Some general results on plane curves with total inflection points

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October 19, 2006

Abstract. — *In this paper we study plane curves of degree d with e total inflection points, for nonzero natural numbers d and e.*

MSC. — *14H10, 14H50, 14N20.*

0 Introduction

Let e and d be nonzero natural numbers and denote by $V_{d,e}$ the set of elements $((L_1, P_1), \ldots, (L_e, P_e))$ with $L_1, \ldots, L_e$ lines in the complex projective plane $\mathbb{P}^2$, P_i a point on the line L_i for each i, $P_i \notin L_j$ for each $i \neq j$ and such that there exists a plane curve C of degree d with contact order d with L_i at P_i for each i. So $P_1, \ldots, P_e$ are total inflection points of C.

In [5], the case $d = 4$ (i.e. quartic curves) has been studied intensively. The main tool used in that thesis is the so-called λ-invariant, which is nothing else than a cross ratio of four points. In [3], the cases $e = 1, 2$ have been handled and also the cases $e = 3, 4$ for some special configurations of the lines L_i and the points P_i.

In Section 1 of this paper, we will first prove a general result (Proposition 1.3). Note that part (a) of this Proposition is already known (see [5, Chapter II, Lemma 2.15] or [3, Theorem A]). In Section 2, we prove a generalization of the main result of a paper of E. Ballico (see [2]). In Section 3, we introduce the notion of expected dimension for a component of $V_{d,e}$.

In a following paper of the authors, the established techniques of this paper will be used to describe all the components of $V_{d,e}$ in case e is equal to 3 or

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4, together with their images in the moduli spaces. Both authors are partially supported by the Fund of Scientific Research - Flanders (G.0318.06).

1 Definition and a general result

Definition 1.1. Let $\mathbb{P}^2$ be the complex projective plane and $\mathcal{P}^2$ be the incidence relation in $(\mathbb{P}^2)^* \times \mathbb{P}^2$, thus $\mathcal{P}^2 = \{(L,P)|P \in L\}$.

If d and e are nonzero natural numbers, we denote by $V_{d,e} \subset (\mathcal{P}^2)^e$ the set of elements $(\mathcal{L},\mathcal{P}) = ((L_1,P_1),\dots,(L_e,P_e))$ with $P_i \notin L_j$ for all $i \neq j$ (hence also $L_i \neq L_j$ for $i \neq j$) and such that there exists a plane curve Γ of degree d, not containing any of the lines L_i, with $i(L_i.\Gamma, P_i) = d$. We write $\overline{V_{d,e}}$ to denote the closure of $V_{d,e}$ in $(\mathcal{P}^2)^e$.

Assume $(\mathcal{L},\mathcal{P}) \in V_{d,e}$. On L_i we have the divisor dP_i, so we can consider the subscheme $dP_i \subset \mathbb{P}^2$. Denote by $d\mathcal{P}$ the subscheme $dP_1 + \dots + dP_e \subset \mathbb{P}^2$ of length ed.

Denote by $V(\mathcal{L},\mathcal{P}) \subset \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ the set $\{s \mid d\mathcal{P} \subset Z(s)\}$, hence $\mathbb{P}(V(\mathcal{L},\mathcal{P}))$ is the associated linear system of plane curves.

Remark 1.2. We can easily see that $V_{d,1} = \mathcal{P}^2$ and that $\dim(V(L,P)) = \binom{d+2}{2} - d$ for all $(L,P) \in \mathcal{P}^2$. This follows from the fact that for each line $L \in (\mathbb{P}^2)^*$ the restriction map $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to \Gamma(L, \mathcal{O}_L(d))$ is surjective and $dP \subset L$ corresponds to a unique element of $\mathbb{P}(\Gamma(L, \mathcal{O}_L(d)))$.

Proposition 1.3. *Let $(\mathcal{L},\mathcal{P}) \in V_{d,e}$.*

(a) $\dim(V(\mathcal{L},\mathcal{P})) = \binom{d-e+2}{2} + 1$, *where* $\binom{n}{2}$ *is defined to be* 0 *if* $n < 2$.

(b) Let L be a line in $\mathbb{P}^2$ with $P_i \notin L$ for $1 \leq i \leq e$. Let $V_L(\mathcal{L},\mathcal{P})$ be the image of the restriction map $V(\mathcal{L},\mathcal{P}) \to \Gamma(L, \mathcal{O}_L(d))$. Let $P_{i0} = L_i \cap L$ for $1 \leq i \leq e$. If $d \geq e$, $\dim(V_L(\mathcal{L},\mathcal{P})) = d-e+2$ and $\mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$ is a linear system g^{d-e+1}_d on L containing $P_{10} + \ldots + P_{e0} + g^{d-e}_{d-e}$. If $d < e$, $\dim(V_L(\mathcal{L},\mathcal{P})) = 1$.

(c) Under the assumptions of (b), for $P \in L$ with $P \neq P_{i0}$ for all $1 \leq i \leq e$ one has $((\mathcal{L},\mathcal{P}),(L,P)) \in V_{d,e+1}$ if and only if $dP \in \mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$.

Proof:
Note that (a) is true in case $e = 1$ by Remark 1.2. Consider $(L_1,P_1) \in \mathcal{P}^2$ and let $L \in (\mathbb{P}^2)^*$ be a line different from L_1. Let ℓ (resp. ℓ_1) be an equation of L (resp. L_1) and let $f \in \mathrm{Ker}[V(L_1,P_1) \to \Gamma(L, \mathcal{O}_L(d))]$, hence ℓ divides f. The divisor $dP_1 + P_{10} \subset L_1$ belongs to $Z(f)$, hence ℓ_1 divides f. This proves

$$\mathrm{Ker}[V(L_1,P_1) \to \Gamma(L, \mathcal{O}_L(d))] = \begin{cases} \ell\ell_1\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-2)) & \text{if } d \geq 2, \\ 0 & \text{if } d = 1, \end{cases}$$

hence the dimension of $\mathrm{Ker}[V(L_1,P_1)\to\Gamma(L,\mathcal{O}_L(d))]$ is equal to $\binom{d}{2}$ and

$$\dim(V_L(L_1,P_1)) = \dim(V(L_1,P_1)) - \binom{d}{2} = d+1.$$

This proves $V_L(L_1,P_1)=\Gamma(L,\mathcal{O}_L(d))$, which is (b) in case $e=1$.

For $P\in L$ with $P\neq P_{10}$ the divisor $dP\subset L$ corresponds to a unique element of $\mathbb{P}(V_L(L_1,P_1))$, hence $((L_1,P_1),(L,P))\in V_{d,2}$ proving (c) in case $e=1$, and we find $\dim(V((L_1,P_1),(L,P)))=\dim(V_L(L_1,P_1))-d=\binom{d}{2}$ ((a) in case $e=2$).

Take $e\geq 2$ and assume that (a) is true for e. Let ℓ_i (resp. ℓ) be an equation for L_i (resp. L). As before we find

$$\mathrm{Ker}[V(\mathcal{L},\mathcal{P})\to\Gamma(L,\mathcal{O}_L(d))] = \begin{cases} \ell\ell_1\ldots\ell_e\Gamma(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d-e-1)) & \text{if } d\geq e,\\ 0 & \text{if } d<e,\end{cases}$$

hence the dimension of $\mathrm{Ker}[V(\mathcal{L},\mathcal{P})\to\Gamma(L,\mathcal{O}_L(d))]$ is equal to $\binom{d-e+1}{2}$ and

$$\begin{aligned}\dim(V_L(\mathcal{L},\mathcal{P})) &= \dim(V(\mathcal{L},\mathcal{P})) - \binom{d-e+1}{2}\\ &= \binom{d-e+2}{2}+1-\binom{d-e+1}{2} = \begin{cases} d-e+2 & \text{if } d\geq e,\\ 1 & \text{if } d<e.\end{cases}\end{aligned}$$

If $d\geq e$, $\ell_1\ldots\ell_e\Gamma(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d-e))\subset V(\mathcal{L},\mathcal{P})$ and we obtain that

$$P_{10}+\ldots+P_{e0}+g^{d-e}_{d-e}\subset\mathbb{P}(V_L(\mathcal{L},\mathcal{P})).$$

This finishes the proof of (b) for e. Moreover, claim (c) holds because of the construction of $\mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$. In particular, if $(\mathcal{L}',\mathcal{P}'):=((\mathcal{L},\mathcal{P}),(L,P))\in V_{d,e+1}$, then dP corresponds to a unique divisor of $\mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$. This implies

$$\begin{aligned}\dim(V(\mathcal{L}',\mathcal{P}')) &= \dim(V(\mathcal{L},\mathcal{P}))-\dim(\mathbb{P}(V_L(\mathcal{L},\mathcal{P})))\\ &= \binom{d-e+2}{2}+1-\begin{cases} d-e+1 & \text{if } d\geq e\\ 0 & \text{if } d<e\end{cases}\\ &= \binom{d-(e+1)+2}{2}+1,\end{aligned}$$

hence (a) holds for $e+1$. ■

Remark 1.4. From the proof of Proposition 1.3 in case $e=2$ we also obtain $V_{d,2}=\{((L_1,P_1),(L_2,P_2))\mid P_i\notin L_j \text{ for } i\neq j\}$, hence $\overline{V_{d,2}}=(\mathcal{P}^2)^2$.

2 A generalization of a result of Ballico

In this section, we will prove the following generalization of a result of E. Ballico (see [2]), where the case $e \leq 2$ has been handled. The main reason why E. Ballico restricted himself to the case $e \leq 2$, is that the analogous statement of Remark 1.4 does not hold for $e \geq 3$. Due to the geometrical clear statements of Proposition 1.3, we are able to make the natural generalization in case $e \geq 3$.

Proposition 2.1. *Assume that $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$ with $d \geq e$ and that z is an integer satisfying $3z < \binom{d-e+2}{2}$ and $z \leq \binom{d-e-1}{2}$. Let $O = \{O_1, \ldots, O_z\}$ be general points in $\mathbb{P}^2$ and let $V(\mathcal{L}, \mathcal{P}, O) = \{s \in V(\mathcal{L}, \mathcal{P}) \mid Z(s)$ is singular at $O_1, \ldots, O_z\}$.*

(a) $\dim(V(\mathcal{L}, \mathcal{P}, O)) = \dim(V(\mathcal{L}, \mathcal{P})) - 3z$ *and a general element of $V(\mathcal{L}, \mathcal{P}, O)$ does not contain some line L_i for $1 \leq i \leq e$.*

(b) If $(d-e, z) \neq (6, 9)$, then a general element of $V(\mathcal{L}, \mathcal{P}, O)$ is smooth outside $O_1, \ldots, O_z$ and has ordinary nodes at $O_1, \ldots, O_z$.

To prove Proposition 2.1, we use the following classical result of E. Arbarello and M. Cornalba (see [1]).

Proposition 2.2. *Let $V_f(O) = \{s \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(f)) \mid Z(s)$ singular at $O_1, \ldots, O_z\}$.*

1. *If $3z < \binom{f+2}{2}$ and $z \leq \binom{f-1}{2}$, then $\dim(V_f(O)) = \binom{f+2}{2} - 3z$.*
2. *In case $(f, z) \neq (6, 9)$, a general element of $V_f(O)$ corresponds to an irreducible curve Γ smooth outside $O_1, \ldots, O_z$ and having an ordinary node at $O_1, \ldots, O_z$.*

Proof of Proposition 2.1: Let $(\mathcal{L}_x, \mathcal{P}_x) = ((L_1, P_1), \ldots, (L_x, P_x))$ for all $1 \leq x \leq e$, $V(\mathcal{L}_0, \mathcal{P}_0) = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ and $V(\mathcal{L}_0, \mathcal{P}_0, O) = V_d(O)$.

Claim i : *For $0 \leq x \leq e$ we have $\dim(V(\mathcal{L}_x, \mathcal{P}_x, O)) = \dim(V(\mathcal{L}_x, \mathcal{P}_x)) - 3z$ and a general element of $\mathbb{P}(V(\mathcal{L}_x, \mathcal{P}_x, O))$ does not contain any of the lines L_i (for $1 \leq i \leq x$).*

Proof of Claim i : Since $3z < \binom{d-e+2}{2} \leq \binom{d+2}{2}$ and $z \leq \binom{d-e-1}{2} \leq \binom{d-1}{2}$, this claim holds in case $x = 0$ (classical case). Assume $x \geq 1$ and that the claim holds for $x - 1$. Clearly

$$\mathrm{Ker}[V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O) \to \Gamma(L_x, \mathcal{O}_{L_x}(d))] = \ell_1 \cdots \ell_x V_{d-x}(O).$$

Since $1 \leq x \leq e$, one has $3z < \binom{d-x+2}{2}$ and $z \leq \binom{d-x-1}{2}$, hence the dimension of that kernel is equal to $\binom{d-x+2}{2} - 3z$, because of Proposition 2.2.

Let $V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O) = \mathrm{Im}[V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O) \rightarrow \Gamma(L_x, \mathcal{O}_{L_x}(d))]$, then we find by using Proposition 1.3 (a) that

$$\begin{aligned}
\dim(V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O)) &= \dim(V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O)) - \binom{d-x+2}{2} + 3z \\
&= \dim(V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) - \binom{d-x+2}{2} \\
&= \{\binom{d-(x-1)+2}{2} + 1 - \delta_{0,x-1}\} - \binom{d-x+2}{2} \\
&= d-x+3-\delta_{0,x-1} = \dim(V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})),
\end{aligned}$$

where $\delta_{0,x-1}$ is a Kronecker delta. This implies that

$$V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O) = V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$$

since $V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O) \subset V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$. So from $(\mathcal{L}_x, \mathcal{P}_x) \in V_{d,x}$ it follows that

$$dP_x \in \mathbb{P}(V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) = \mathbb{P}(V_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O)),$$

hence

$$\begin{aligned}
\dim(V(\mathcal{L}_x, \mathcal{P}_x, O)) &= \dim(V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}, O)) - (d-x+2-\delta_{0,x-1}) \\
&= \dim(V(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) - (d-x+2-\delta_{0,x-1}) - 3z \\
&= \dim(V(\mathcal{L}_x, \mathcal{P}_x)) - 3z
\end{aligned}$$

and a general element of $\mathbb{P}(V(\mathcal{L}_x, \mathcal{P}_x, O))$ does not contain L_x. If it would contain L_i for some $1 \leq i \leq x-1$ then it should contain the divisor $P_{ix} + dP_x \subset L_x$ (here $P_{ix} = L_i \cap L_x$), hence it would contain L_x, a contradiction. This finishes the proof of Claim i. $\square$

Since $(d-e, z) \neq (6, 9)$, there exists an irreducible curve $\Gamma_0 \in V_{d-e}(O)$ smooth outside $O_1, \ldots, O_z$ having ordinary nodes at $O_1, \ldots, O_z$.

Claim ii : *We can assume that $P_i \notin \Gamma_0$ for $1 \leq i \leq e$.*

Proof of Claim ii : Varying $O_1, \ldots, O_z$, the closure of the union of the loci $V_{d-e}(O)$ contains the union of $d-e$ general lines in $\mathbb{P}^2$ (see for example [4]). Such union does not contain any of those points P_i, hence for $O_1, \ldots, O_z$ general we can assume $P_i \notin \Gamma_0$ for $1 \leq i \leq e$. $\square$

It follows that the curve $L_1 + \ldots + L_e + \Gamma_0 \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ is smooth at $P_1, \ldots, P_e$.

From Claim i we know that $\Gamma \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}, O))$ general does not contain any of the lines L_i $(1 \leq i \leq e)$ hence $\Gamma \cap L_i = \{P_i\}$. It follows that the fixed locus of $\mathbb{P}(V(\mathcal{L}, \mathcal{P}, O))$ is contained in the finite set $\{P_1, \ldots, P_e\} \cap (\Gamma \cap \Gamma_0)$.

A general element of $\mathbb{P}(V(\mathcal{L},\mathcal{P},O))$ is singular at one of the points if and only if each element of $\mathbb{P}(V(\mathcal{L},\mathcal{P},O))$ is singular at that point. We already knew this does not hold for $P_1,\ldots,P_e$. Also, since Γ_0 is smooth outside $O_1,\ldots,O_z$ and $\Gamma\cap\Gamma_0\cap L_i=\emptyset$, it follows that a general element of $\mathbb{P}(V(\mathcal{L},\mathcal{P},O))$ is smooth outside $O_1,\ldots,O_z$. Since $L_1+\ldots+L_e+\Gamma_0$ has ordinary nodes at $O_1,\ldots,O_z$, a general element Γ of $\mathbb{P}(V(\mathcal{L},\mathcal{P},O))$ has ordinary nodes at $O_1,\ldots,O_z$. ■

3 Expected dimension of a component of $V_{d,e}$

Let $(\mathcal{P}^2)^{e,0}$ be the open subset of $(\mathcal{P}^2)^e$ of elements $(\mathcal{L},\mathcal{P})$ satisfying $P_i\notin L_j$ for $i\neq j$. For $(\mathcal{L},\mathcal{P})\in(\mathcal{P}^2)^{e,0}$ and $i\neq j$ we write $P_{i,j}=L_i\cap L_j$ and we write g'_i to denote the linear system $P_{1,i}+\ldots+P_{i-1,i}+g^{d-i+1}_{d-i+1}$ on L_i in case $2\leq i\leq d+1$. We take $g'_i=\emptyset$ if $i>d+1$.

Assume $e\geq 3$ and let $\mathcal{G}^e$ be the space of pairs $(g,(\mathcal{L},\mathcal{P}))$ with $(\mathcal{L},\mathcal{P})\in(\mathcal{P}^2)^{e,0}$ and g being an $(e-2)$-tuple $(g_3,\ldots,g_e)$ of linear systems as follows. For $3\leq i\leq e$, g_i is a linear system $g^{\max\{d-i+2,0\}}_d$ on L_i containing g'_i.

Let $\tau:\mathcal{G}^e\rightarrow(\mathcal{P}^2)^{e,0}$ be the natural projection. It is a smooth morphism of relative dimension $\sum_{i=1}^{e-2}i=\frac{(e-1)(e-2)}{2}$.

Claim 3.1: *There exist 2 sections S_1 and S_2 of τ such that $V_{d,e}=\tau(S_1\cap S_2)$.*

Corollary 3.2. *Each component of $V_{d,e}$ has codimension at most $\frac{(e-1)(e-2)}{2}$ inside $(\mathcal{P}^2)^e$.*

Definition 3.3. Let V be a component of $V_{d,e}$. We say V has the expected dimension if $\dim(V)=3e-\frac{(e-1)(e-2)}{2}$ and we say that V is of exceptional dimension if $\dim(V)>3e-\frac{(e-1)(e-2)}{2}$. In case $e\geq 9$ (this is a bound independent of d) then all components of $V_{d,e}$ have exceptional dimension.

Proof of Claim 3.1: The first section S_1 is defined by $S_1(\mathcal{L},\mathcal{P})=g$ with $g_i=\langle g'_i\cup\{dP_i\}\rangle\subset g^d_d$ on L_i. In order to define $S_2(\mathcal{L},\mathcal{P})=h$, we are going to make a well-defined construction imitating the construction of the linear space $V(\mathcal{L}_x,\mathcal{P}_x)$ for $1\leq x\leq e$.

We are going to define h inductively starting with $h_3=\mathbb{P}(V_{L_3}(\mathcal{L}_2,\mathcal{P}_2))$. Inductively we also are going to define a sequence of subspaces

$$V'(\mathcal{L}_e,\mathcal{P}_e)\subset V'(\mathcal{L}_{e-1},\mathcal{P}_{e-1})\subset\cdots\subset V'(\mathcal{L}_3,\mathcal{P}_3)\subset V(\mathcal{L}_2,\mathcal{P}_2)$$

such that the dimension of $V'(\mathcal{L}_x,\mathcal{P}_x))$ is equal to $\binom{d-x+2}{2}+1$, the space $V'(\mathcal{L}_x,\mathcal{P}_x)$ contains $\ell_1\ldots\ell_x\Gamma(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d-x))$ if $x\leq d$ and if $s\in V'(\mathcal{L}_x,\mathcal{P}_x)$ and $2\leq i\leq$

$\min\{x, d+1\}$, then either $\ell_i \subset Z(s)$ or the multiplicity of $Z(s) \cap L_i$ at P_i is more then $d-i+1$.

Let $\mathcal{D}_3 \in h_3$ be the divisor in h_3 having maximal multiplicity at P_3 (being at least $d-1$) and let $\mathcal{W}_3 \subset V_{L_3}(\mathcal{L}_2, \mathcal{P}_2)$ be the corresponding 1-dimensional linear subspace. Then $V'(\mathcal{L}_3, \mathcal{P}_3)$ is the inverse image of $\mathcal{W}_3$ in $V(\mathcal{L}_2, \mathcal{P}_2)$.

Since

$$\operatorname{Ker}[V(\mathcal{L}_2, \mathcal{P}_2) \to V_{L_3}(\mathcal{L}_2, \mathcal{P}_2)] = \begin{cases} \ell_1\ell_2\ell_3\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) & \text{if } d \geq 3, \\ 0 & \text{if } d < 3, \end{cases}$$

one has $\ell_1\ell_2\ell_3\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \subset V'(\mathcal{L}_3, \mathcal{P}_3)$ if $d \geq 3$ and $\dim(V'(\mathcal{L}_3, \mathcal{P}_3)) = \binom{d-1}{2} + 1$.

Now let $3 < x \leq e$ and assume h_{x-1} and $V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$ are defined. Let $s \in \operatorname{Ker}[V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}) \to \Gamma(L_x, \mathcal{O}_{L_x}(d))]$, hence ℓ_x divides s. We are going to prove that ℓ_i divides s for $1 \leq i \leq x-1$ too. Since $Z(s)$ contains $P_{1,x} + dP_1 \subset L_1$ we find ℓ_1 divides s. Let $1 < i_0 \leq x-1$ and assume ℓ_i divides s for $1 \leq i < i_0$. If $i_0 \geq d+2$, $\ell_1 \ldots \ell_{i_0-1}$ divides s, so $s = 0$ and a fortiori ℓ_{i_0} divides s. If $i_0 \leq d+1$, $Z(s)$ contains $P_{1,i_0} + \ldots + P_{i_0-1,i_0} \subset L_{i_0}$. If ℓ_{i_0} does not divide s then $Z(s) \cap L_{i_0}$ is a divisor with multiplicity at most $d - i_0 + 1$ at P_{i_0}, a contradiction. Hence ℓ_{i_0} divides s. Since $\ell_1 \ldots \ell_{x-1}\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-x+1)) \subset V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$ if $x \leq d+1$, we find

$$\operatorname{Ker}[V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}) \to \Gamma(L_x, \mathcal{O}_{L_x}(d))] = \begin{cases} \ell_1 \ldots \ell_x\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-x)) & \text{if } x \leq d, \\ 0 & \text{if } x > d. \end{cases}$$

Let $V'_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}) = \operatorname{Im}[V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}) \to \Gamma(L_x, \mathcal{O}_{L_x}(d))]$ then we find

$$\begin{aligned} \dim(V'_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) &= \dim(V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) - \binom{d-x+2}{2} \\ &= \begin{cases} d-x+3 & \text{if } x \leq d+1, \\ 1 & \text{if } x > d+1. \end{cases} \end{aligned}$$

We define $h_x = \mathbb{P}(V'_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}))$. Since $\ell_1 \ldots \ell_{x-1}\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-x+1)) \subset V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$ if $x \leq d+1$, we obtain $g'_x \subset h_x$ on L_x. Let $\mathcal{D}_x$ be the divisor in h_x having maximal multiplicity at P_x (being at least $d-x+2$) and let $\mathcal{W}_x \subset V'_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$ be the associated 1-dimensional linear subspace. Then $V'(\mathcal{L}_x, \mathcal{P}_x)$ is the inverse image of $\mathcal{W}_x$ in $V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})$. One has

$$\begin{aligned} \dim(V'(\mathcal{L}_x, \mathcal{P}_x)) &= \dim(V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) - \dim(V'_{L_x}(\mathcal{L}_{x-1}, \mathcal{P}_{x-1})) + 1 \\ &= \binom{d-x+2}{2} + 1. \end{aligned}$$

Also $V'(\mathcal{L}_x, \mathcal{P}_x)$ contains the kernel of $V'(\mathcal{L}_{x-1}, \mathcal{P}_{x-1}) \to \Gamma(L_x, \mathcal{O}_{L_x}(d))$, hence it contains $\ell_1 \ldots \ell_x\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-x))$ if $x \leq d$.

Of course $S_1(\mathcal{L},\mathcal{P}) = S_2(\mathcal{L},\mathcal{P})$ if and only if $h_i = g_i$ for all $3 \leq i \leq e$. The condition $h_3 = g_3$ is clearly equivalent to $dP_3 \in \mathbb{P}(V_{L_3}(\mathcal{L}_2,\mathcal{P}_2))$, hence it is equivalent to $(\mathcal{L}_3,\mathcal{P}_3) \in V_{d,e}$. Under these equivalence we also have $V(\mathcal{L}_3,\mathcal{P}_3) = V'(\mathcal{L}_3,\mathcal{P}_3)$. Let $3 < i \leq e$ and assume that $h_j = g_j$ for $3 \leq j < i$ is equivalent to $(\mathcal{L}_{i-1},\mathcal{P}_{i-1}) \in V_{d,i-1}$ and that under this equivalence one has $V(\mathcal{L}_{i-1},\mathcal{P}_{i-1}) = V'(\mathcal{L}_{i-1},\mathcal{P}_{i-1})$. Assume $h_j = g_j$ for $3 \leq j \leq i$. Since $V(\mathcal{L}_{i-1},\mathcal{P}_{i-1}) = V'(\mathcal{L}_{i-1},\mathcal{P}_{i-1})$ we have $V'_{L_i}(\mathcal{L}_{i-1},\mathcal{P}_{i-1}) = V_{L_i}(\mathcal{L}_{i-1},\mathcal{P}_{i-1})$. Then the condition $g_i = h_i$ is equivalent to $dP_i \in \mathbb{P}(V_{L_i}(\mathcal{L}_{i-1},\mathcal{P}_{i-1}))$, which is equivalent to $(\mathcal{L}_i,\mathcal{P}_i) \in V_{d,i}$ and by construction we get $V'(\mathcal{L}_i,\mathcal{P}_i) = V(\mathcal{L}_i,\mathcal{P}_i)$. This finishes the proof of the claim. ■

Example 3.4. : case $e = 3$. All components V of $V_{d,3} \subset (\mathcal{P}^2)^3$ are of expected dimension 8. In order to prove this, it is enough to show that $V \subsetneq (\mathcal{P}^2)^3$, since $\dim((\mathcal{P}^2)^3) = 9$ and $\dim V \geq 8$ by Corollary 3.3. Now since for $(\mathcal{L},\mathcal{P}) \in V$ general, we have $V_{L_3}(\mathcal{L}_2,\mathcal{P}_2) \neq \Gamma(L_3,\mathcal{O}_{L_3}(d))$ (see Proposition 1.3(b)) and thus $((\mathcal{L}_2,\mathcal{P}_2),(L_3,P_3)) \notin V_{d,3}$ for general $(L_3,P_3) \in \mathcal{P}^2$, so the claim follows immediately.

Example 3.5. Let d be even and consider

$$V = \{(\mathcal{L},\mathcal{P}) \in (\mathcal{P}^2)^e \,|\, P_i \notin L_j \text{ if } i \neq j \text{ and } \exists C \in \Gamma(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2)) : \mathbb{T}_{P_i}(C) = L_i\}.$$

It is clear that $V \subset V_{d,e}$ has dimension $5+e$ and thus V is of unexpected dimension if and only if $e > 4$.

Example 3.6. Let $e \geq 3$ and consider

$$V = \{\,(\mathcal{L},\mathcal{P}) \in (\mathcal{P}^2)^e \,|\, P_i \notin L_j \text{ if } i \neq j \text{ and } P_1,\ldots,P_e \text{ collinear}\}.$$

It is clear that V is a component of $V_{d,e}$ of dimension $2e+2$, so it is of unexpected dimension if $e > 3$. In fact, in a following paper of the authors, it will be proved that this is the only component of unexpected dimension if $e = 4$.

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