead of all covers of L.

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Anneliese Schauerte
Anneliese.Schauerte@uct.ac.za

REFERENCES

John Frith
John.Frith@uct.ac.za

Department of Mathematics and Applied Mathematics
University of Cape Town
Private Bag Rondebosch
7701
South Africa