On high $\mathbf{G_{k-1,k}}$-defective varieties

Filip Cools *

August 11, 2006

Abstract.— *We give a rough characterization for n-dimensional varieties with $G_{k-1,k}$-defect equal to $a > 0$ if $k \geq n$. Then we apply this in the case that $a \geq n-2$ to become a fine classification.*

MSC.— *[14N15],[14J40],[14M15]*

1 Introduction

Let X be a non-degenerate irreducible variety in $\mathbb{P}^N$ of dimension n and let $0 \leq h \leq k$ be integers. Then we denote by $G_{h,k}(X)$ the closure of the set of h-dimensional linear subspaces H of $\mathbb{P}^N$ that are contained in the span of $k+1$ independent points of X. We call $G_{h,k}(X)$ the h-Grassmannian of $(k+1)$-secant k-planes of X. The subvariety $G_{h,k}(X)$ of $\mathbb{G}(h, N)$ has an expected dimension, equal to

$$\min\{(k+1)n + (k-h)(h+1), (h+1)(N-h)\}.$$

This is an upper bound of the dimension of $G_{h,k}(X)$. Now we say that X is $G_{h,k}$-defective if the dimension of $G_{h,k}(X)$ is smaller than the expected dimension. In this case we call the difference of both dimensions the $G_{h,k}$-defect of X.

If $h = 0$, the variety $G_{h,k}(X)$ is just the kth secant variety $S_k(X)$ of X and in this case the terminology $G_{h,k}$-defectivity is usually replaced by k-defectivity. This case is for example studied in [14].

In case $h > 0$, things are much more complicated, mainly because of the lack of a Terracini Lemma. Nevertheless, there are some results. For example in [4] is shown that irreducible curves are not $G_{h,k}$-defective, in [5] is proved a classification of surfaces with $G_{1,2}$-defect and in [7] one can find a classification of $G_{k-1,k}$-defective surfaces and threefolds in case $k > n$.

One of the applications of the study of $G_{h,k}$-defective varieties is found in the study of the Waring problem for forms (see [2, 6, 11]). Here one is mainly

*K.U.Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, Belgium, email: Filip.Cools@wis.kuleuven.ac.be

interested in the case where X is a Veronese embedding of projective spaces.

Another extrinsic reason of studying $G_{h,k}$-defective varieties is their strange behaviour under projections (especially varieties with high $G_{k-1,k}$-defect).

In this paper we will prove that the $G_{k-1,k}$-defect of a n-dimensional variety is at most $n-1$ if $k \geq n$. This can be seen as an extension of Zak's result (case $k=1$) giving a bound on the dimension of the secant varieties (see [14]). Moreover we characterize varieties X^n with a certain $G_{k-1,k}$-defect $a > 0$, again in the case $k \geq n$.

Theorem 1.1. *Let X^n ($n \geq 2$) be a non-degenerate variety in $\mathbb{P}^N$ with $N \geq n+k+1$ for an integer $k \geq n$ and suppose that $G_{k-1,k}(X) = (k+1)n + k - a$ with $a > 0$. Then $a \leq n-1$ and one of the following two properties hold for $k+1$ general points $P_0, \ldots, P_k$ on X:*

1. *For each $i \in \{0, \ldots, k\}$, there exists a linear subspace L_i of dimension a on X containing P_i and so that $\dim\langle L_0, \ldots, L_k\rangle = k+a$.*
2. *There exists a variety Υ^a of minimal degree $k+1$ on X that contains the points $P_0, \ldots, P_k$. In this case there exists an $(a-1)$-dimensional family of rational normal curves Γ of degree $k+1$ on X containing $P_0, \ldots, P_k$.*

Furthermore, if X satisfies one of the two properties, X is $G_{k-1,k}$-defective with defect at least a.

Using the above Theorem, we will be able to classify the two most extremal cases ($G_{k-1,k}$-defect equal to $n-2$ or $n-1$) if X^n is smooth and $k \geq n$. Since the case $n \leq 3$ has been handled in [4, 7, 8], we will focus on the case $n \geq 4$.

Theorem 1.2. *Let $X^n \subset \mathbb{P}^N$ be a smooth non-degenerate $G_{k-1,k}$-defective variety with defect $a \geq n-2$ for an integer $k \geq n$. Then X is one of the following varieties:*

1. *X^n is a rational normal scroll of (minimal) degree $k+2$ in $\mathbb{P}^{n+k+1}$ ($a = n-1$);*
2. *$n \geq 3$ and X^n is a rational normal scroll of (minimal) degree $k+3$ in $\mathbb{P}^{n+k+2}$ ($a = n-2$);*
3. *$n \geq 3$ and X^n is the projection in $\mathbb{P}^{n+k+1}$ of a n-dimensional rational normal scroll of (minimal) degree $k+3$ in $\mathbb{P}^{n+k+2}$ ($a = n-2$);*
4. *$n = k = 3$ and X^3 is a hyperplane section of the Segre-embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ ($a = 1$);*

5. $n=3$, $k=4$ *and* X^3 *is a (linearly normal) embedding of the blowing-up of* $\mathbb{P}^3$ *in a point in* $\mathbb{P}^8$ $(a=1)$*;*

6. $n=3$, $k=5$ *and* X^3 *is the image of the* 2*-Veronese-embedding of* $\mathbb{P}^3$ *in* $\mathbb{P}^9$ $(a=1)$.

2 Some conventions, definitions and generalities

2.1. Conventions. We write $\mathbb{P}^N$ to denote the N-dimensional projective space over the field $\mathbb{C}$. A variety $X \subset \mathbb{P}^N$ is an irreducible reduced Zariski-closed subset of $\mathbb{P}^N$ and we say that X is non-degenerate in $\mathbb{P}^N$ if X is not contained in a hyperplane of $\mathbb{P}^N$.

Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of dimension n. Then we call a closed subscheme $Y \subset X$ a m-dimensional section of X if Y is the scheme-theoretical intersection of X with a linear subspace $\mathbb{P}^{N-n+m} \subset \mathbb{P}^N$ such that all components have dimension m.
If $X \subset \mathbb{P}^N$ is a variety and $P=(P_1,\ldots,P_r)$ is a point of X^r, we write $\langle P\rangle$ to denote the linear span $\langle P_1,\ldots,P_r\rangle \subset \mathbb{P}^N$.

2.2. Definition of $\mathbf{G_{k-1,k}(X)}$ and some results. Let $X \subset \mathbb{P}^N$ be a non-degenerate n-dimensional variety and $k \leq N$ an integer. We can consider the rational map $\omega : X^{k+1} \dashrightarrow \mathbb{G}(k,N)$ that maps a point $P=(P_0,\ldots,P_k)$ in X^{k+1} to the span $\langle P\rangle$ if $P_i \neq P_j$ for all $i \neq j$ and $\dim(\langle P\rangle)=k$. Such a span is called a $(k+1)$-secant k-plane of X. Now consider the incidence diagram

$$\begin{tikzcd} & I \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\ \mathbb{G}(k-1,N) & & \mathbb{G}(k,N) & X^{k+1} \arrow[l, dashed, "\omega"'] \end{tikzcd}$$

with $I=\{(H,G)\,|\, H \subset G\} \subset \mathbb{G}(k-1,N)\times\mathbb{G}(k,N)$ and projection maps α and β. Now we define $G_{k-1,k}(X)$ as $\alpha(\beta^{-1}(\overline{\mathrm{im}(\omega)}))$. So $G_{k-1,k}(X)$ is equal to the closure of the set of $(k-1)$-dimensional subspaces H of $\mathbb{P}^N$ contained in some $(k+1)$-secant k-plane G of X.

From [5, Prop. 1.1] it follows that the dimension of $G_{k,k}(X) := \overline{\mathrm{im}(\omega)} \subset \mathbb{G}(k,N)$ is equal to $\min\{(k+1)(N-k),(k+1)n\}$. Since the fibers of β are k-dimensional, we have that the dimension of $G_{k-1,k}(X)$ is smaller or equal than $(k+1)n+k$. Hence we define the expected dimension of $G_{k-1,k}(X) \subset \mathbb{G}(k-1,N)$ as

$$\mathrm{expdim}(G_{k-1,k}(X)) = \min\{(k+1)n+k, k(N-k+1)\}.$$

If $\dim(G_{k-1,k}(X))$ is smaller than this expected dimension, we say that X has $G_{k-1,k}$-defect.

In case $k \geq n$ the expected dimension of $G_{k-1,k}(X)$ is equal to $(k+1)n+k$ if and only if $N \geq n+k+1$. If $\dim(G_{k-1,k}(X)) = (k+1)n+k-a$ and $N \geq n+k+1$, for a general element $H \in G_{k-1,k}(X)$ the set of $(k+1)$-secant k-planes of X containing H has dimension a.

Remark. If $N \leq n+k$, $\overline{\mathrm{im}(\omega)}$ is equal to $\mathbb{G}(k,N)$. Hence, in this case X is not $G_{k-1,k}$-defective if $N \leq n+k$ since $G_{k-1,k}(X) := \alpha(\beta^{-1}(\overline{\mathrm{im}(\omega)})) = \mathbb{G}(k-1,N)$.

2.3. Let $X \subset \mathbb{P}^N$ be a non-degenerate n-dimensional variety with $N \geq n+k+1$ for some integer k and let $P_0, \ldots, P_k$ be general points on X. Since these $k+1$ points are contained in a general curve section of X, the uniform position lemma for curves (see [1] and [3, Proposition 2.6] for the argument) implies that $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$ as a scheme.

2.4. Polarized varieties. A polarized variety is a pair $(V, \mathcal{S})$ such that V is an abstract projective variety and $\mathcal{S}$ is an ample invertible sheaf on V.

For polarized varieties, the notion of sectional genus (for a general definition, see [12]) exists. If $\mathcal{S}$ is very ample on V and $X \subset \mathbb{P}^N$ is the embedding of V using the global sections of $\mathcal{S}$, then the sectional genus of $(V, \mathcal{S})$ is defined as being the arithmetic genus of a general curve section of $X \subset \mathbb{P}^N$.

The classification of smooth polarized varieties $(V, \mathcal{S})$ of sectional genus at most one is given in [12, Section 12]. We only consider the case where $V = X^n \subset \mathbb{P}^N$ and $\mathcal{S} = \mathcal{O}_X(1)$ with $n \geq 4$ and $N \geq 9$.

Since $n \geq 4$, if the sectional genus is 0 (see [12, Section 5]), X is a scroll of a vector bundle on $\mathbb{P}^1$. Moreover, if X is embedded using the complete linear system then X is a rational normal scroll of (minimal) degree $N-n+1$. We can obtain all smooth rational normal scrolls $X^{n \geq 4} \subset \mathbb{P}^N$ in this way.

If the sectional genus is equal to 1, X has to be a scroll of a vector bundle on a elliptic curve, since there are no Del Pezzo varieties in the considered case (see [12, Section 8]).

3 A characterization

Proof of Theorem 1.1.:
Let $X \subset \mathbb{P}^{N \geq n+k+1}$ be a non-degenerate n-dimensional variety with $G_{k-1,k}$-defect equal to $a > 0$ for some $k \geq n$.

Take $H \in G_{k-1,k}(X)$ general and consider the closure in X^{k+1} of the set of points $(P_0, \ldots, P_k)$ with $P_i \neq P_j$ for all $i \neq j$ and $H \subset \langle P_0, \ldots, P_k \rangle$. The dimension of this set is equal to a. Let $\Omega_{H,k}$ be an a-dimensional component of that set. Take a general element $P = (P_0, \ldots, P_k)$ of $\Omega_{H,k}$. Since we have chosen $H \in G_{k-1,k}(X)$ generally, $(P_0, \ldots, P_k)$ is a general element of X^{k+1}. In particular, $\langle P_0, \ldots, P_k \rangle \cap X = \{P_0, \ldots, P_k\}$ as a scheme. Now let $Q^{(1)} =$

$(Q_0^{(1)},\dots,Q_k^{(1)}),\dots,Q^{(a)}=(Q_0^{(a)},\dots,Q_k^{(a)})$ be other general elements of $\Omega_{H,k}$. Denote $\Psi_{H,k}=\bigcup_{i=0}^k p_i(\Omega_{H,k})$ and $\langle P,Q^{(1)},\dots,Q^{(b)}\rangle$ by M_b for each $b\in\{1,\dots,a\}$, with p_i the $(i+1)$th projection map from X^{k+1} to X.

Claim. $\dim M_a=k+a$ *and* $\dim(X\cap M_a)=a$.

Proof of the Claim:
We will proof by induction on b the following subclaim: $\dim M_b=k+b$ and $\dim(X\cap M_b)=b$ for each $b\in\{0,\dots,a\}$. The Claim will then follow directly from the subclaim by taking b equal to a.

For b equal to 0, the subclaim follows from Section 2.3. Now let b be an integer so that $0<b\le a$. We know that

$$\begin{aligned}\dim M_b &= \dim M_{b-1}+\dim\langle Q^{(b)}\rangle-\dim(M_{b-1}\cap\langle Q^{(b)}\rangle)\\ &\le (k+b-1)+k-(k-1)=k+b\end{aligned}$$

since $H\subset M_{b-1}\cap\langle Q^{(b)}\rangle$. Suppose that $\dim M_b<k+b$, thus $M_b=M_{b-1}$ and $\dim M_b=k+b-1<k+a$. Since we have chosen $Q^{(b)}$ generically, we may assume that $\Psi_{H,k}\subset M_b$. Hence $\dim\Psi_{H,k}\le b-1<a$ because $\Psi_{H,k}\subset X\cap M_b=X\cap M_{b-1}$ and $\dim(X\cap M_{b-1})=b-1$. So the map $p_0:\Omega_{H,k}\to\Psi_{H,k}$ is not generically finite. Let $P_0'=p_0(P')$ be a general element in the image of the map. Then $\dim(X\cap\langle P'\rangle)=\dim(X\cap\langle P_0',H\rangle)\ge 1$, hence $H\cap X\neq\emptyset$. This gives us a contradiction since H is a general element of $G_{k-1,k}(X)$.

Now we are going to show that $\dim(X\cap M_b)=b$. Take a general hyperplane π of M_b through H but not through $\langle P\rangle$, so $\pi\cap\langle P,Q^{(i)}\rangle$ is a hyperplane of $\langle P,Q^{(i)}\rangle$ for each $i\in\{1,\dots,b\}$. Then it follows from [7, Prop. 1.3] that there exists a point $R^{(i)}\in X^{k+1}$ such that $H\subset\langle R^{(i)}\rangle\subset\langle P,Q^{(i)}\rangle$ for each i. Since $\langle P,R^{(i)}\rangle=\langle P,Q^{(i)}\rangle$, the span $\langle R^{(1)},\dots,R^{(b)}\rangle$ is equal to π. By induction ($R^{(1)},\dots,R^{(b)}$ are also general in $\Omega_{H,k}$) we have $\dim(X\cap\langle R^{(1)},\dots,R^{(b)}\rangle)=b-1$, so $\dim(X\cap M_b)=b$. $\square$

The Claim immediately implies that $a<n$. Let Υ be an a-dimensional component of $X\cap M_a$.

Claim. *Either* $\Upsilon\cap\{P_0,\dots,P_k\}=\{P_0,\dots,P_k\}$ *or* $\Upsilon\cap\{P_0,\dots,P_k\}$ *is only one point. In the second case* $X\cap M_a$ *contains an* a*-dimensional linear subspace* L_i *with* $L_i\cap\{P_0,\dots,P_k\}=\{P_i\}$ *for each* $i\in\{0,\dots,k\}$.

Proof of the Claim:
Analogous to the proof of [7, Prop. 1.3, Claim 3]. $\square$

If $\Upsilon\cap\{P_0,\dots,P_k\}=\{P_0,\dots,P_k\}$, we find $\Upsilon\cap\langle P\rangle=\{P_0,\dots,P_k\}$ as a scheme because $X\cap\langle P\rangle=\{P_0,\dots,P_k\}$ as a scheme and $\Upsilon\subset X$. Hence $\deg(\Upsilon)=k+1=$

$\operatorname{codim}_{M_a}(\Upsilon)+1$ and so Υ is of minimal degree. In this case, we find that $k+1$ general points on X are contained in an a-dimensional variety of minimal degree $k+1$.

It is easy to see that both properties give rise $G_{k-1,k}$-defectivity with defect at least a. ■

4 The first case of the characterization

Let $X^n \subset \mathbb{P}^N$ be a non-degenerate irreducible variety with $N \geq n+k+1$ for an integer $k \geq n$ such that for $k+1$ general points $P_0, \ldots, P_k$ on X there exist a-dimensional ($a > 0$) linear subspaces $L_0, \ldots, L_k \subset X$ of $\mathbb{P}^N$ with $P_i \in L_i$ for each i and $\dim\langle L_0, \ldots, L_k\rangle = k+a$. Note that by a monodromy argument, each property that holds for a subset of $\{L_0, \ldots, L_k\}$ also holds for another subset with the same cardinality.

Suppose that $\dim\langle L_0, \ldots, L_{k-1}\rangle < k+a-1$. Then one can easily prove by induction that $\dim\langle L_0, \ldots, L_{l-1}\rangle < k+a-1-2(k-l)$. For $l=1$ we have $a < k+a-1-2(k-1)$ and so $k < 1$, a contradiction.

Now suppose that $\dim\langle L_0, \ldots, L_{k-1}\rangle = k+a-1$. Then it is easy to prove by induction that $\dim\langle L_0, \ldots, L_i\rangle = a+i$, in particular we have $\dim\langle L_0, L_1\rangle = a+1$ and so $\dim(L_0 \cap L_1) = a-1$. By monodromy we have $\dim(L_i \cap L_j) = a-1$ for all $i \neq j$. Hence the Linear Lemma (see [5]) implies that $M := L_0 \cap \cdots \cap L_k$ is of dimension $a-1$. Note that $P_i \notin M$ since otherwise $\langle P_0, \ldots, P_k\rangle \subset M$ and so $k \leq a-1 \leq n-1$, a contradiction. Consider $T \subset X \times X^{k+1}$ with $(S, (P_0, \ldots, P_k)) \in T$ if and only if there exist a-dimensional subspaces $L_0, \ldots, L_k \subset X$ with $S, P_i \in L_i$ for all i. We know that $\dim T \geq (k+1)n + a - 1$ and so there exists a point $S \in X$ so that the fibre above S is at least $nk+a-1$-dimensional. Let T' be a component of that fibre and let X_i be its image under the i-th projection map to X ($i \in \{0, \ldots, k\}$). If $X_i \neq X$ for each i, then $nk+a-1 \leq \dim T' \leq (k+1)(n-1)$, so $a \leq n-k \leq 0$, a contradiction. Hence there exists an i so that $X_i = X$ and thus X is a cone with center S.

If $a > 1$, let $\mathbb{P}^{N-1}$ be a hyperplane of $\mathbb{P}^N$ not through S. Then for $k+1$ general points $P'_0, \ldots, P'_k$ on $X' := X \cap \mathbb{P}^{N-1}$, there exist $(a-1)$-dimensional linear subspaces $L'_0, \ldots, L'_k$ on X' with $P'_i \in L'_i$ such that $\dim\langle L'_0, \ldots, L'_k\rangle = k+a-1$ and $\dim\langle L'_0, \ldots, L'_{k-1}\rangle = k+a-2$. With the same arguments as above, we see that X' is again a cone. So, by induction, we find that X is a cone with center a linear subspace of dimension $a-1$.

Assume that $\dim\langle L_0, \ldots, L_{k-1}\rangle > k+a-1$, so $\langle L_0, \ldots, L_{k-1}\rangle = \langle L_0, \ldots, L_k\rangle$. Take $i \in \{1, \ldots, k\}$ and consider $\langle L_i, P_1 \ldots, P_k\rangle$. Clearly, the dimension of this

set is smaller than $k+a$ since $P_i \in L_i$. Suppose that $\dim\langle L_i, P_1 \ldots, P_k\rangle < k+a-1$, then $\dim(L_i \cap \langle P_1, \ldots, P_k\rangle) \geq 1$ but $X \cap \langle P_1, \ldots, P_k\rangle = \{P_1, \ldots, P_k\}$ as a scheme, a contradiction. So we have that $\dim\langle L_i, P_1 \ldots, P_k\rangle = k+a-1$.
Now, let $1 \leq i \leq j \leq n$. If $\dim\langle L_i, L_j, P_1 \ldots, P_k\rangle < k+a$ we have $L_j \subset \langle L_i, P_1 \ldots, P_k\rangle$ and by monodromy $\langle L_0, \ldots, L_k\rangle \subset \langle L_i, P_1 \ldots, P_k\rangle$, a contradiction. Hence $\dim\langle L_i, L_j, P_1 \ldots, P_k\rangle = k+a$ and $\langle L_i, L_j, P_1 \ldots, P_k\rangle = \langle L_0, \ldots, L_k\rangle$.
Now fix $P_1, \ldots, P_k$ on X and consider a 1-parameter family $P_0(t)$ on X with $P_0(0) = P_0$. Consider also a 1-parameter family $H(t) \subset \langle P_0(t), P_1, \ldots, P_k\rangle$ of linear subspaces of dimension $k-1$ with $H(0) = H$ and 1-parameter families $Q_i^{(j)}(t)$ on X with $Q_i^{(j)}(0) = Q_i^{(j)}$ for each $i \in \{0, \ldots, k\}$ and each $j \in \{1, \ldots, a\}$ with $H(t) \subset \langle Q_0^{(j)}(t), \ldots, Q_k^{(j)}(t)\rangle$ for each j. These families imply the existence of 1-parameter families $L_0(t), \ldots, L_k(t)$ of a-dimensional subspaces with $L_i(0) = L_i$ for all i, $P_i \in L_i(t)$ for all $i \in \{1, \ldots, k\}$, $P_0(t) \in L_0(t)$ and $\dim\langle L_0(t), \ldots, L_k(t)\rangle = k+a$ for every general value of the parameter t.
We may assume that in general $P_0(t) \notin \langle L_0, \ldots, L_k\rangle$. If $L_i(t) = L_i$ for each i and for a general value of t, then $P_0(t) \in \langle L_0(t), \ldots, L_k(t)\rangle = \langle L_0, \ldots, L_k\rangle$, a contradiction. Thus by monodromy we may assume that $L_i(t) \neq L_i$ for a general value of t. So there exists a family of a-dimensional subspaces on X through a general point of X. Consider $\Sigma = \{(\ell, P) | P \in \ell \subset X\} \subset \mathbb{G}(1, N) \times X$. Above a general point P of X there is at least an a-dimensional family of lines, so $\dim(\Sigma) \geq n+a$. Since a general not-empty fibre of $p : \Sigma \to \mathbb{G}(1, N)$ is 1-dimensional, we have $\dim(p(\Sigma)) \geq n+a-1$. If $a \geq n-2$ we find by using [13] that $a = n-2$ and X is a scroll in $\mathbb{P}^{n-1}$, so X is embedded in $\mathbb{P}^N$ as a $\mathbb{P}^{n-1}$-bundle over a curve K. Let K_P be the $(n-1)$-dimensional component of the union of all lines on X through a (general) point $P \in X$. We know that K_P is a $(n-1)$-dimensional linear subspace of $\mathbb{P}^N$. Using a 1-parameter family $P_0(t)$ on X we find 1-parameter families $L_1(t)$ and $L_2(t)$ in respectively K_{P_1} and K_{P_2}. We have

$$\langle P_0(t), P_1, \ldots, P_k\rangle \subset \langle L_1(t), L_2(t), P_3, \ldots, P_k\rangle \subset \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k\rangle.$$

Since

$$\dim(\langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k\rangle) \leq k+a+2 = k+n$$

and thus $X \not\subset \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k\rangle$, we can choose the parameter family $P_0(t)$ such that $P_0(t) \notin \langle K_{P_1}, K_{P_2}, P_3, \ldots, P_k\rangle$ for general values of the parameter t, a contradiction.

5 The second case of the characterization

Proposition 5.1. *Let $X^n \subset \mathbb{P}^{N \geq n+k+1}$ $(k \geq n)$ be a non-degenerate variety so that for each $k+1$ general points $P_0, \ldots, P_k$ on X there exists an $(a-1)$-dimensional family of rational normal curves Γ of degree $k+1$ on X through*

$P_0, \ldots, P_k$ ($a > 0$). Then the genus of a general curve section of X is at most $n - a - 1$.

Proof:
Let $\{\Gamma\}$ be the family of rational normal curves Γ on X of degree $k+1$. By assumption, we have $\dim\{\Gamma\} \geq n(k+1) - (k+1) + (a-1)$. Now take a general curve section $C' = X \cap G'_0$ (G'_0 is a linear subspace of dimension $N - n + 1 \geq k + 2$) and take $k+1$ general points $P_0, \ldots, P_k$ on C' (these points are also general on X). From Bertini's theorems it follows that C' is irreducible and smooth at $P_0, \ldots, P_k$. Consider a $(k-1)$-dimensional linear subspace H of $\langle P_0, \ldots, P_k \rangle$ and take $(Q_0, \ldots, Q_k) \in \Omega_{H,k}$. Hence, $G' = \langle G'_0, Q_0 \rangle$ is a $(N - n + 2)$-dimensional linear subspace of $\mathbb{P}^N$ which defines an irreducible surface section $S' = X \cap G'$ of X that is smooth in $P_0, \ldots, P_k$.

Consider the inclusion relation $I' \subset \{\Gamma\} \times \mathbb{G}(N - n + 2, N)$. The dimension of a general fibre of $I' \to \{\Gamma\}$ is $(N - n - k + 1)(n - 2)$. Hence, we obtain a irreducible component I of I' containing (Γ, G') of dimension greater than or equal to $(N - n - k + 1)(n - 2) + (k + 1)(n - 1) + (a - 1)$, with Γ the rational normal curve contained in $X \cap \langle P_0, \ldots, P_k, Q_0, \ldots, Q_k \rangle$. Consider the projection $\nu : I \to \mathbb{G}(N - n + 2, N)$. The dimension of a general non-empty fibre of ν is at least

$$(N-n-k+1)(n-2)+(k+1)(n-1)+(a-1)-(N-n+3)(n-2) = k-n+a+2.$$

If we consider the fibre above G', we find that $S' = X \cap G'$ contains a subfamily of $\{\Gamma\}$ of dimension at least $k - n + a + 2$. Let S be the minimal resolution of singularities of S'. We become a family $\{\gamma\}$ of rational curves on S of dimension at least $k - n + a + 2$ by considering the strict transforms of the curves in $\{\Gamma\}$ on S'. Denote the strict transforms on S of Γ and C' by resp. γ and C''. Any two points of S can be connected by means of a rational curve in $\{\gamma\}$. This implies $h^1(S, \mathcal{O}_S) = 0$, so the family $\{\gamma\}$ is contained in a linear system $\{\gamma\}$ of dimension at least $k - n + a + 2$. This linear system induces a linear system $|g|$ on the normalization C of C''. Since S' is smooth at $P_0, \ldots, P_k$, we find that S and S' are isomorphic above neighborhoods of those points. Since $\dim(|C'' - \gamma|) \geq 1$ (C'' is a divisor corresponding to the morphism $S \to G' \cong \mathbb{P}^{N-n+2}$ and γ corresponds to Γ with $\dim(\langle \Gamma \rangle) = k + 1$), no curve of $|\gamma|$ contains C'', hence $\dim(|g|) \geq k - n + a + 2$. Since $\Gamma \cap C' = \{P_0, \ldots, P_k\}$ as a scheme, we find $\gamma \in |\gamma|$ gives rise to $P_0 + \ldots + P_k \in |g|$. Since $P_0, \ldots, P_k$ are general points of C, we see that $|g|$ is non-special and $\dim(|g|) = \deg(g) - g(C) = k + 1 - g(C)$. Thus, $k + 1 - g(C) \geq k - n + a + 2$, so $g(C) \leq n - a - 1$. ■

6 Examples

Example 6.1. *Let $X^n \subset \mathbb{P}^N$ be a n-dimensional smooth rational normal scroll. If $k \geq n$ and $n+k+1 \leq N \leq 2n+k-1$, X is $G_{k-1,k}$-defective with defect at least $2n+k-N$.*
Proof:
Take $k+1$ general points $P_0, \dots, P_k$ on X.

Assume first that $N = n+k+1$. Denote the family of rational normal curves of degree $k+1$ on X through $P_0, \dots, P_k$ by $\{\Gamma\}$ and the inclusion relation $\{(G, \Gamma) \mid \Gamma \subset G\} \subset \mathbb{G}(k+3, n+k+1) \times \{\Gamma\}$ by I. From the proof of [7, Prop. 6.3] it follows that for a general surface section $S \subset \mathbb{P}^{k+3}$ of X containing $P_0, \dots, P_k$ there exists a rational normal curve $\Gamma \subset S$ of degree $k+1$ through $P_0, \dots, P_k$. So we know that $\dim I \geq \dim\{G \in \mathbb{G}(k+3, n+k+1) \mid P_0, \dots, P_k \in G\} = 3(n-2)$. Since a general fibre of the (surjective) projection map $I \to \{\Gamma\}$ has dimension $2(n-2)$, we know that $\dim\{\Gamma\} \geq n-2$. So X is $G_{k-1,k}$-defective with defect at least $n-1$.

Suppose now that $N > n+k+1$. Consider a linear subspace $\mathbb{P}^{N-k-1} \subset \mathbb{P}^N$ disjoint with $\langle P_0, \dots, P_k \rangle$. Let Y be the closure of the image of the projection map $X \to \mathbb{P}^{N-k-1}$. From the proof of [7, Prop. 6.1] it follows that a general point in the singular locus of Y gives rise to a rational normal curve Γ on X of degree $k+1$ through $P_0, \dots, P_k$, hence X has $G_{k-1,k}$-defect $\delta_{k-1,k}(X)$ at least $\dim(\mathrm{Sing}(Y)) + 1$. In case $N = n+k+2$ the variety Y is a quadric in $\mathbb{P}^{n+1}$ that contains linear subspaces of dimension $n-1$. From [10, Chap. 6, Sec. 1] it follows that the rang of the quadric is at most 4 and so $\mathrm{Sing}(Y)$ is at least $(n-3)$-dimensional, hence X has $G_{k-1,k}$-defect at least $n-2$. If $N > n+k+2$, Y is a scroll with $\dim(\mathrm{Sing}(Y)) \geq 2n+k-1-N \geq 0$, so $\delta_{k-1,k}(X) \geq 2n+k-N \geq 1$. ■

Remark 6.2. If we take $N = n+k+2$, we see that a smooth n-dimensional rational normal scroll $X \subset \mathbb{P}^N$ is $G_{k-1,k}$-defective with defect at least $n-2$. One can see that the image of X under a projection with center a point in $\mathbb{P}^N \setminus X$ will also have $G_{k-1,k}$-defect at least $n-2$.

7 A fine classification

The case of curves, surfaces and threefolds has been handled in [7] ($k > n$ and $n \in \{2,3\}$), [5] ($n = k = 2$), [8] ($n = k = 3$) or [4] ($n = 1$). So we have only to prove Theorem 1.2. in case $n \geq 4$. From Section 4 follows that there are no smooth varieties that satisfy condition 1 of Theorem 1.1, so we only have to consider condition 2. The proof will also imply that only in the case that X is rational normal scroll of (minimal) degree in $\mathbb{P}^{n+k+1}$ the $G_{k-1,k}$-defect is equal to $n-1$.

Theorem 7.1. *Let $X^{n\geq 4} \subset \mathbb{P}^{N\geq n+k+1}$ be a smooth non-degenerate variety with $a = \delta_{k-1,k}(X) \geq n-2$ that satisfies condition 2 of the characterization for $k \geq n$. Then X is one of the following varieties:*

1. *X^n is a rational normal scroll of (minimal) degree $k+2$ in $\mathbb{P}^{n+k+1}$;*
2. *X^n is a rational normal scroll of (minimal) degree $k+3$ in $\mathbb{P}^{n+k+2}$;*
3. *X^n is the projection in $\mathbb{P}^{n+k+1}$ of a n-dimensional rational normal scroll of (minimal) degree $k+3$ in $\mathbb{P}^{n+k+2}$.*

Proof:
Suppose that $X^{n\geq 4} \subset \mathbb{P}^{N\geq n+k+1}$ satisfies the conditions of the Theorem. Let $P_0, \ldots, P_k$ be $k+1$ general points of X, contained in a general curve section $C = X \subset G_0$ (G_0 is a linear subspace of dimension $N-n+1$) of X. Note that C is smooth and that there exists a family of rational normal curves of degree $k+1$ on X through $P_0, \ldots, P_k$ of dimension $a-1 \geq n-3$. Let $H \subset \langle P_0, \ldots, P_k \rangle$ be a $k-1$-dimensional linear subspace, $(Q_0, \ldots, Q_k) \in \Omega_{H,k}$ a general point and $S' = X \subset G'$ with $G' = \langle G_0, Q_0 \rangle$ (S' is an irreducible surface section of X that is smooth in the points $P_0, \ldots, P_k$). There is a $(n-2)$-dimensional family of $(N-n+2)$-dimensional linear subspaces of $\mathbb{P}^N$ containing G_0, so we have two possibilities (we use the notations of the proof of Prop. 5.1):

(a) G' is a general element in this family; i.e. the projection morphism ν is surjective. In this case S' is smooth since X is smooth and S' is a general surface section of X. The surface S' contains a subfamily of $\{\Gamma\}$ of dimension at least k.

(b) G' is a special element in this family; i.e. the projection morphism ν is not surjective. In this case S' contains a subfamily of $\{\Gamma\}$ of dimension at least $k+1$. In particular the linear system $|g|$ on C has degree $k+1$ and dimension at least $k+1$. Hence S' has sectional genus 0, but S' does not need to be smooth.

Case (a).

Let $\mathcal{L}$ be the linear system defining $S \subset \mathbb{P}^{N-1\geq k+3}$ and write $\mathcal{L}(-\Gamma)$ to denote $\{D \in \mathcal{L} \,|\, D - \Gamma \geq 0\}$. Since $\dim(\langle\Gamma\rangle) = k+1$, we have $\dim(\mathcal{L}(-\Gamma)) \geq 1$. Notice that $\mathcal{L} - \Gamma = \{D - \Gamma \,|\, D \in \mathcal{L}(-\Gamma)\}$ does not have fixed components because Γ is the only curve in $X \cap \langle\Gamma\rangle$ and $X \cap \langle\Gamma\rangle$ smooth in a general point of Γ. Let C' be a general element of $\mathcal{L} - \Gamma$, then $\Gamma.(\Gamma + C') = k+1$. Since $\Gamma \cup C'$ is connected we have $\Gamma.C' \geq 1$. On the other hand, since S' contains a subfamily of $\{\Gamma\}$ of dimension at least k we find $\Gamma.\Gamma \geq k-1$. So we obtain two possibilities: $\Gamma.C' = 1$ and $\Gamma.\Gamma = k$ or $\Gamma.C' = 2$ and $\Gamma.\Gamma = k-1$.

Case $\Gamma.C' = 2$ and $\Gamma.\Gamma = k-1$.
If $\mathcal{L} - \Gamma$ is composed by a pencil, we can prove using exactly the same arguments as in the proof of [7, Th. 1.2, p. 216] that X^n is minimal of degree $k+3$ in $\mathbb{P}^{n+k+2}$ or a projection of such a variety in $\mathbb{P}^{n+k+1}$ from a point. By [9], the only n-dimensional varieties with minimal degree $k+3$ for some $k \geq n \geq 4$ are rational normal scrolls, so we get the cases 2 and 3 of the Theorem.

Assume now that $\mathcal{L} - \Gamma$ is not composed by a pencil, hence in general C' is irreducible. Since $\Gamma.C' = 2$, we have

$$g(C) = p_a(\Gamma + C') = 1 + \frac{1}{2}(\Gamma + C').(\Gamma + C' + K) = p_a(C') + p_a(\Gamma) + 1 \leq 1.$$

Since $g(\Gamma) = p_a(\Gamma) = 0$, we find $g(C') = p_a(C') = 0$ and $g(C) = 1$, so X has sectional genus 1. From Section 2.4 it follows that X has to be a scroll over an elliptic curve. This gives us a contradiction since $k+1$ general points of X are contained in a rational normal curve of degree $k+1$.

Case $\Gamma.C' = 2$ and $\Gamma.\Gamma = k-1$.
Because $|C'|$ has no fixed components, $|C'|$ is not composed by a pencil and so C' is in general irreducible. Since $\dim|\Gamma| \geq k$ and $\Gamma.C' = 1$, it follows that $|\Gamma - C'| \neq \emptyset$ and we can write $\Gamma \sim \alpha C' + C''$ for some $\alpha \geq 1$ and $C'' \geq 0$ with $|C'' - C'| = \emptyset$. If $C'' = 0$ we have $\Gamma \sim \alpha C'$ and thus $\alpha^2(C'.C') = \Gamma.\Gamma = k$ and $\alpha(C'.C') = \Gamma.C' = 1$, which is in contradiction with $k \geq n \geq 4$, hence $C'' \neq 0$. We have

$$\alpha(C'.C') + C'.C'' = (\alpha C' + C'').C' = \Gamma.C' = 1.$$

Since $C'.C' \geq 0$ ($\dim(|C'|) \geq 1$) and $C'.C'' \geq 1$ ($C' \cup C''$ is connected) we obtain $C'.C' = 0$ and so $\deg(X) = C.C = k+2$. Because $\mathrm{codim}(X)+1 = N-n+1 \geq k+2$ we find that X^n is a smooth variety in $\mathbb{P}^{n+k+1}$ of minimal degree $k+2$. Using [9], we find that X is a rational normal scroll (Case 1 of the Theorem).

Case (b).

Since C is a smooth hyperplane section of S', S' is smooth along C and thus $\mathrm{Sing}(S') \cap C = \emptyset$. So $\mathrm{Sing}(S')$ is finite and S' is irreducible.

Claim. *If $s \in Sing(S')$ and $\Gamma \subset S'$ is a general curve in $\{\Gamma\}$, then $s \notin \langle\Gamma\rangle$.*

Proof Claim: First we are going to prove that $s \notin \Gamma$. Assume $s \in \Gamma$. Since $\mathrm{Sing}(S')$ is finite, $s \in \Gamma$ for all curves Γ on S'. So a general curve Γ on S' is completely determined by $k+1$ points $P_0, \ldots, P_k$ on C as being the only 1-dimensional component of $X \cap \langle P_0, \ldots, P_k, s\rangle$. The uniqueness follows from $X \cap \langle P_0, \ldots, P_k\rangle = \{P_0, \ldots, P_k\}$ as a scheme. Now take $k+2$ general points $P_0, \ldots, P_{k-1}, Q, Q'$ on C and let Γ (respectively Γ') be the curve in the family corresponding with $P_0, \ldots, P_{k-1}, Q$ (respectively $P_0, \ldots, P_{k-1}, Q'$). Because

$\dim(\langle P_0,\ldots,P_{k-1},Q,Q'\rangle)=k+1$, we can consider a deformation of C on S' to another curve C' containing $P_0,\ldots,P_{k-1},Q,Q'$. Since Γ and Γ' are contained in $\langle C'\cup\{s\}\rangle$, the surface S' is deformed into $S''=X\cap\langle C'\cup\{s\}\rangle$. Because $\Gamma\cap\Gamma'$ is finite it follows $s\in \mathrm{Sing}(S'')$. So for a general linear subspace $\mathbb{P}^{N-n+2}\subset\mathbb{P}^N$ with $\langle P_0,\ldots,P_{k-1},Q,Q',s\rangle\subset\mathbb{P}^{N-n+2}$ we find $\dim(\mathbb{T}_s(X)\cap\mathbb{P}^{N-n+2})>2$, so we may assume that

$$\mathbb{T}':=\mathbb{T}_s(X)\subset\langle P_0,\ldots,P_{k-1},Q,Q',s\rangle=\mathbb{T}_s(X)\subset\mathbb{P}^{N-n+2}$$

with $\mathbb{P}^{N-n+2}=\langle S'\rangle=\langle C\cup\{s\}\rangle$. Denote $\mathbb{T}_s(X)\subset\langle P_0,\ldots,P_{k-1},Q,Q'\rangle$ by $\mathbb{T}$. Since $s\notin C=X\cap\langle C\rangle$ and $\langle P_0,\ldots,P_{k-1},Q,Q'\rangle\subset\langle C\rangle\subsetneq\langle C\cup\{s\}\rangle$, we have $\mathbb{T}=\mathbb{T}_s(X)\cap\langle C\rangle$ and so

$$\mathbb{T}=\mathbb{T}_s(X)\cap\langle C\rangle\subset\langle P_0,\ldots,P_{k-1},Q,Q'\rangle\subset\langle C\rangle.$$

Since $P_0,\ldots,P_{k-1},Q,Q'$ are generally chosen on C and $k+1<N-n+1$, we may assume that those points are contained in a general hyperplane of $\langle C\rangle$ (not containing $\mathbb{T}$), a contradiction.

If $s\in\langle\Gamma\rangle\backslash\Gamma$ then s is one of the finitely many points in $\langle\Gamma\rangle\cap X$ not on Γ. So a general curve Γ is again completely determined by $k+1$ points $P_0,\ldots,P_k$ on C. Take a deformation of C on X to another curve C' containing $P_0,\ldots,P_k$. Since Γ is contained in $\langle C'\cup\{s\}\rangle$ and $s\in\langle\Gamma\rangle$, the surface S' deforms to $S''=\langle C'\cup\{s\}\rangle\cap X$ with $s\in\mathrm{Sing}(S'')$. As before we find $\mathbb{T}_s(X)\subset\langle P_0,\ldots,P_k,s\rangle$ and thus $\dim(\mathbb{T}_s(X)\cap\langle P_0,\ldots,P_k\rangle)\geq 2$ for general points $P_0,\ldots,P_k$ on C. Since $s\notin\langle P_0,\ldots,P_k\rangle\subset\langle C\rangle$ (otherwise $s\in C=X\cap\langle C\rangle$ and so $s\notin\mathrm{Sing}(S')$) we obtain $\mathbb{T}:=\mathbb{T}_s(X)\cap\langle P_0,\ldots,P_k\rangle=\mathbb{T}_s(X)\cap\langle C\rangle$. But we may again assume that $P_0,\ldots,P_k$ are contained in a general hyperplane of $\langle C\rangle$ since $k<N-n+1$. So we get a contradiction. □

By using the same arguments as in the proof of [7, Theorem 1.2, p. 218-219] we can prove that in this case X is of minimal degree $k+2$ in $\mathbb{P}^{n+k+1}$. Hence X is a rational normal scroll by [9]. ■

Acknowledgements

The author wishes to thank Marc Coppens for suggesting the subject of the paper and some interesting ideas. His research was partially supported by the Fund of Scientific Research - Flanders (G.0318.03).

References

[1] E. Arbarello, M. Cornalba, P.A. Griffith, J. Harris *Geometry of Algebraic Curves Volume I*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Vol. **267** (1985).

[2] C. Carlini, J. Chipalkatti, *On Waring's problem for several algebraic forms*, Comment. Math. Helv. **78** (2003), 494-517.

[3] L. Chiantini, C. Ciliberto, *Weakly defective varieties*, Trans. Amer. Soc. **354** (2001), 151-178.

[4] L. Chiantini, C. Ciliberto, *The Grassmannians of secant varieties of curves are not defective*, Indag. Math. 13 (2002) 23-28.

[5] L. Chiantini, M. Coppens, *Grassmannians of secant varieties*, Forum Math. 13 (2001) 615-628.

[6] C. Ciliberto, *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics, Barcelona, Vol. *I* (2000), Progress in Mathematics, Vol. **201** (2001), 289-316.

[7] F. Cools, *On $G_{k-1,k}$-defectivity of smooth surfaces and threefolds*, Journal of Pure and Applied Algebra 203 (2005) 204-219.

[8] M. Coppens, *Smooth Threefolds with $G_{2,3}$-defect*, Int. J. of Math., Vol. **15**, No. 7 (2004), 651-671.

[9] D. Eisenbud, J. Harris, *On varieties of minimal degree*, in Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, Vol. 46 (1987), 3-14.

[10] Ph. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience Publ. (1978).

[11] C. Fontanari, *On Waring's problem for many forms and Grassmann defective varieties*, Journal of Pure and Applied Algebra, Vol. **74**, No. 3 (2002), 243-247.

[12] T. Fujita, *Classifiaction theories of polarized varieties*, London Math. Soc. Lecture Note Series, vol. 155, Cambrigde University Press, 1990.

[13] E. Rogora, *Varieties with many lines.*

[14] F. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs 127 (1993).