On $\mathbf{G_{k-1,k}}$-defectivity of smooth surfaces and threefolds

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Abstract.— *In this paper, we prove a rough characterization for $G_{k-1,k}$-defective n-dimensional non-degenerate varieties $X \subset \mathbb{P}^N$ if $k \geq n$. In the case of smooth surfaces or threefolds, we give a fine classification.*

Keywords.— *Grassmann-defectivity; secant varieties; surfaces; threefolds.*

1 Introduction

Let X be an irreducible non-degenerate projective variety of dimension n in $\mathbb{P}^N$ and let h and k be integers such that $0 \leq h \leq k \leq N$. Then $G_{h,k}(X)$ is the closure in $\mathbb{G}(h, N)$ of the set of h-dimensional linear subspaces contained in the span of $k+1$ different points of X and is called the h-Grassmannian of $(k+1)$-secant k-planes of X. We say that X is $G_{h,k}$-defective if the dimension of $G_{h,k}(X)$ is smaller then the expected dimension, which is the minimum between $(h+1)(N-h)$ and $(k+1)n + (k-h)(h+1)$.

In case $h = 0$, the variety $G_{0,k}(X)$ is just the k-th secant variety $S_k(X)$ of X. A variety X is called k-defective if it is $G_{0,k}$-defective. Such varieties are intensively studied in [16].

In case $h > 0$, little is known. The most important reason for this is the lack of a so-called Terracini lemma, which in case $h = 0$ gives a description for the tangent space on $S_k(X)$ in a general point. Nevertheless, for example in [4] is shown that irreducible curves are not $G_{h,k}$-defective and in [5] there is given a classification of surfaces with $G_{1,2}$-defect. There is also a rough classification for varieties having $G_{n-1,n}$-defect together with a fine classification for $G_{2,3}$-defective smooth threefolds (see [7]).

Beside the intrinsic importance of $G_{h,k}$-defective varieties, defective varieties

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are also important for some extrinsic reasons. For example, varieties with $G_{h,k}$-defect have a strange behaviour under projections. Waring's problem for forms (see [2, 6, 9]) gives us another extrinsic reason for studying defective varieties. This problem is in connection with the $G_{h,k}$-behaviour of Veronese embeddings of projective spaces.

In this paper we will classify the smooth surfaces X in $\mathbb{P}^N$ with $G_{k-1,k}$-defect for $k > 2$.

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate surface and $k > 2$. Then X is $G_{k-1,k}$-defective if and only if $N = k+3$ and X is of minimal degree $k+2$.*

We will also give a full classification of smooth threefolds $X \subset \mathbb{P}^N$ with $G_{k-1,k}$-defect for $k > 3$.

Theorem 1.2. *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate threefold and $k > 3$. Then X is $G_{k-1,k}$-defective if and only if X is one of the following varieties:*

1. *X is a threefold of minimal degree $k+2$ in $\mathbb{P}^{k+4}$;*
2. *X is a threefold of minimal degree $k+3$ in $\mathbb{P}^{k+5}$;*
3. *X is the projection in $\mathbb{P}^{k+4}$ of a threefold of minimal degree $k+3$ in $\mathbb{P}^{k+5}$;*
4. *$k = 4$ and X is the (linearly normal) embedding in $\mathbb{P}^8$ of the blowing-up of $\mathbb{P}^3$ at a point.*
5. *$k = 5$ and X is the image of the 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$.*

Compared with the classification of smooth $G_{2,3}$-defective varieties with $N \geq 7$ (see [7]), the first three cases are totally analogous.

Before proving Theorem 1.1 and Theorem 1.2 we will first give a rough characterization for $G_{k-1,k}$-defective n-dimensional varieties with $k \geq n$. Here we don't require that X needs to be smooth.

Proposition 1.3. *Let X be an n-dimensional variety in $\mathbb{P}^N$ and let $k \geq n$ be an integer. Then X is $G_{k-1,k}$-defective if and only if $N \geq n+k+1$ and one of the following properties hold for $k+1$ general points $P_0, \dots, P_k$ on X:*

1. *For each $i \in \{0, \dots, k\}$, there exists a line L_i on X containing P_i such that the linear span of the lines has dimension $k+1$.*
2. *There exists a rational normal curve Γ of degree $k+1$ on X containing $P_0, \dots, P_k$.*

We can see that both properties are enough for $G_{k-1,k}$-defectivity. In case n is equal to 2 or 3, we will prove that the first property is the same as saying that X is a cone (see Section 4). If X satisfies the second property, we will prove that X has sectional genus at most $n-2$ (see Section 5).

2 Some conventions and generalities

2.1. Conventions. We denote the N-dimensional projective space over the field $\mathbb{C}$ of the complex numbers by $\mathbb{P}^N$. We write $\mathbb{G}(h,N)$ to denote the Grassmannian of h-dimensional linear subspaces of $\mathbb{P}^N$.

An n-dimensional variety X in $\mathbb{P}^N$ is an irreducible reduced n-dimensional Zariski-closed subset of $\mathbb{P}^N$. We say that a variety $X \subset \mathbb{P}^N$ is non-degenerate if X is not contained in a hyperplane of $\mathbb{P}^N$.

Let X be a non-degenerate n-dimensional variety in $\mathbb{P}^N$. We say that a closed subscheme $Y \subset X$ is a m-dimensional section of X if Y is the scheme-theoretical intersection of X with a linear subspace $\mathbb{P}^{N-n+m}$ of $\mathbb{P}^N$ such that all irreducible components have dimension m. We will often use the notions of curve section, surface section and hyperplane section in case m is equal to respectively 1, 2 and $n-1$.

The linear span $\langle Y \rangle$ of a closed subscheme Y of $\mathbb{P}^N$ is the intersection of all hyperplanes $H \subset \mathbb{P}^N$ containing Y as a closed subscheme. This linear span is always a linear subspace of $\mathbb{P}^N$. If $P_0, \ldots, P_r$ are different points of $\mathbb{P}^N$, we write $\langle P_0, \ldots, P_r \rangle$ to denote the linear span of the reduced subscheme of $\mathbb{P}^N$ supported by those points.

Let Y be a closed subscheme of $\mathbb{P}^N$ and let $P \in Y$. We can take a hyperplane $H \subset \mathbb{P}^N$ such that $P \notin H$ and identify $\mathbb{P}^N \backslash H$ with the affine space $\mathbb{A}^N$ and $Y \backslash (Y \cap H)$ with a closed subscheme of $\mathbb{A}^N$ (containing P). We can define the Zariski-tangent space $T_P(Y \backslash (Y \cap H)) \subset \mathbb{A}^N$ by using the equations of the subscheme $Y \backslash (Y \cap H)$. Its closure in $\mathbb{P}^N$ is called the embedded tangent space $\mathbb{T}_P(Y)$ in $\mathbb{P}^N$ of Y at P.

If D_1 and D_2 are divisors on a smooth surface S, we will write $D_1.D_2$ to denote the intersection number of those divisors. If D is an effective divisor on S, then saying D is irreducible means D is integral (i.e. also reduced) by convention.

2.2. Definition of $\mathbf{G_{k-1,k}(X)}$. Let $X \subset \mathbb{P}^N$ be a non-degenerate n-dimensional variety and let $k \leq N$ be an integer. The set of points $(P_0, \ldots, P_k)$ in X^{k+1} with $\dim(\langle P_0, \ldots, P_k \rangle) = k$ is non-empty and open; so we have a rational map $\omega : X^{k+1} \dashrightarrow \mathbb{G}(k,N)$. An element of the image of ω is called a $(k+1)$-secant

k-plane of X. Consider the incidence diagram

$$\begin{array}{ccc} & I & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{G}(k-1,N) & & \mathbb{G}(k,N) \end{array}$$

with $I = \{(A,B) |\, A \subset B\} \subset \mathbb{G}(k-1,N) \times \mathbb{G}(k,N)$. Now we define $G_{k-1,k}(X)$ as being $\alpha(\beta^{-1}(\overline{\mathrm{im}(\omega)}))$ (this is equal to the closure of the set of $(k-1)$-dimensional subspaces of $\mathbb{P}^N$ contained in some $(k+1)$-secant k-plane of X). Since the fibers of β are irreducible and k-dimensional, we find that the expected dimension of $G_{k-1,k}(X)$ is equal to

$$\mathrm{expdim}(G_{k-1,k}(X)) = \min\{(k+1)n + k, k(N-k+1)\}.$$

If $\dim(G_{k-1,k}(X))$ is smaller then this expected dimension, we say that X has $G_{k-1,k}$-defect.

It is easy to see that in case $k \geq n$ the expected dimension of $G_{k-1,k}(X)$ is equal to $(k+1)n + k$ if and only if $N \geq n + k + 1$.

If $\dim(G_{k-1,k}(X)) = (k+1)n + k - a$ and $N \geq n + k + 1$, for a general element $H \in G_{k-1,k}(X)$ the set of $(k+1)$-secant k-planes of X containing H has dimension a.

2.3. Let X be a non-degenerate variety in $\mathbb{P}^N$ and let $k \leq N$ be an integer. From Proposition 1.1 in [5] it follows that $G_{k,k}(X) := \overline{\mathrm{im}(\omega)}$ is equal to $\mathbb{G}(k,N)$ if $N \leq n + k$. Hence, X is not $G_{k-1,k}$-defective if $N \leq n + k$ since in this case $G_{k-1,k}(X) := \alpha(\beta^{-1}(\overline{\mathrm{im}(\omega)})) = \mathbb{G}(k-1,N)$. If $k > n$, this also follows from [9].

2.4. Let $X \subset \mathbb{P}^N$ be a non-degenerate n-dimensional variety with $N \geq n+k+1$ for some integer k and let $P_0, \ldots, P_k$ be general points on X. Then these $k+1$ points are contained in a general curve section of X in some $\mathbb{P}^{N-n+1 \geq k+2}$. So the uniform position lemma for curves (see [1] and [3, Proposition 2.6] for the argument) implies that $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$ as a scheme. This implies that $\omega : X^{k+1} \dashrightarrow \mathbb{G}(k,N)$ is generically injective.

2.5. Polarized varieties. A polarized variety is a pair $(V, \mathcal{S})$ such that V is an abstract projective variety and $\mathcal{S}$ is an ample invertible sheaf on V.

2.5.1. Examples. If $X \subset \mathbb{P}^N$ is a variety and $\mathcal{O}_X(1)$ is the restriction to X of the twisting sheaf of Serre $\mathcal{O}_{\mathbb{P}^N}(1)$, the pair $(X, \mathcal{O}_X(1))$ is a polarized variety. Another important example can be given by taking an abstract projective variety V and a locally free sheaf $\mathcal{E}$ on V. Let $\mathbb{P}(\mathcal{E})$ be the projective bundle associated to $\mathcal{E}$ and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the associated tautological sheaf (see [12, p. 162]). If this sheaf is ample then $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is a polarized variety and is called a scroll on V.

2.5.2. Sectional genus. For a polarized variety we can define the notion of sectional genus (for a general definition, see [10]). If $\mathcal{S}$ is very ample on V and $V \subset \mathbb{P}^N$ is the embedding of V using the global sections of $\mathcal{S}$, then the sectional genus of $(V, \mathcal{S})$ is defined as being the arithmetic genus of a general curve section of $V \subset \mathbb{P}^N$.

The classification of smooth polarized varieties $(V, \mathcal{S})$ of sectional genus at most one is given in [10, Section 12]. We only consider the case where $V = X \subset \mathbb{P}^N$ and $\mathcal{S} = \mathcal{O}_X(1)$ with $n = \dim(X) = 3$ and $N \geq 8$.

If the sectional genus is 0 we only have scrolls of vectorbundles on $\mathbb{P}^1$ as possibilities. Moreover, if X is embedded using the complete linear system then X is of minimal degree, so $\deg(X) = N - 2$. We can obtain all smooth threefolds $X \subset \mathbb{P}^N$ of minimal degree in this way.

If the sectional genus is equal to 1, the only possibilities are scrolls of vectorbundles on elliptic curves and Del Pezzo varieties. In our situation a Del Pezzo variety is one of the following possibilities (see [10, Section 8]):

i. $\deg(X) = 7$; X is isomorphic to the blowing-up $\sigma : Bl_Q(\mathbb{P}^3) \to \mathbb{P}^3$ at one point Q and $\mathcal{O}_X(1) \cong \sigma^*(\mathcal{O}_{\mathbb{P}^3}(2)) \otimes \mathcal{O}_{Bl_Q(\mathbb{P}^3)}(-E)$ where E is the exceptional divisor.

ii. $\deg(X) = 8$ and $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

2.6. Theorems of Bertini. Let $\mathcal{L}$ be a linear system on a smooth projective variety V without fixed components. Then, for a general element $D \in \mathcal{L}$ the singular locus $\mathrm{Sing}(D)$ is contained in the locus of fixed points of $\mathcal{L}$ on V and D is irreducible unless $\mathcal{L}$ is composed with a pencil. For the proofs of this properties, see [13, 17, 18].

A linear system $\mathcal{L}$ is composed by a pencil if and only if there exists a morphism $f : W \to C$ with $\sigma : W \to V$ the blowing-up of V at the fixed points of $\mathcal{L}$ and C a curve such that the following holds. There is a linear system $\mathcal{L}'$ on C with $\dim(\mathcal{L}) = \dim(\mathcal{L}')$ such that for all $D \in \mathcal{L}$ there exists a $D' \in \mathcal{L}'$ such that $D = \sigma(f^{-1}(D'))$. Using a Stein factorization and a desingularization for W, one can see that we can assume that the general fibre of f is irreducible.

2.7. If D_1 is an irreducible reduced divisor on a smooth projective variety V and D_2 is an effective divisor on V linear equivalent to D_1, then D_2 is connected. For an argument, see Section 2.6 in [7].

3 A rough characterization

Proof of Proposition 1.3: Let $X \subset \mathbb{P}^N$ be an n-dimensional variety with $G_{k-1,k}$-defect for some $k \geq n$. From Sec. 2.3 it follows that $N \geq n+k+1$, hence $\dim(G_{k-1,k}(X)) < (k+1)n + k$ (Sec. 2.2).

Take $H \in G_{k-1,k}(X)$ general and consider the closure in X^{k+1} of the set of points $(P_0, \dots, P_k)$ with $P_i \neq P_j$ for all $i \neq j$ and $H \subset \langle P_0, \dots, P_k \rangle$. Let a be its dimension and let $\Omega_{H,k}$ be an a-dimensional component of that set. We know that $a \geq 1$. Take a general element $(P_0, \dots, P_k)$ of $\Omega_{H,k}$. Since we have chosen $H \in G_{k-1,k}(X)$ generally, $(P_0, \dots, P_k)$ is a general element of X^{k+1}. In particular, $\langle P_0, \dots, P_k \rangle \cap X = \{P_0, \dots, P_k\}$ as a scheme. Now let $(Q_0, \dots, Q_k)$ be another general element of $\Omega_{H,k}$.

Claim 1. *For each $i \in \{0, \dots, k\}$ one has $Q_i \notin \{P_0, \dots, P_k\}$.*

Proof Claim 1: analogous to the proof of Claim 3.1 in [7]. □

Write $L = \langle P_0, \dots, P_k \rangle$ and $M = \langle Q_0, \dots, Q_k \rangle$. Since $L \neq M$, $\dim(L) = \dim(M)$ and $H \subset L \cap M$; one has $H = L \cap M$ and $\dim(\langle L \cup M \rangle) = k+1$. Write $\mathbb{P}^{k+1} = \langle L \cup M \rangle$.

Claim 2. *$\mathbb{P}^{k+1} \cap X$ is not finite.*

Proof Claim 2: Assume $\mathbb{P}^{k+1} \cap X$ is finite.

Subclaim 2.1. *A general linear subspace of $\mathbb{P}^N$ of dimension $N-n+1$ containing $\mathbb{P}^{k+1} \cap X$ gives rise to an irreducible curve section of X smooth at $P_0, \dots, P_k$.*

Proof Subclaim 2.1: analogous to the proof of Subclaim 3.3 in [7]. □

Denote by Ψ_0' the closure of the set of elements $(P_0, \dots, P_k; Q_0, \dots, Q_k)$ in $X^{k+1} \times X^{k+1}$ such that $\dim(\langle P_0, \dots, P_k \rangle) = k$, $P_i \neq P_j$ and $Q_i \neq Q_j$ for $i \neq j$, $\{P_0, \dots, P_k\} \neq \{Q_0, \dots, Q_k\}$ and $H \subset \langle Q_0, \dots, Q_k \rangle$ for some $(k-1)$-dimensional linear subspace H of $\langle P_0, \dots, P_k \rangle$.

Subclaim 2.2. *There exists an irreducible component Ψ_0 of Ψ_0' of dimension $(k+1)n + k + a$ dominating the first factor X^{k+1}.*

Proof Subclaim 2.2: analogous to the proof of Subclaim 3.4 in [7]. □

Now consider the closure $\Psi_1 \subset \Psi_0 \times \mathbb{G}(N-n+1, N)$ of the set of pairs $(P_0, \dots, P_k; Q_0, \dots, Q_k; G)$ with the dimension of $\langle P_0, \dots, P_k, Q_0, \dots, Q_k \rangle$ equal to $k+1$ and $\langle P_0, \dots, P_k, Q_0, \dots, Q_k \rangle \subset G$. The dimension of a general fibre of the projection $\Psi_1 \to \Psi_0$ is $(N-n-k)(n-1)$, hence $\dim(\Psi_1) = (k+1)n + k + a + (N-n-k)(n-1)$. This implies that a general non-empty fiber of $\tau : \Psi_1 \to \mathbb{G}(N-n+1, N)$ has dimension at least $(k+1)n + k + a + (N-n-k)(n-1) - (N-n+2)(n-1) = 2k - n + 2 + a$.

For $G \in \tau(\Psi_1)$ general we have by Subclaim 2.1 that $G \cap X$ is an irreducible curve $C \subset \mathbb{P}^{N-n+1}$ spanning $\mathbb{P}^{N-n+1}$. So we find a subset $S \subset C^{2k+2}$ of dimension $2k - n + 2 + a \geq k + 3$ such that for $(P_0, \ldots, P_k, Q_0, \ldots, Q_k) \in S$ the points impose at most $k+2$ conditions on hyperplanes. Since we can choose $k+3$ of those points general on C, we conclude that $k+3$ general points of C do not impose independent conditions on hyperplanes. Hence, $N - n + 1 \leq k + 1$ and so $N \leq n + k$. This gives us a contradiction. □

Now we know that $\dim(\mathbb{P}^{k+1} \cap X) \geq 1$. Since $\dim(L \cap X) = 0$ and L is a hyperplane in $\mathbb{P}^{k+1}$, we find $\dim(\mathbb{P}^{k+1} \cap X) = 1$. Denote by Γ an irreducible curve in $\mathbb{P}^{k+1} \cap X$.

Claim 3. *Either $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_k\}$ or $\Gamma \cap \{P_0, \ldots, P_k\}$ is only one point. In the second case $\mathbb{P}^{k+1} \cap X$ contains a line L_i with $L_i \cap \{P_0, \ldots, P_k\} = \{P_i\}$ for each $i \in \{0, \ldots, k\}$.*

Proof Claim 3: Assume that $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_l\}$ for some $0 \leq l < k$. Let m be an integer such that $l < m \leq k$. We will now prove using a monodromy argument that there exists another component $\Gamma' \subset \mathbb{P}^{k+1} \cap X$ such that $\Gamma' \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_{l-1}, P_m\}$.

Let $\Theta_1 \subset X^{k+1} \times \mathbb{G}(k-1, N)$ be the closure of the set of points $((P_0, \ldots, P_k), H)$ such that $P_i \neq P_j$ for $i \neq j$, $\dim(\langle P_0, \ldots, P_k \rangle) = k$ and $H \subset \langle P_0, \ldots, P_k \rangle$. Consider the projections $p_{1,1} : \Theta_1 \to X^{k+1}$ and $p_{1,2} : \Theta_1 \to \mathbb{G}(k-1, N)$. Since $p_{1,1}$ is surjective with irreducible general fibers of dimension k, we see that Θ_1 also is irreducible and of dimension $(k+1)n + k$. The fibers of $p_{1,2}$ have dimension at least a. Denote $\Theta_1 \times_{\mathbb{G}(k-1,N)} \Theta_1$ by Θ_2 and consider the projections $p_{2,i} : \Theta_2 \to \Theta_1$ onto the i-th factor for $i \in \{1, 2\}$. Let Δ be the diagonal of Θ_1 in Θ_2. If $((P_0, \ldots, P_k), H)$ is a general element of Θ_1 then $p_{2,2}(p_{2,1}^{-1}((P_0, \ldots, P_k), H))$ contains $\Omega_{H,k}$ as an irreducible component; more precisely, $\Omega_{H,k}$ corresponds to the irreducible component of $p_{2,1}^{-1}((P_0, \ldots, P_k), H)$ intersecting Δ. It follows that Δ is contained in a unique irreducible component Θ of Θ_2. If $p_1 : \Theta \to \Theta_1$ denotes the restriction of the projection $p_{2,1}$ to Θ, we obtain $p_1^{-1}((P_0, \ldots, P_k), H) = \Omega_{H,k}$. Consider $\Theta \subset X^{k+1} \times X^{k+1} \times \mathbb{G}(k-1, N)$ and let $\Theta_3 \subset \Theta \times X$ be the set of elements $(((P_0, \ldots, P_k), (Q_0, \ldots, Q_k), H), R)$ with $R \in \langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$. By assumption, there is a curve Γ in the fibre of $p_3 : \Theta_3 \to \Theta$ with $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_l\}$. Let Θ_4 be the irreducible component of the Hilbert scheme parameterizing curves in fibres of the projection p_3 containing the point that parameterizes Γ. Let $q : \Theta_4 \to \Theta$ be the natural morphism. Let $\Xi \subset \Theta_4 \times X$ be the universal curve and let $q' : \Theta_4 \times X \to \Theta_4$ be the projection. Consider the sections $\mathcal{S}_i : \Theta_4 \to \Theta_4 \times X$ with $\mathcal{S}_i(z) = (z, P_i)$ if $q(z) = ((P_0, \ldots, P_k), (Q_0, \ldots, Q_k), H)$. For a general point z of Θ_4 we have $\mathcal{S}_i(z) \in \Xi$ if and only if $i \in \{0, \ldots, l\}$. By construction and assumption, Θ_4 is irreducible and q is surjective. Let $z' \in \Theta_4$ with $q(z') = ((P_0, \ldots, P_{l-1}, P_{l+1}, P_l, \ldots, P_k), (Q_0, \ldots, Q_k), H)$. The point $q(z')$ belongs to Θ because $\Omega_{H,k}$ is determined by H and $\{P_0, \ldots, P_k\}$, thus independent of the order of the points $P_0, \ldots, P_k$. Hence, $z' \in \Theta_4$ corresponds to a curve $\Gamma' \subset \mathbb{P}^{k+1} \cap X$ with $P_0, \ldots, P_{l-1}, P_{l+1} \in \Gamma'$. So, we have proved the statement above for $m = l+1$; analogous we can prove the statement for other values of m.

When we take $l = 0$ we immediately get the second part of the statement of the Claim. If $l > 0$, $P_0 \in \Gamma \cap \Gamma' \subset \mathbb{P}^{k+1} \cap X$ hence $\dim(\mathbb{T}_{P_0}(\mathbb{P}^{k+1} \cap X)) \geq 2$. Thus we get a contradiction because $\dim(\mathbb{T}_{P_0}(L \cap X)) = 0$. So we proved also the first part of the statement of the Claim. $\square$

If $\Gamma \cap \{P_0, \ldots, P_k\} = \{P_0, \ldots, P_k\}$, we find $\Gamma \cap L = \{P_0, \ldots, P_k\}$ as a scheme because $X \cap L = \{P_0, \ldots, P_k\}$ as a scheme and $\Gamma \subset X$. Hence $\deg(\Gamma) = k+1 = \mathrm{codim}_{\mathbb{P}^{k+1}}(\Gamma) + 1$ and so Γ is a rational normal curve. In this case, we find that $k+1$ general points on X are contained in a rational normal curve of degree $k+1$ on X. $\blacksquare$

4 The first case of the characterization

Here we will study the first case occurring in the Proposition: for general points $P_0, \ldots, P_k \in X$ there exist lines L_i on X containing P_i for each $i \in \{0, \ldots, k\}$ such that $\dim\langle L_0, \ldots, L_k\rangle = k+1$. Remember that a generally chosen element $(P_0, \ldots, P_k, Q_0, \ldots, Q_k, H) \in \Theta$ determines $L_0, \ldots, L_k$ uniquely. By monodromy on Θ, a property that holds for some subset of $\{L_0, \ldots, L_k\}$ holds for each subset of the same cardinality.

Claim 4.1. *If k lines of $\{L_0, \ldots, L_k\}$ span a linear subspace of dimension k, then X is a cone.*

Proof: analogous to the proof of Claim 3.6 in [7]. $\square$

Assume that X is not a cone. From Claim 4.1, we know $\dim(\langle L_1, \ldots, L_k\rangle) \neq k$, hence $\langle L_1, \ldots, L_k\rangle = \mathbb{P}^{k+1}$. Notice that $\dim(\langle L_i, P_1 \ldots, P_k\rangle) = k$ for all $i \in \{1, \ldots, k\}$ because $L_i \not\subset \langle P_1, \ldots, P_k\rangle$.

Now, let $1 \leq i < j \leq k$. If $\dim(\langle L_i, L_j, P_1 \ldots, P_k\rangle) = k$, then it follows that $L_j \subset \langle L_i, P_1 \ldots, P_k\rangle$ and thus $L_l \subset \langle L_i, P_1 \ldots, P_k\rangle$ for each $l \in \{1, \ldots, k\}$ by monodromy. Hence, $\dim(\langle L_1,\ldots,L_k\rangle)=k$, a contradiction. So $\langle L_i,L_j,P_1\ldots,P_k\rangle=\mathbb{P}^{k+1}$.

Now fix $P_1,\ldots,P_k$ on X and let $P_0(t)$ be a 1-parameterfamily on X with $P_0(0)=P_0$. Consider also a 1-parameterfamily $H(t)\subset\langle P_0(t),P_1,\ldots,P_k\rangle$ of linear subspaces of dimension $k-1$ with $H(0)=H$ and 1-parameterfamilies $Q_0(t),\ldots,Q_k(t)$ on X with $Q_i(0)=Q_i$ for each i and $H(t)\subset\langle Q_0(t),\ldots,Q_k(t)\rangle$. Those families imply the existence of 1-parameterfamilies $L_0(t),\ldots,L_k(t)$ of lines on X with $L_i(0)=L_i$ for $i\in\{0,\ldots,k\}$, $P_i\in L_i(t)$ for all $i\in\{1,\ldots,k\}$, $P_0(t)\in L_0(t)$ and $\dim(\langle L_0(t),\ldots,L_k(t)\rangle)=k+1$ for each value of the parameter t. We may assume that $P_0(t)\notin\mathbb{P}^{k+1}$ for general values of t. If $L_i(t)=L_i$ for all $i\in\{1,\ldots,k\}$ and for a general value of t, then $P_0(t)\in\langle P_0(t),P_1,\ldots,P_k\rangle\subset\langle L_1(t),\ldots,L_k(t)\rangle=\mathbb{P}^{k+1}$, a contradiction. By monodromy we can assume that $L_i(t)\neq L_i$ for all $i\in\{0,\ldots,k\}$.

So there is a family of lines on X through each general point of X.

Remark 4.2. If X is a surface, one can easily see that this situation cannot occur.

Proposition 4.3. *Let $X\subset\mathbb{P}^N$ ($N\geq k+4, k\geq 3$) be a threefold such that for $k+1$ general points $P_0,\ldots,P_k$ on X there exist lines $L_0,\ldots,L_k$ on X such that $P_i\in L_i$ for $i\in\{0,\ldots,k\}$ and dim$(\langle L_0,\ldots,L_k\rangle)=k+1$, then X is a cone.*

Proof: Assume that X is not a cone. For a general point P on X there exists a 1-dimensional family of lines on X through P. Hence, X contains a 3-dimensional family of lines. By [14] or [15], X is embedded in $\mathbb{P}^N$ as a $\mathbb{P}^2$-bundle over a curve K. Let K_P be the 2-dimensional component of the union of all lines on X through P. We know that K_P is a plane. Using a 1-parameterfamily $P_0(t)$ we find 1-parameterfamilies $L_1(t)$ and $L_2(t)$ in respectively K_{P_1} and K_{P_2}. We have

$$\langle P_0(t),P_1,\ldots,P_k\rangle\subset\langle L_1(t),L_2(t),P_3,\ldots,P_k\rangle\subset\langle K_{P_1},K_{P_2},P_3,\ldots,P_k\rangle.$$

Since $\dim(\langle K_{P_1},K_{P_2},P_3,\ldots,P_k\rangle)\leq k+3$ and thus $X\not\subset\langle K_{P_1},K_{P_2},P_3,\ldots,P_k\rangle$, we can choose the parameterfamily $P_0(t)$ such that $P_0(t)\notin\langle K_{P_1},K_{P_2},P_3,\ldots,P_k\rangle$ for general values of the parameter t. This gives us a contradiction and finishes the proof. ■

5 The second case of the characterization

Proposition 5.1. *Let $X\subset\mathbb{P}^N$ ($N\geq n+k+1, k\geq n$) be an n-dimensional variety such that for $k+1$ general points $P_0,\ldots,P_k$ on X there exists a rational normal curve Γ on X of degree $k+1$ containing $P_0,\ldots,P_k$. Then, the geometric genus of a general curve section of X is at most $n-2$.*

Proof: Denote the family of rational normal curves of degree $k+1$ on X by $\{\Gamma\}$. By assumption, $\dim(\{\Gamma\}) \geq (k+1)n - (k+1) = (n-1)(k+1)$.

Because $k \leq N - n + 1$, $k+1$ general points on X are contained in a curve section of X. So, taking $k+1$ general points $P_0, \ldots, P_k$ on X can be done by first taking a general curve section C' of X and then considering $k+1$ general points on C'. Bertini's theorems imply that C' is irreducible and smooth at $P_0, \ldots, P_k$. Write $C' = X \cap G'_0$ with G'_0 a linear subspace of $\mathbb{P}^N$ of dimension $N - n + 1$. Consider a general linear subspace $H \subset L = \langle P_0, \ldots, P_k \rangle$ of dimension $k-1$ and let $(Q_0, \ldots, Q_k)$ be a general element of $\Omega_{H,k}$. Hence, $G' = \langle G'_0 \cup \{Q_0\} \rangle \subset \mathbb{P}^N$ is a linear subspace of dimension $N - n + 2$. Consider $S' = X \cap G'$. Since C' is a irreducible curve and G'_0 is a hyperplane of G', we find that S' is an irreducible surface. Since C' is smooth at $P_0, \ldots, P_k$ we see that S' is smooth at $P_0, \ldots, P_k$.

Let $I' \subset \{\Gamma\} \times \mathbb{G}(N - n + 2, N)$ be the inclusion relation. The dimension of a general fibre of $I' \to \{\Gamma\}$ is $(N - n - k + 1)(n - 2)$. Hence, we obtain a irreducible component I of I' containing (Γ, G') of dimension greater than or equal to $(N-n-k+1)(n-2)+(k+1)(n-1)$, with Γ the rational normal curve contained in $X \cap \langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$. Consider the projection $\nu : I \to \mathbb{G}(N-n+2, N)$. The dimension of a general non-empty fibre of ν is at least

$$(N - n - k + 1)(n - 2) + (k + 1)(n - 1) - (N - n + 3)(n - 2) = k - n + 3.$$

If we consider the fibre above G', we find that S' contains a subfamily of $\{\Gamma\}$ of dimension at least $k - n + 3$. Let S be the minimal resolution of singularities of S'. We become a family $\{\gamma\}$ of rational curves on S of dimension at least $k - n + 3$ by considering the strict transforms of the curves in $\{\Gamma\}$ on S'. Denote the strict transforms on S of Γ and C' by resp. γ and C''. Any two points of S can be connected by means of a rational curve in $\{\gamma\}$. This implies $h^1(S, \mathcal{O}_S) = 0$, so the family $\{\gamma\}$ is contained in a linear system $\{\gamma\}$ of dimension at least $k - n + 3$. This linear system induces a linear system $|g|$ on the normalization C of C''. Since S' is smooth at $P_0, \ldots, P_k$, we find that S and S' are isomorphic above neighborhoods of those points. Since $\dim(|C'' - \gamma|) \geq 1$ (C'' is a divisor corresponding to the morphism $S \to G' \cong \mathbb{P}^{N-n+2}$ and γ corresponds to Γ with $\dim(\langle \Gamma \rangle) = k + 1$), no curve of $|\gamma|$ contains C'', hence $\dim(|g|) \geq k - n + 3$. Since $\Gamma \cap C' = \{P_0, \ldots, P_k\}$ as a scheme, we find $\gamma \in |\gamma|$ gives rise to $P_0 + \ldots + P_k \in |g|$. Since $P_0, \ldots, P_k$ are general points of C, we see that $|g|$ is non-special and $\dim(|g|) = \deg(g) - g(C) = k + 1 - g(C)$. Thus, $k + 1 - g(C) \geq k - n + 3$, so $g(C) \leq n - 2$. ■

6 Some examples

Proposition 6.1. *Let $X \subset \mathbb{P}^N$ be an n-dimensional smooth variety of minimal degree. If $k \geq n$ and $n + k + 1 < N \leq 2n + k - 1$ then X has $G_{k-1,k} - defect$.*

Proof: Notice that $n \geq 3$ because $n+k+1 < 2n+k-1$.

Take $k+1$ general points $P_0, \ldots, P_k$ on X and choose a linear subspace $\mathbb{P}^{N-k-1} \subset \mathbb{P}^N$ disjoint with $\langle P_0, \ldots, P_k \rangle$. Consider the projection of X on $\mathbb{P}^{N-k-1}$ with center $\langle P_0, \ldots, P_k \rangle$ and let Y be the closure of the image of that projection. Then Y is also an n-dimensional variety of minimal degree.

From the classification of varieties of minimal degree (see [8]) follows that X is a smooth rational normal scroll. In particular X has a bundle structure $\pi : X \to \mathbb{P}^1$ such that $L(P) := \pi^{-1}(P) \subset X \subset \mathbb{P}^N$ is a linear subspace of dimension $n-1$. For $P \in \mathbb{P}^1$ general $L(P) \cap \langle P_0, \ldots, P_k \rangle = \emptyset$ because $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$. Hence, on Y the image of $L(P)$ is again a linear subspace of dimension $n-1$ of $\mathbb{P}^{N-k-1}$. So Y cannot be a cone over a Veronese surface. If $N = n+k+2$ it follows that Y is a quadric in $\mathbb{P}^{n+1}$. This quadric contains linear subspaces of dimension $n-1$, so Y is singular ([11, Chapter 6, Section 1]). Let s be a general point of the singular locus of Y, which is a linear subspace of $\mathbb{P}^{n+1}$. The image of $L(P)$ on Y contains s, for $P \in \mathbb{P}^1$ general. Let $G = \langle P_0, \ldots, P_k, s \rangle$ then $\dim(G) = k+1$ and $\langle P_0, \ldots, P_k \rangle$ is a hyperplane in G. Since $L(P) \cap G \neq \emptyset$ for $P \in \mathbb{P}^1$ general, $\dim(X \cap G) \geq 1$. Let Γ be a curve in $X \cap G$ intersecting $L(P)$ for general $P \in \mathbb{P}^1$. Since $\langle P_0, \ldots, P_k \rangle$ is a hyperplane in G and $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$, we find two possibilities by similar monodromy arguments as in the proof of Claim 3 of Section 3. If Γ is a rational normal curve of degree $k+1$ through $P_0, \ldots, P_k$; the proof is finished. The second possibility is that Γ is a line. Then there exist lines $\Gamma_0, \ldots, \Gamma_k$ on X such that $P_i \in \Gamma_i$ for all $i \in \{0, \ldots, k\}$. If we denote $\pi(P_i)$ by P_i', then $P_i \in L(P_i')$. The line Γ_0 intersects $L(P_1')$ at a point P_1'' different from P_1. We have $\langle P_1, P_1'' \rangle \cup \Gamma_1 \subset X \cap G$ and $\langle P_1, P_1'' \rangle \neq \Gamma_1$ (Γ_1 is not contained in $L(P_1')$). This contradicts $\dim(\mathbb{T}_{P_1}(X \cap G)) \leq 1$. Hence the second possibility cannot occur.

If $n+k+2 < N \leq 2n+k-1$, it follows that Y is a scroll with $\dim(\mathrm{Sing}(Y)) \geq 2n+k-1-N \geq 0$. So we can finish this proposition by taking the same arguments as in the case $N = n+k+2$. ■

Remark 6.2. If $n = 3$ this proposition says that minimal threefold $X \subset \mathbb{P}^{k+5}$ is $G_{k-1,k}$-defective for $k \geq 3$. Let $\tilde{X} \subset \mathbb{P}^{k+4}$ be the image of $X \subset \mathbb{P}^{k+5}$ under the projection with center $P \in \mathbb{P}^{k+5} \backslash X$. The curve Γ of the proof of the proposition above gives rise to a rational normal curve $\tilde{\Gamma} \subset \tilde{X}$ of degree $k+1$ containing $k+1$ general points on $\tilde{X}$. So, $\tilde{X}$ is also $G_{k-1,k}$-defective.

Proposition 6.3. *Let $X \subset \mathbb{P}^{n+k+1}$ be an n-dimensional smooth variety of minimal degree $k+2$, not being the Veronese surface in $\mathbb{P}^5$. If $k \geq n$ then X has $G_{k-1,k}$-defect.*

Proof: Consider a general surface section $S \subset \mathbb{P}^{k+3}$ of X. Then S is smooth and of minimal degree $k+2$. Since S is not the Veronese surface (X is smooth), it is a smooth rational normal scroll surface.

We will use some results on smooth rational normal scroll surfaces. We know that they are isomorphic to a Hirzebruch surface $\mathbb{F}_r = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ for some $r \in \mathbb{N}$. If $r \geq 1$, those surfaces contain a curve B with negative self-intersection $B^2 = -r$ and have a 1-dimensional linear system of curves F with $F^2 = 0$ and $F.B = 1$. In case $r = 0$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and we can take $B = \mathbb{P}^1 \times \{0\}$ ($B^2 = 0$) and $F = \{p\} \times \mathbb{P}^1$ for $p \in \mathbb{P}^1$.

Let b (respectively f) be the element of $\mathrm{Pic}(\mathbb{F}_r)$ corresponding to the curve B (respectively F). Write $h = b + r\, f$. It is well-known that $\mathrm{Pic}(\mathbb{F}_r) = \mathbb{Z}h \oplus \mathbb{Z}f$. For each $l > 0$, the linear system $|h + l\, f|$ is very ample on $\mathbb{F}_r$. For $l \geq 0$, one has $\dim(|h + l\, f|) = r + 2l + 1$ and $(h + l\, f)^2 = r + 2l$. Hence for $l \geq 1$ the linear system $|h + l\, f|$ gives rise to a surface $S \subset \mathbb{P}^{r+2l+1}$ of minimal degree. Those surfaces are the smooth rational normal scroll surfaces.

Let Γ be an element of $|h + (l-1)\, f|$ for $l \geq 1$. We have $\dim(|(h + l\, f) - \Gamma|) = \dim(|f|) = 1$ hence $\dim(\langle\Gamma\rangle) = r + 2l - 1$ for $\Gamma \subset S \subset \mathbb{P}^{r+2l+1}$. On the other hand $\deg(\Gamma) = (h + (l-1)\, f).(h + l\, f) = r + 2l - 1$, hence $\Gamma \subset S \subset \mathbb{P}^{r+2l+1}$ is a rational normal curve of degree $r + 2l - 1$. Since $\dim(|h + (l-1)\, f|) = r + 2l - 1$, any $r + 2l - 1$ general points on S contain such a curve.

Now take X as above and take $k + 1$ general points $P_0, \ldots, P_k$ on X. The points $P_0, \ldots, P_k$ can be considered as $k + 1$ general points on a general surface section $S \subset \mathbb{P}^{k+3}$ of X. Since S is a smooth rational normal scroll surface, the points $P_0, \ldots, P_k$ are contained in a rational normal curve $\Gamma \subset S \subset \mathbb{P}^{k+3}$ of degree $k + 1$. This implies that X is $G_{k-1,k}$-defective. ■

Proposition 6.4. *Let X be the 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$. Then X is $G_{4,5}$-defective.*

Proof: Denote the 2-uple embedding $\mathbb{P}^3 \to X \subset \mathbb{P}^9$ by ν_2. Let $P_0, \ldots, P_5$ be six general points on X and denote their inverse images in $\mathbb{P}^3$ under ν_2 by $Q_0, \ldots, Q_5$. These points are contained in a rational normal curve $\tilde{\Gamma} \subset \mathbb{P}^3$ of degree 3 (see [11, p. 530]). The image of $\tilde{\Gamma}$ under ν_2 is a rational normal curve Γ of degree 6 in $\mathbb{P}^9$ through $P_0, \ldots, P_5$ that is contained in X since $\tilde{\Gamma}$ is cut out by quadrics in $\mathbb{P}^3$ (see again [11, p. 530]), so X is $G_{4,5}$-defective. ■

Proposition 6.5. *Let X be the blowing-up of $\mathbb{P}^3$ in a point Q linearly normal embedded in $\mathbb{P}^8$. Then X is $G_{3,4}$-defective.*

Proof: Let $P_0, \ldots, P_4$ be five general points of X. We may assume that non of those points is contained in the exceptional divisor $E \subset X$. We can consider X as a subset of $\mathbb{P}^3 \times \mathbb{P}^2 \subset \mathbb{P}^{11}$ (with $\mathbb{P}^8 \subset \mathbb{P}^{11}$). Let $p : X \to \mathbb{P}^3$ be the projection to the first factor and let $Q_0, \ldots, Q_4$ be the images under p of respectively $P_0, \ldots, P_4$. Hence there exists a rational normal curve $\tilde{\Gamma}$ in $\mathbb{P}^3$ containing $Q, Q_0, \ldots, Q_4$. The inverse image of $\tilde{\Gamma}$ under p contains a rational normal curve Γ in X of degree 5 containing $P_0, \ldots, P_4$, so X is $G_{3,4}$-defective. ■

7 What for smooth surfaces?

Proof of Theorem 1.1: We have already proved that smooth surfaces $X \subset \mathbb{P}^{k+3}$ of minimal degree are $G_{k-1,k}$-defective (see Prop. 6.3).

So let X be a smooth $G_{k-1,k}$-defective surface in P^N. Now we can use Proposition 1.3. It follows that $N \geq k+3$ and (since X is smooth) that for $k+1$ general points of X there exists a rational normal curve of degree $k+1$ on X through those points. Take $k+1$ general points $P_0, \ldots, P_k$ on X. One can assume that $P_0, \ldots, P_k$ are general points on a general (smooth) curve section C of X. Write $\Gamma \subset X$ to denote the rational normal curve of degree $k+1$ through $P_0, \ldots, P_k$. Since $\dim(\langle C \rangle) = N - 1 \geq k+2$, we find $\dim(|C - \Gamma|) \geq 1$. Let C' be a general element of $|C - \Gamma|$. The linear system $|C'| = |C - \Gamma|$ has no fixed component because Γ is the only curve in $X \cap \langle \Gamma \rangle$ and $X \cap \langle \Gamma \rangle$ is smooth in a general point of Γ. Either C' is irreducible or it is the sum of irreducible curves in a pencil on X. So, if C' would contain a curve Γ, then $C' \sim (\alpha - 1)\Gamma$ for some $\alpha \geq 2$ and so $C \sim \alpha\Gamma$. So from $\Gamma.C = k+1$ it would follow that $\alpha(\Gamma.\Gamma) = k+1$. But this would contradict $\alpha \geq 2$, $k > 2$ and $\Gamma.\Gamma \geq k$ ($\dim|\Gamma| \geq k+1$). Since $\Gamma \cup C'$ is connected, we get $\Gamma.C' \geq 1$. Hence $\Gamma.\Gamma + \Gamma.C' = \Gamma.C = k+1$ implies $\Gamma.\Gamma = k$ and $\Gamma.C' = 1$. Since $\dim|\Gamma| \geq k+1 \geq 2$ we find $|\Gamma - C'| \neq \emptyset$. So we can write $\Gamma \sim \beta.C' + C''$ for some $\beta \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$.

If $C'' = 0$, then $\beta(C'.C') = \Gamma.C' = 1$ implies $\beta = 1$ and $C'.C' = 1$. Since $\beta^2(C'.C') = \Gamma.\Gamma = k$, this gives us a contradiction with $k > 2$, so $C'' \neq 0$. Since $C' \cup C''$ is connected, we find $C'.C'' \geq 1$. From $1 = \Gamma.C' = \beta(C'.C') + C'.C''$ it follows that $C'.C' = 0$ and $C'.C'' = 1$ because $C'.C' \geq 0$ ($|C'|$ is 1-dimensional and has no fixed components). Thus,

$$\deg(X) = C.C = (\Gamma + C').(\Gamma + C') = \Gamma.\Gamma + 2(\Gamma.C') + C'.C' = k+2.$$

Since $\operatorname{codim}(X) + 1 = N - 1 \geq k+2$ it follows that $N = k+3$ and that X is of minimal degree. ■

8 What for smooth threefolds?

Proof of Theorem 1.2: We have already proved that the threefolds of the statement are $G_{k-1,k}$-defective (see Sec. 6), so we only have to prove that there are no other threefolds with $G_{k-1,k}$-defect. Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate threefold with $G_{k-1,k}$-defect. From Proposition 1.3 and Section 4, it follows that $N \geq n+k+1$ and that any $k+1$ general points on X are contained in a rational normal curve of degree $k+1$ on X. Now fix $k+1$ general points $P_0, \ldots, P_k$ on X. We may assume that $P_0, \ldots, P_k$ are contained in a general curve section C'

of X. Using the notations of the proof of Proposition 5.1, since X is smooth and $\dim(X) = 3$ we have $C = C' = X \cap G'_0$ for some linear subspace $G'_0 \subset \mathbb{P}^N$ of dimension $N-2$ and $S' = X \cap G'$ for some hyperplane $G' \subset \mathbb{P}^N$ containing G'_0. There is a 1-dimensional family of hyperplanes of $\mathbb{P}^N$ containing G'_0 and we distinguish two possibilities:

(a) The hyperplane G' is a general element in this family; i.e. the projection morphism ν in the proof of Proposition 5.1 is surjective. In this case S' is smooth since X is smooth and S' is a general surface section of X (Sec. 2.6). The surface S' contains a subfamily of $\{\Gamma\}$ of dimension at least k.

(b) The hyperplane G' is a special element in this family; i.e. the projection morphism ν in the proof of Proposition 5.1 is not surjective. In this case S' contains a subfamily of $\{\Gamma\}$ of dimension at least $k+1$. In particular the linear system $|g|$ on C has degree $k+1$ and dimension at least $k+1$. Hence S' has sectional genus 0, but S' does not need to be smooth.

Case (a).

Write $\mathcal{L}$ to denote the linear system defining $S \subset \mathbb{P}^{N-1 \geq k+3}$. If $\mathcal{L}(-\Gamma)$ is defined as being $\{D \in \mathcal{L} \,|\, D - \Gamma \geq 0\}$, then $\dim(\mathcal{L}(-\Gamma)) \geq 1$ since $\dim(\langle\Gamma\rangle) = k+1$. Notice that $\mathcal{L} - \Gamma = \{D - \Gamma \,|\, D \in \mathcal{L}(-\Gamma)\}$ does not have fixed components because Γ is the only curve in $X \cap \langle\Gamma\rangle$ and $X \cap \langle\Gamma\rangle$ smooth in a general point of Γ. Let C' be a general element of $\mathcal{L} - \Gamma$, then $\Gamma.(\Gamma + C') = k+1$. Since $\Gamma \cup C'$ is connected we have $\Gamma.C' \geq 1$. On the other hand, since S' contains a subfamily of $\{\Gamma\}$ of dimension at least k we find $\Gamma.\Gamma \geq k-1$. So we obtain two possibilities: $\Gamma.C' = 1$ and $\Gamma.\Gamma = k$ or $\Gamma.C' = 2$ and $\Gamma.\Gamma = k-1$.

Case $\Gamma.C' = 2$ and $\Gamma.\Gamma = k-1$.

First assume that $\mathcal{L} - \Gamma$ is composed with a pencil, so there is a morphism $f : \tilde{S} \to T$ with T a curve and $\tilde{S}$ a blowing-up of S at the fixed points of $\mathcal{L} - \Gamma$ such that $C' = f^{-1}(c_1) + f^{-1}(c_2)$ for $c_1 + c_2$ moving in a linear system on T. Indeed, C' cannot be contained in a fibre of f and each fibre of f intersects Γ otherwise $\Gamma.C'$ would be 0. Since Γ dominates T, we find $T \cong \mathbb{P}^1$. So the fibres of f form a linear system on S. Thus $C' \in |2C_0|$ for a irreducible curve C_0 with $\dim|C_0| = 1$ and $\Gamma.C_0 = 1$. Because $\dim|\Gamma| \geq k$, there are curves in $|\Gamma|$ that contain C_0. Suppose that $\Gamma \sim \alpha C_0 + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C_0| = \emptyset$. If $C'' = 0$, it would follow $\Gamma \sim \alpha C_0$, hence $\alpha^2(C_0.C_0) = \Gamma.\Gamma = k-1$ and $2\alpha(C_0.C_0) = \Gamma.C' = 2$, a contradiction (with $k > 3$).

So $C'' \neq 0$. Since $\alpha C_0 + C''$ is connected (Sec. 2.7) and C_0 irreducible, we find $C_0.C'' \geq 1$. We know that $2 = \Gamma.C' = \alpha(C_0.C') + C''.C' = 2\alpha(C_0.C_0) + 2(C''.C_0)$. Hence $C_0.C_0 = 0$ and $C''.C_0 = 1$, since $C_0.C_0 \geq 0$ ($\dim|C_0| = 1$ and $|C_0|$ has no fixed components). This implies that $C'.C' = 0$ and so

$$\deg(X) = \deg(S) = C.C = (\Gamma + C').(\Gamma + C') = k + 3.$$

Hence $N \in \{k+4, k+5\}$, because $\mathrm{codim}(X)+1 = N-2 \geq k+2$. Since $g(C) \leq 1$ and $C \sim \Gamma + C'_0 + C''_0$ for C'_0 and C''_0 general on S, we find $p_a(\Gamma + C'_0 + C''_0) \leq 1$ and since $g(C'_0) = g(C''_0)$ it follows $g(C) = p_a(\Gamma + C'_0 + C''_0) = 0$. So the sectional genus of X is 0. Now it follows from Theorema 12.1 in [10] that the polarized variety (X, L) has Δ-genus equal to 0. From the classification theory of polarized varieties (Section 2.5.2) it follows that $(X, L) = (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. A linearly normal embedding of (X, L) gives rise to a threefold $\bar{X} \subset \mathbb{P}^{k+5}$ of minimal degree $k+3$. So $X = \bar{X}$ or X is the projection of $\bar{X}$ in $\mathbb{P}^{k+4}$ with center $P \in \mathbb{P}^{k+5} \backslash \bar{X}$. This gives rise to possibilities 2 and 3.

Assume now that $\mathcal{L} - \Gamma$ is not composed with a pencil. Hence in general C' is irreducible (Sec. 2.6). Since $\Gamma.C' = 2$ we have

$$g(C) = p_a(\Gamma + C') = 1 + \frac{1}{2}(\Gamma + C').(\Gamma + C' + K) = p_a(C') + p_a(\Gamma) + 1 \leq 1.$$

Since $g(\Gamma) = 0$, we find $C' \cong \mathbb{P}^1$ and X has sectional genus equal to 1. From $\dim|\Gamma| \geq k$, it follows $|\Gamma - C'| \neq \emptyset$. Now write $\Gamma \sim \alpha C' + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$.

If $C'' = 0$, we have $\Gamma \sim \alpha C'$ and so

$$k - 1 = \Gamma.\Gamma = \alpha^2(C'.C') = \alpha(\Gamma.C') = 2\alpha.$$

Hence $\alpha = \frac{k-1}{2}$ and so $C'.C' = \frac{2}{\alpha} = \frac{4}{k-1}$. Since $k > 3$ it follows $k = 5$, $\alpha = 2$, $\Gamma.\Gamma = 4$, $C'.C' = 1$ and $\Gamma.C' = 2$; so $\deg(X) = C.C = 9(C'.C') = 9$. From the classification of polarized varieties (X, L) with sectional genus 1 (Sec. 2.5.2) follows that X has to be a scroll over an elliptic curve. This gives us a contradiction because $k+1$ general points on X are contained in a rational normal curve on X.

So we find $C'' \neq 0$. We have $\Gamma.C' \geq 0$ and $\Gamma.C'' \geq 0$ since Γ has no fixed component. On the other hand, $C'.C' \geq 0$ since $\dim(|C'|) \geq 1$ and C' has no fixed component. We also have

$$k - 1 = \Gamma.\Gamma = \alpha(\Gamma.C') + \Gamma.C'' = 2\alpha + \Gamma.C''$$

and

$$\deg(X) = C.C = (\Gamma + C').(\Gamma + C') = k + 3 + C'.C'.$$

First consider the case $k = 4$. Then $2\alpha + \Gamma.C'' = 3$ and so $\alpha = 1$ and $\Gamma.C'' = 1$. Since $2 = \Gamma.C' = C'.C' + C''.C'$ and $C'.C'' \geq 1$ ($C' \cup C''$ connected) we have two possibilities: $C'.C' = 0$ and $C''.C' = 2$ or $C'.C' = 1 = C''.C'$.

Consider the first possibility. It follows $\deg(X) = C.C = 7$ and $C''.C'' = -1$ (since $\Gamma.\Gamma = 3$). So (X, L) is a smooth 3-dimensional variety with sectional genus 1 of degree 7. From the classification of polarized varieties with sectional genus 1 (see Sec. 2.5.2) follows that $(X, L) \cong (Bl_Q(\mathbb{P}^3), \sigma^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E)$ with $\sigma : Bl_Q(\mathbb{P}^3) \to \mathbb{P}^3$ the blowing-up of $\mathbb{P}^3$ at Q and E the exceptional divisor. This gives rise to a linearly normal embedding $\bar{X} \subset \mathbb{P}^8$ of $Bl_Q(\mathbb{P}^3)$ and hence case 4 of the Theorem.

Now consider the second possibility. We find $\deg(X) = C.C = 8$ and $C''.C'' = 0$ (since $\Gamma.\Gamma = 3$). So we obtain a 3-dimensional smooth variety with sectional genus 1 of degree 8, thus $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ using the classification of polarized varieties with sectional genus 1 (see Sec. 2.5.2). This implies that S needs to be a smooth quadric in $\mathbb{P}^3$ embedded by $|2C' + C''| = |C| = |\mathcal{O}_S(2, 2)|$. This gives us a contradiction since $C'.C' = 1 = C''.C'$ and $C''.C'' = 0$.

Now let $k = 5$, thus $2\alpha + \Gamma.C'' = 4$. Hence we again have two possibilities: $\alpha = 2$ and $\Gamma.C'' = 0$ or $\alpha = 1$ and $\Gamma.C'' = 2$.

We start with the first possibility. Since $\Gamma \sim 2C' + C''$, we have $2 = \Gamma.C' = 2(C'.C') + C'.C''$, hence $C'.C' = 0$ and $C'.C'' = 2$. It follows that $\deg(X) = C.C = 8$ and $C''.C'' = -4$. From Section 2.5.2 we see that $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

Now we take a look at the second possibility. Since $\Gamma \sim C' + C''$, we have $2 = \Gamma.C' = C'.C' + C'.C''$. Notice that $C'.C' \leq 0$ since there are no 3-dimensional smooth Del Pezzo varieties $\bar{X}$ with $\deg(\bar{X}) > 8$. It follows $C'.C' = 0$, $C'.C'' = 2$, $\deg(X) = C.C = 8$ and $C''.C'' = 0$. From Section 2.5.2 we see that $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

So, in both cases we end up with $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. This gives rise to the 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$, which is case 5 of the Theorem.

If $k > 5$ it follows $\deg(X) = k + 3 + C'.C' > 8$ since $C'.C' \geq 0$. This immediately gives us a contradiction since there are no 3-dimensional smooth Del Pezzo varieties $\bar{X}$ with $\deg(\bar{X}) > 8$ (see Sec. 2.5.2).

Case $\Gamma.C' = 1$ and $\Gamma.\Gamma = k$.

In particular, since $|C'|$ has no fixed components, $|C'|$ cannot be composed by a pencil and it follows that in general C' is irreducible (Bertini's theorem, see Sec. 2.6). Since $\dim|\Gamma| \geq k$ and $\Gamma.C' = 1$, we can write $\Gamma \sim \alpha C' + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$. If $C'' = 0$ it follows $\Gamma \sim \alpha C'$ and thus $\alpha^2(C'.C') = \Gamma.\Gamma = k$ and $\alpha(C'.C') = \Gamma.C' = 1$, a contradiction with $k > 3$. Hence $C'' \neq 0$. We have

$$\alpha(C'.C') + C'.C'' = (\alpha C' + C'').C' = \Gamma.C' = 1.$$

Since $C'.C' \geq 0$ and $C'.C'' \geq 1$ we obtain $C'.C' = 0$ and so $\deg(X) = C.C = k+2$. Because $\text{codim}(X) + 1 = N - 2 \geq k + 2$ we find that X is a smooth threefold in $\mathbb{P}^{k+4}$ of minimal degree $k + 2$. From Proposition 6.3, it follows that such a threefold X has $G_{k-1,k}$-defect. This gives rise to case 1 of the Theorem.

Case (b).

Because C is a smooth hyperplane section of S', S' is smooth along C, hence $\text{Sing}(S') \cap C = \emptyset$. It follows that $\text{Sing}(S')$ is a finite set and so S' is irreducible.

Claim. *If $s \in Sing(S')$ and Γ is a general curve in the set of curves $\{\Gamma\}$ in S', then $s \notin \langle \Gamma \rangle$.*

Proof Claim: First we are going to prove that $s \notin \Gamma$. Assume $s \in \Gamma$. Since $\text{Sing}(S')$ is finite, $s \in \Gamma$ for all curves Γ on S'. So a general curve Γ on S' is completely determined by $k + 1$ points $P_0, \ldots, P_k$ on C as being the only 1-dimensional component of $X \cap \langle P_0, \ldots, P_k, s \rangle$. The uniqueness follows from $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$ as a scheme. Now take $k + 2$ general points $P_0, \ldots, P_{k-1}, Q, Q'$ on C and let Γ (respectively Γ') be the curve in the family corresponding with $P_0, \ldots, P_{k-1}, Q$ (respectively $P_0, \ldots, P_{k-1}, Q'$). Because $\dim(\langle P_0, \ldots, P_{k-1}, Q, Q' \rangle) = k + 1$, we can consider a deformation of C on S' to another curve C' containing $P_0, \ldots, P_{k-1}, Q, Q'$. Since Γ and Γ' are contained in $\langle C' \cup \{s\} \rangle$, the surface S' is deformed into $S'' = X \cap \langle C' \cup \{s\} \rangle$. Because $\Gamma \cap \Gamma'$ is finite it follows $s \in \text{Sing}(S'')$. So for a general hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ with $\langle P_0, \ldots, P_{k-1}, Q, Q', s \rangle \subset \mathbb{P}^{N-1}$ we find $\mathbb{T}_s(X) \subset \mathbb{P}^{N-1}$, hence $\mathbb{T}_s(X) \subset \langle P_0, \ldots, P_{k-1}, Q, Q', s \rangle$. Since $s \notin C = X \cap \langle C \rangle$ and $\langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset \langle C \rangle$, we have $\dim(\mathbb{T}) = n - 1 = 2$ with $\mathbb{T} = \mathbb{T}_s(X) \cap \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle$. If $s \in \langle C \rangle$ then $s \in C = \langle C \rangle \cap X$ and thus $s \notin \text{Sing}(S')$, a contradiction. So we have

$$\mathbb{T} = \mathbb{T}_s(X) \cap \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset \mathbb{T}_s(X) \cap \langle C \rangle \subsetneq \mathbb{T}_s(X),$$

hence $\mathbb{T} = \mathbb{T}_s(X) \cap \langle C \rangle$ since $\dim(\mathbb{T}) = 2$. This implies

$$\mathbb{T} = \mathbb{T}_s(X) \cap \langle C \rangle \subset \langle P_0, \ldots, P_{k-1}, Q, Q' \rangle \subset \langle C \rangle.$$

Since $P_0, \ldots, P_{k-1}, Q, Q'$ are generally chosen on C and $k + 1 < N - 2$, we may assume that those points are contained in a general hyperplane of $\langle C \rangle$ (not containing $\mathbb{T}$), a contradiction.

If $s \in \langle \Gamma \rangle \backslash \Gamma$ then s is one of the finitely many points in $\langle \Gamma \rangle \cap X$ not on Γ. So a general curve Γ is again completely determined by $k + 1$ points $P_0, \ldots, P_k$ on C. Take a deformation of C on X to another curve C' containing $P_0, \ldots, P_k$. Since Γ is contained in $\langle C' \cup \{s\} \rangle$ and $s \in \langle \Gamma \rangle$, the surface S' deforms to $S'' = \langle C' \cup \{s\} \rangle \cap X$ with $s \in \text{Sing}(S'')$. As before we find $\mathbb{T}_s(X) \subset \langle P_0, \ldots, P_k, s \rangle$ and thus $\dim(\mathbb{T}_s(X) \cap \langle P_0, \ldots, P_k \rangle) \geq 2$ for general points $P_0, \ldots, P_k$ on C. Since

$s \notin \langle P_0, \ldots, P_k \rangle \subset \langle C \rangle$ (otherwise $s \in C = X \cap \langle C \rangle$ and so $s \notin \mathrm{Sing}(S')$) we obtain $\mathbb{T} := \mathbb{T}_s(X) \cap \langle C \rangle = \mathbb{T}_s(X) \cap \langle P_0, \ldots, P_k \rangle$ and $\dim(\mathbb{T}) = 2$. On the other hand, we may assume that $P_0, \ldots, P_k$ are contained in a general hyperplane of $\langle C \rangle$ since $k < N - 2$. So we get a contradiction. □

Now take a minimal resolution of singularities $\chi : S \to S'$. General curves C and Γ can be considered as curves on S and Γ is contained in a linear system on S of dimension at least $k+1$. Since $\Gamma.C = k+1$ and $|\Gamma - C| = \emptyset$ the linear system of curves Γ is complete and induces a g^{k+1}_{k+1} on C, so C is rational. We have $\dim(|C - \Gamma|) \geq 1$, since $\dim(\langle C \rangle) = N - 2 \geq k+2$ and $\dim(\langle \Gamma \rangle) = k+1$. Let C' be a general element of $|C - \Gamma|$. The linear system $|C'| = |C - \Gamma|$ has no fixed component since Γ is the only curve contained in $X \cap \langle \Gamma \rangle$ and $\mathrm{Sing}(S') \cap \langle \Gamma \rangle = \emptyset$. So C' is irreducible or it is the sum of irreducible curves in a pencil. Hence, if C' would contain a curve Γ, then $C' \sim (\alpha - 1)\Gamma$ and $C \sim \alpha\Gamma$ for some $\alpha \geq 2$. This would imply that $k + 1 = \Gamma.C = \alpha(\Gamma.\Gamma)$, but $\Gamma.\Gamma \geq k$ since $\dim(|\Gamma|) \geq k+1$, a contradiction. So C' is irreducible. Since $\Gamma \cup C'$ is connected, $\Gamma.C' \geq 1$. From $k + 1 = \Gamma.C = \Gamma.\Gamma + \Gamma.C'$ then follows $\Gamma.\Gamma = k$ and $\Gamma.C' = 1$. Since $\dim(|\Gamma|) \geq k + 1$ this also implies $|\Gamma - C'| \neq \emptyset$.

We can write $\Gamma \sim \beta C' + C''$ for some $\beta \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$. If $C'' = 0$ then $\Gamma \sim \beta C'$, hence $\beta(C'.C') = \Gamma.C' = 1$ and so $\beta = 1$ and $C'.C' = 1$. This would imply $k = \Gamma.\Gamma = \beta^2(C'.C') = 1$, a contradiction. So $C'' \neq 0$. We know $C'.C' \geq 0$ ($|C'|$ has dimension at least 1) and $C'.C'' \geq 1$ ($C' \cup C''$ connected), so $\beta(C'.C') + C'.C'' = \Gamma.C' = 1$ implies $C'.C' = 0$ and $C'.C'' = 1$. Hence

$$\deg(X) = C.C = (\Gamma + C').(\Gamma + C') = k + 2.$$

Since $\mathrm{codim}(X) + 1 = N - 2 \geq k + 2$ this implies $N = k + 4$ and X is a smooth threefold in $\mathbb{P}^N$ with minimal degree $k + 2$. This case corresponds to case 1 of the Theorem. ■

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