On Grassmann secant extremal varieties

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Abstract. — *In this paper we give a sharp lower bound on the dimension of Grassmann secant varieties of a given variety and we classify varieties for which the bound is attained.*

MSC. — *14M07, 14M15, 14N15*

1 Introduction

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n and let h and k be integers such that $0 \leq h \leq k \leq N$. Denote by $\mathbb{G}_{h,k}(X) \subset \mathbb{G}(h,N)$ the (h,k)–*Grassmann secant variety* of X, i.e. the closure of the set of h-dimensional linear subspaces contained in the span of $k+1$ independent points of X. We will say that X is (h,k)-*Grassmann defective*, or simply (h,k)-*defective*, if the dimension $g_{h,k}(X)$ of $\mathbb{G}_{h,k}(X)$ is smaller than the expected one, which is

$$\gamma_{h,k}(X) := \min\{(k+1)n + (k-h)(h+1), (N-h)(h+1)\}. \qquad (1)$$

The difference $\delta_{h,k}(X) = \gamma_{h,k}(X) - g_{h,k}(X)$ is called the (h,k)-*defect* of X.

The case $h = 0$ is the most studied case (see for example [14]), where the variety $\mathbb{G}_{h,k}(X)$ coincides with the k-th secant variety $S_k(X)$. In this case one speaks of k–defective, rather than $(0,k)$–defective variety, of k-defect $\delta_k(X)$, rather that $(0,k)$–defect, etc.

If $h > 0$, little is known. One of the most important reasons for this is the lack of an analogue of Terracini's Lemma (see [14, Chapter V, Prop. 1.4]). Nevertheless, in [6] it is shown that curves are never Grassmann defective, in [7] there is the classification of $(1,2)$-defective surfaces and in [9] one gives a classification of $(k-1,k)$-defective surfaces and threefolds with $k > n$.

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Beside the intrinsic interest of Grassmann defective varieties, there are also some extrinsic reasons to study them. For example, Grassmann defective varieties behave strangely under projections and they are related to interesting aspects of the Waring problem (see [3, 8, 11]).

In this paper we will prove the following lower bound for the dimension of $\mathbb{G}_{h,k}(X)$:

$$\begin{aligned} g_{h,k}(X) \ &\geq f(n,h,k) := (k+1)n + (k-h)(h+1) - (k-h)(n-1) \\ &\qquad\qquad \text{if} \quad g_{h,k}(X) < (h+1)(N-h). \end{aligned} \tag{2}$$

Varieties for which $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h,N)$ and the bound (2) is attained are called (h,k)-*extremal varieties*. The main aim of this paper is to classify them (see Theorem 4.1). The case $(h,k) = (0,1)$ was already treated in [4, Section 3]. In view of the result in [6], curves in $\mathbb{P}^N$ are (h,k)–extremal as soon as $(h+1)(N-h) > f(1,h,k)$, i.e. $N > k + \frac{k+1}{h+1}$.

In Section 2, we start with some definitions and basic properties and then we prove the bound (2). In Section 3, we give examples of extremal varieties. In Section 4, we prove that these examples are the only extremal varieties.

2 Basic properties and the proof of the bound.

We start this section with some conventions. We denote the N-dimensional projective space over the field of the complex numbers $\mathbb{C}$ by $\mathbb{P}^N$ and we write $\mathbb{G}(h,N)$ to denote the Grassmannian of h-dimensional linear subspaces of $\mathbb{P}^N$.

A variety $X \subset \mathbb{P}^N$ is an irreducible reduced Zariski closed subset of $\mathbb{P}^N$. We say that $X \subset \mathbb{P}^N$ is *non-degenerate* if X is not contained in any hyperplane of $\mathbb{P}^N$.

Let X be a non-degenerate n-dimensional variety in $\mathbb{P}^N$. We say that a closed subscheme $Y \subset X$ is a *m-dimensional section* of X if Y is the scheme-theoretical intersection of X with a linear subspace of dimension $N-n+m$ of $\mathbb{P}^N$ such that all irreducible components of Y have dimension m. When m equals 1, 2 or $n-1$, we will use the terminology of *curve section, surface section* and *hyperplane section*.

Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of dimension n and let $k \leq N$ be an integer. The set of points $(P_0,\dots,P_k) \in X^{k+1}$ with $\dim(\langle P_0,\dots,P_k\rangle) = k$ is a non-empty Zariski open subset of X^{k+1}. So we have the rational map $\omega : X^{k+1} \dashrightarrow \mathbb{G}(k,N)$ sending $(P_0,\dots,P_k)$ to $\langle P_0,\dots,P_k\rangle$. An element of the image is called a $(k+1)$-*secant k-space* of X.

Consider, for all integers $h \leq k$, the closed, irreducible variety $I = \{(H,G) | \, H \subset G\} \subset \mathbb{G}(h,N) \times \mathbb{G}(k,N)$ with projections α, β to the first and second factor respectively. Define $\mathbb{G}_{h,k}(X)$ as the variety $\alpha(\beta^{-1}(\overline{\mathrm{Im}(\omega)}))$.

In case $N \leq n+k$, the general k-dimensional subspace G meets X along a subvariety of dimension $n+k-N$ which is non–degenerate in G. Therefore the map ω is dominant, thus $\mathbb{G}_{h,k}(X) = \mathbb{G}(h,N)$.

From now on we assume that $N > n+k$. In this case, the map ω is generically injective (see Lemma 2.1 below), hence $\dim(\overline{\mathrm{Im}(\omega)}) = (k+1)n$. Since the fibers of β are Grassmannians $\mathbb{G}(h,k)$, then $\beta^{-1}(\overline{\mathrm{Im}(\omega)})$ is irreducible of dimension $(k+1)n+(k-h)(h+1)$. This explains the definition given in (1) of the expected dimension $\gamma_{h,k}(X)$ of $\mathbb{G}_{h,k}(X) \subset \mathbb{G}(h,N)$ versus the true dimension $g_{h,k}(X) = \dim(\mathbb{G}_{h,k}(X)) \leq \gamma_{h,k}(X)$. As we said, $\delta_{h,k}(X) = \gamma_{h,k}(X) - g_{h,k}(X)$ is called the (h,k)-defect of X, and we we say that X is (h,k)-defective if $\delta_{h,k}(X) > 0$.

Note that, if $h = k$ and $N > n+k$, then clearly $\delta_{k,k}(X) = 0$.

Lemma 2.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate n-dimensional variety with $N \geq n+k+1$ for some integer k and let $P_0, \ldots, P_k$ be general points of X. Then $X \cap \langle P_0, \ldots, P_k \rangle = \{P_0, \ldots, P_k\}$ as a scheme.*

Proof. The points $P_0, \ldots, P_k$ are contained in a general curve section of X in some $\mathbb{P}^{k+2}$. The uniform position lemma for curves (see [1] or [5, Prop. 2.6] for the argument) implies the assertion. □

We will need the following classical result. We sketch its proof for sake of completeness.

Proposition 2.2. *Let $X \subset \mathbb{P}^N$ be a variety and let X' be its general hyperplane section. Then:*

(i) if X is a cone with vertex of dimension $m > 1$, then X' is a cone with vertex of dimension $m-1$ which is the intersection of the vertex of X with the span of X';

(ii) if X' is a cone with vertex of dimension m, then X is a cone with vertex of dimension $m+1$.

Proof. Part (i) is trivial. As for part (ii), note that X is a cone if and only if it has a point of multiplicity equal to the degree of X. Since degree and multiplicity of points are preserved by a general hyperplane section, X' being a cone implies that X is a cone. Then part (i) applies and part (ii) follows. □

The following proposition is well known in the case $h = 0$ (see [14]).

Proposition 2.3. *If $\mathbb{G}_{h,k}(X) = \mathbb{G}_{h,k+1}(X)$, then $\mathbb{G}_{h,k}(X) = \mathbb{G}(h,N)$.*

Proof. Let k_h be the smallest integer l such that $\mathbb{G}_{h,l}(X) = \mathbb{G}(h,N)$. We will show by induction that for all $i \in \{k, \ldots, k_h - 1\}$ we have $\mathbb{G}_{h,i}(X) = \mathbb{G}_{h,i+1}(X)$. This gives us a contradiction if $k < k_h$.

For $i = k$ there is nothing to prove. Let us suppose that $\mathbb{G}_{h,i}(X) = \mathbb{G}_{h,i+1}(X)$ for some $k \leq i < k_h - 1$. Let $\overline{H}$ be a general element of $\mathbb{G}_{h,i+2}(X)$, so $\overline{H} \subset \langle P_0, \ldots, P_{i+2} \rangle$ with $P_0, \ldots, P_{i+2} \in X$ and $\dim(\langle P_0, \ldots, P_{i+2} \rangle) = i + 2$. We may assume that $P_{i+2} \notin \overline{H}$. Let $\overline{H}'$ be the image of the projection of $\overline{H}$ from P_{i+2} to $\langle P_0, \ldots, P_{i+1} \rangle$. Now $\overline{H}' \subset \langle P_0, \ldots, P_{i+1} \rangle$ has dimension h, so $\overline{H}'$ is an element of $\mathbb{G}_{h,i+1}(X) = \mathbb{G}_{h,i}(X)$. Since $\overline{H} \subset \langle \overline{H}', P_{i+2} \rangle$, we have $\overline{H} \in \mathbb{G}_{h,i+1}(X)$. This implies $\mathbb{G}_{h,i+1}(X) = \mathbb{G}_{h,i+2}(X)$. □

Since ω is generically injective if $N > n + k$, there is a slightly different way of describing $\mathbb{G}_{h,k}(X)$. Consider the closure $I_{h,k}(X) \subset X^{k+1} \times \mathbb{G}(h, N)$ of the set of points $(P_0, \ldots, P_k, H)$ with $\dim(\langle P_0, \ldots, P_k \rangle) = k$ and $H \subset \langle P_0, \ldots, P_k \rangle$. Note that $I_{h,k}(X)$ is irreducible of dimension $(k+1)n + (h+1)(k-h)$. Consider the projection map $q : I_{h,k}(X) \to \mathbb{G}(h, N)$. The closure of the image of q is $\mathbb{G}_{h,k}(X)$. Let H be a general element of $\mathbb{G}_{h,k}(X)$. The fibre of q above H is pure dimensional, of dimension

$$\begin{aligned} \xi_{h,k}(X) &:= \dim(I_{h,k}(X)) - \dim(\mathbb{G}_{h,k}(X)) \\ &= \delta_{h,k}(X) + \max\{0, (k+1)n - (h+1)(N-k)\} \end{aligned} \tag{3}$$

(see [12, II, §3, Ex. 3.22]). Let $\Omega_{H,k}$ be a component of this fibre, which can be naturally viewed as a subvariety of X^{k+1}, and denote the i-th projection map to X by p_i.

Now we come to the proof of the bound (2). We will do this in two different ways, both useful for the classification result we will prove in §4. By taking into account (3), one way is to bound $\xi_{h,k}(X)$ form above. This we will do first. The following two lemmas are useful.

Lemma 2.4. *If $h < k$ then the map $p := p_0 \times \cdots \times p_{k-h-1} : \Omega_{H,k} \to X^{k-h}$ is generically finite.*

Proof. Let $(P_0, \ldots, P_k)$ be a general element of $\Omega_{H,k}$. Since $H \cap \langle P_0, \ldots, P_{k-h-1} \rangle = \varnothing$, we have $\langle P_0, \ldots, P_{k-h-1}, H \rangle = \langle P_0, \ldots, P_k \rangle$, so we have the following equality of intersection schemes

$$\langle P_0, \ldots, P_{k-h-1}, H \rangle \cap X = \langle P_0, \ldots, P_k \rangle \cap X = \{P_0, \ldots, P_k\}.$$

Thus if $(P_0, \ldots, P_{k-h-1}, Q_{k-h}, \ldots, Q_k)$ is another element of $p^{-1}(P_0, \ldots, P_{k-h-1})$, we have $Q_i \in \{P_0, \ldots, P_k\}$ for all $i \in \{k-h, \ldots, k\}$. This proves the assertion. □

Lemma 2.5. *If $h < k$ and $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h, N)$, then the map $p_i : \Omega_{H,k} \to X$ is not surjective for any $i = 0, \ldots, k$.*

Proof. Let $\widetilde{H}$ be a general element of $\mathbb{G}_{h,k+1}(X)$. So $\widetilde{H} \subset \langle P_0, \ldots, P_{k+1} \rangle$ with $P_i \in X$ for all i and $\dim(\langle P_0, \ldots, P_{k+1} \rangle) = k+1$. We may assume that $P_{k+1} \notin \widetilde{H}$. Consider the projection of $\widetilde{H}$ to $\langle P_0, \ldots, P_k \rangle$ from P_{k+1}. Since this is a general element of $\mathbb{G}_{h,k}(X)$, we may assume that it is equal to H. If p_i is surjective, then $P_{k+1} \in \mathrm{Im}(p_i)$, so there is a $G \in \mathbb{G}_{k,k}(X)$ with $\widetilde{H} \subset \langle H, P_{k+1} \rangle \subset G$, hence $\widetilde{H} \in \mathbb{G}_{h,k}(X)$ and $\mathbb{G}_{h,k}(X) = \mathbb{G}_{h,k+1}(X)$. Proposition 2.3 implies that $\mathbb{G}_{h,k}(X) = \mathbb{G}(h, N)$, and this yields a contradiction. □

Next we prove the promised bound.

Proposition 2.6. *If $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h, N)$, then*

$$\xi_{h,k}(X) \leq (k-h)(n-1)$$

and

$$\delta_{h,k}(X) \leq \min\{(k-h)(n-1), (h+1)(N-k-n) - (k-h)\}.$$

Proof. The assertion is clear if $h = k$. So assume $h < k$. Denote $p_i(\Omega_{H,k})$ by X_i. Lemma 2.5 implies that $X_i \neq X$ and thus $\dim(X_i) < n$. By Lemma 2.4 we have

$$\begin{aligned} \xi_{h,k}(X) &= \dim(\Omega_{H,k}) = \dim(\mathrm{Im}(p_0 \times \cdots \times p_{k-h-1})) \\ &\leq \dim(X_0) \times \ldots \times \dim(X_{k-h-1}) \leq (k-h)(n-1). \end{aligned}$$

The rest of the assertion follows by (3). □

As an immediate consequence we have:

Theorem 2.7. *If $X \subset \mathbb{P}^N$ is a non–denerate variety then* (2) *holds.*

Corollary 2.8. *If X is (h,k)-defective, then $N > k+n+\frac{k-h}{h+1}$, $n > 1$ and $h < k$.*

Proof. If X is (h,k)-defective, then $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h, N)$ and from Proposition 2.6 we see that that

$$0 < \delta_{h,k}(X) \leq (h+1)(N-k-n) - (k-h)$$

thus the first part of the statement follows. Moreover we also have

$$0 < \delta_{h,k}(X) \leq (k-h)(n-1)$$

yielding the rest of the assertion. □

Note that this provides an alternative proof of the main result of [6].

Definition 2.9. *Let $X \subset \mathbb{P}^N$ be a non-degenerate variety of dimension n. We say that X is (h,k)-*extremal *if $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h, N)$ and equality holds in* (2) *or, equivalently, if $\xi_{h,k}(X) = (k-h)(n-1)$.*

Note that if $X \subset \mathbb{P}^N$ is (h,k)–extremal, then $N \geq n+k+1$. In this case any variety is (k,k)–extremal (see Corollary 2.8). Moreover, any (h,k)–extremal variety is also (h,k)–defective, unless either $h=k$ or $n=1$. Indeed, all curves in $\mathbb{P}^N$ are (h,k)–extremal if and only if $N > k + \frac{k+1}{h+1}$ (see [6]).

We finish this section by giving another proof of Proposition 2.6, which will be useful for the proof of the classification Theorem 4.1 below.

Alternative proof. We want to prove (2). In view of the results in [6], this is true if $n=1$. We will assume $n>1$ and proceed by induction.

Note that $f(n,h,k) - f(n-1,h,k) = h+1$. Let X' be a general hyperplane section of X. Since $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h,N)$, then clearly $\mathbb{G}_{h,k}(X') \neq \mathbb{G}(h,N-1)$. Therefore, by induction, it suffices to prove that

$$g_{h,k}(X) \geq g_{h,k}(X') + h + 1. \tag{4}$$

It is well known that one can flatly degenerate X to the cone over X' with vertex a point, with possibly some embedded component at the vertex, as follows. Consider $\mathbb{P}^N$ as a hyperplane in $\mathbb{P}^{N+1}$ and take a point $T \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$. Let $C(X)$ (resp. $C(X')$) be the cone over X (resp. X') with vertex T. Let $\mathcal{L}$ be a general pencil of hyperplanes in $\mathbb{P}^{N+1}$. A general $L \in \mathcal{L}$ cuts out on $C(X)$ a copy of X. If $L' \in \mathcal{L}$ contains T then its intersection with $C(X)$ has support on $C(X')$. Note however that this intersection might have an embedded component at T due to the failure of the S_2 property for $C(X)$ at T, namely to the lack of projective normality of X (see [12], p. 185). Such an embedded component however will not play any role in our argument.

Let J be the closure of the set of elements $(P_0,\ldots,P_k,L,H)$ with $L \in \mathcal{L}$, $P_0,\ldots,P_k$ general points of $C(X) \cap L$ and H a h-dimensional linear subspace of $\langle P_0,\ldots,P_k\rangle$. Let π be the projection of J to the last two components. For a general $(L,H) \in \pi(J)$, we get

$$\dim(\pi^{-1}(L,H)) = \xi_{h,k}(C(X) \cap L) = \xi_{h,k}(X).$$

If $H \in \mathbb{G}_{h,k}(C(X'))$ is general, then $(L',H) \in \pi(J)$, and we get

$$\dim(\pi^{-1}(L',H)) = \xi_{h,k}(C(X')).$$

Hence $\xi_{h,k}(X) \leq \xi_{h,k}(C(X'))$ and thus

$$g_{h,k}(X) \geq g_{h,k}(C(X')). \tag{5}$$

The projection from the vertex T gives a rational dominant map

$$p : \mathbb{G}_{h,k}(C(X')) \dashrightarrow \mathbb{G}_{h,k}(X').$$

Let $H' \in \mathbb{G}_{h,k}(X')$ be general. We claim that the closure of the fiber of p above H' equals the set of hyperplanes of $\langle H', T\rangle$. Indeed, let H be a general hyperplane of

$\langle H', T\rangle$. There exist points $P'_0, \dots, P'_k \in X'$ with $H' \subset \langle P'_0, \dots, P'_k\rangle$, hence $H \subset \langle P'_0, \dots, P'_k, T\rangle$. A hyperplane of $\langle P'_0, \dots, P'_k, T\rangle$ through H intersects each line $\langle P'_i, T\rangle$ at a point $P''_i \in C(X')$. Hence $H \subset \langle P''_0, \dots, P''_k\rangle$ and $H \in \mathbb{G}_{h,k}(C(X'))$. We conclude that

$$g_{h,k}(C(X')) = g_{h,k}(X') + h + 1.$$

By (5), (4) follows, proving our result. □

From the above argument, we can draw an interesting conclusion about (h, k)–extremal varieties.

Proposition 2.10. *Let X be a (h, k)-extremal variety of dimension n. Then:*

(i) if $n \geq 2$, a general hyperplane section of X is again (h, k)-extremal;

(ii) a cone over X is again (h, k)-extremal.

3 Some examples

Example 3.1. *Let $X \subset \mathbb{P}^N$ be a cone over a curve with vertex a linear subspace of dimension $n - 2$. Then X is (h, k)-extremal, as soon as $N > k + n + \frac{k-h}{h+1}$.*

Proof. Note that X is a cone over its general curve section C which is non–degenerate in $\mathbb{P}^{N-n+1}$. Thus C is (h, k)–extremal as soon as $N > k + n + \frac{k-h}{h+1}$. Then the statement follows from part (ii) of Proposition 2.10. □

Example 3.2. *Let $X \subset \mathbb{P}^{n+3}$ be a cone over the Veronese surface of degree 4 in $\mathbb{P}^5$ with vertex a linear subspace of dimension $n - 3$, and $n \geq 2$. Then X is $(0, 1)$-extremal.*

Proof. By Proposition 2.10, it suffices to remark that the Veronese surface is (0, 1)-extremal, equivalently that it is 1-defective, which is classical (see [13]). □

Example 3.3. *Let $X \subset \mathbb{P}^{n+k+1}$ be a rational normal scroll of dimension n and degree $k + 2$. If $k \leq 2h$ then X is (h, k)-extremal.*

Proof. The assertion is clear for $n = 1$. So we may assume $n \geq 2$. Since singular rational normal scrolls are cones over smooth rational normal scrolls of the same degree, part (ii) of Proposition 2.10 implies that we may assume that X is smooth, and at least 2-dimensional (see Example 3.1).

One has $X = \mathbb{P}(\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(r_i))$, with $r_n \geq \ldots \geq r_1 \geq 1$ integers such that $\sum_{i=1}^n r_i = k + 2$, and X is embedded in $\mathbb{P}^{n+k+1}$ via the sections of the $\mathcal{O}_X(1)$ bundle (see [10]). We will denote by L the hyperplane class of X. Let Π be a general $(n - 1)$–dimensional ruling of the scroll. The linear system $|L - \Pi|$ corresponds to sections of the bundle $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(r_i - 1)$, hence $\dim(|L - \Pi|) = k + 1$, and, since $r_i \geq 1$, $i = 1, \dots, n$, then $|L - \Pi|$ is base point free and therefore its general element X' is a smooth rational normal scroll of dimension $n-1$ and degree $k+1$, which spans a $\mathbb{P}^{n+k-1}$. Indeed the hyperplanes through X' cut out on X the pencil $|\Pi|$.

Let $P_0, \ldots, P_k$ be general points of X and let H be a h–dimensional subspace of $\langle P_0, \ldots, P_k \rangle$. Then there is a unique element $X' \in |L - \Pi|$ containing $P_0, \ldots, P_k$. Consider a general subspace G of dimension k containing H and contained in $\langle X' \rangle$. Any such a G is $(k+1)$–secant to X', hence to X, and contains H. Since these subspaces G fill up a family of dimension $(k-h)(n-1)$, we see that $\xi_{h,k}(X) \geq (k-h)(n-1)$ and therefore $g_{h,k}(X) \leq f(n,h,k)$.

We claim that $\mathbb{G}_{h,k}(X) \neq \mathbb{G}(h, n+k+1)$. To see this, note that $(h+1)(n+k-h+1) > f(n,h,k)$ because $k \leq 2h$. Thus we may apply Proposition 2.6 and conclude that $\xi_{h,k}(X) = (k-h)(n-1)$ hence $g_{h,k}(X) = f(n,h,k)$. □

4 The classification theorem

In this section we prove the classification theorem for (h,k)–extremal varieties.

Theorem 4.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate, n–dimensional, (h,k)-extremal variety for integers $0 \leq h \leq k$. Then either $n = 1$ and $N > k + \frac{k+1}{h+1}$, or $h = k$ and $N > n + k$, or X is one of the examples considered in §3, i.e.:*

(i) *$N > n + k + \frac{k-h}{h+1}$ and $X \subset \mathbb{P}^N$ is a cone over a curve with vertex a linear subspace $\mathbb{P}^{n-2}$;*

(ii) *$(h,k) = (0,1)$ and X is a cone over the Veronese surface of degree 4 in $\mathbb{P}^5$ with vertex a linear subspace of dimension $n-3$;*

(iii) *$N = n + k + 1$, $k \leq 2h$ and X is a rational normal scroll of degree $k+2$.*

Proof. Assume $n > 1$ and $h < k$. From Corollary 2.8, we have $N > n + k + \frac{k-h}{h+1}$. Consider a general surface section Y which spans a projective space of dimension $M := N - n + 2$. By Proposition 2.10, Y is also (h,k)-extremal.

If $(h,k) = (0,1)$, the surface Y is 1-defective and [13] implies Y is either a cone over a curve or the Veronese surface in $\mathbb{P}^5$. By Proposition 2.2 and non–extendability of the Veronese surface (see e.g. [2]), we are either in case (i) or (ii).

Assume that $(h,k) \neq (0,1)$, in particular $k \geq 2$. Let $H \in \mathbb{G}_{h,k}(Y)$ be a general element. Consider $I_{h,k}(Y) \subset Y^{k+1} \times \mathbb{G}(h,M)$. One has the rational map $p : I_{h,k}(Y) 1 \mathbb{G}(k,M)$, sending a general point $(P_0, \ldots, P_k, H)$ to $\langle P_0, \ldots, P_k \rangle$. Note that the general fibre of p has degree $(k+1)!$ and the monodromy, or Galois, group of p is the full symmetric group on $\{P_0, \ldots, P_k\}$ (see [1] for a similar situation).

Consider also the projection q of $I_{h,k}(Y)$ to $\mathbb{G}(h,M)$. Since Y is (h,k)-extremal, any component $\Omega_{H,k}$ of $q^{-1}(H)$ has dimension $k-h$. Let $(P_0, \ldots, P_k, H)$ be a general element of $\Omega_{H,k}$, thus $P_0, \ldots, P_k$ are general points of Y. Denote the i–th projection map of $\Omega_{H,k}$ to Y by p_i and the image of p_i by Γ_i. Since the monodromy group of p is the full symmetric group, $\Gamma_0, \dots, \Gamma_k$ all have the same dimension (see [9, p. 209-210] for a similar argument). The proof of Proposition 2.5 and Proposition 2.6 imply that $\Gamma_0, \dots, \Gamma_k$ are curves.

Write Γ to denote the curve $\Gamma_0 \cup \dots \cup \Gamma_k$. We claim that Γ spans a $\mathbb{P}^{k+1}$. Indeed, let r be the dimension of this span. If we consider the projection in $\langle\Gamma\rangle$ from H to a $\mathbb{P}^{r-h-1}$, the curve Γ projects to a curve Γ' which has a $(k-h)$-dimensional family of $(k+1)$-secant $\mathbb{P}^{k-h-1}$. Choose any $k-h-2$ among the curves $\Gamma_0, \dots, \Gamma_k$, say $\Gamma_{i_1}, \dots, \Gamma_{i_{k-h-2}}$. By imposing to such a $\mathbb{P}^{k-h-1}$ to contain a general point each of these $k-h-2$ curves, and projecting down from these points, we find a curve Γ'' in $\mathbb{P}^{r-k+1}$, which contain the images $\Gamma''_{j_1}, \dots, \Gamma''_{j_{h+3}}$ of the curves different from $\Gamma_{i_1}, \dots, \Gamma_{i_{k-h-2}}$. Now any line joining general points of two of the curves $\Gamma'_{j_1}, \dots, \Gamma'_{j_{h+3}}$, meets all the other curves of this set. This is only possible if $r-k+1=2$, i.e. $r=k+1$.

Next, for every $i=0,\dots,k$, we consider the intersection $\Gamma_i \cap \{P_0, \dots, P_k\}$, which contains at least the point P_i. We claim that either $\Gamma_i \cap \{P_0, \dots, P_k\} = \{P_i\}$ for all $i=0,\dots,k$, or $\{P_0, \dots, P_k\} \subset \Gamma_i$ for all $i=0,\dots,k$. Suppose indeed that there is a $j \neq i$ such that $P_i \in \Gamma_j$. By monodromy (see a similar argument above), then $\{P_0, \dots, P_k\} \subset \Gamma_j$, and, again by monodromy, $\{P_0, \dots, P_k\} \subset \Gamma_i$ for all $i=0,\dots,k$.

Suppose first that $\Gamma_i \cap \{P_0, \dots, P_k\} = \{P_i\}$ for all $i=0,\dots,k$. Note that $\langle P_0, \dots, P_k\rangle$ intersects Y in $\{P_0, \dots, P_k\}$ as a scheme (see Lemma 2.1). Hence it intersects each curve Γ_i at P_i transversally. Thus Γ_i is a curve in $\mathbb{P}^{k+1}$ intersecting transversally the hyperplane $\langle P_0, \dots, P_k\rangle \subset \mathbb{P}^{k+1}$ at one point, hence each Γ_i is a line. So for general points $P_0, \dots, P_k$ on Y, there exist lines $\Gamma_0, \dots, \Gamma_k \subset Y$ with $P_i \in \Gamma_i$ and $\dim(\langle\Gamma_1, \dots, \Gamma_k\rangle) = k$. Now [9, Prop. 1.3 and Section 4] implies that Y is a cone over a curve. By Proposition 2.2, we are in case (i).

Suppose finally that $\{P_0, \dots, P_k\} \subset \Gamma_i$ for all $i=0,\dots,k$. Arguing as above we see that each Γ_i is a rational normal curve of degree $k+1$ in $\langle\Gamma\rangle$. Consider a minimal resolution of singularities $S \to Y$ and let C be the proper transform on S of one of the curves $\Gamma_0, \dots, \Gamma_k$. Let $\mathcal{L}$ be the continuous system of curves in which C moves. Notice that $\dim(\mathcal{L}) \geq k+1 \geq 3$, since $k+1$ general points on Y belong to a curve $C \in \mathcal{L}$. Since C is rational, the surface S is rational, $\dim(|C|) \geq k+1$ and $C^2 \geq k$. Let G be the pull-back to S of a hyperplane section of Y. The index theorem tells us that $(k+1)^2 \geq kG^2$, hence $G^2 \leq k+2$ because $k>1$. Since Y is non–degenerate in $\mathbb{P}^M$, we have $G^2 \geq M-1 = N-n+1 \geq k+2$. Therefore $G^2 = k+2$ and Y is a surface of minimal degree $k+2$. By the way, this yields $C^2 = k$ and $\Gamma_0 = \dots \Gamma_k = \Gamma$ in this case. In any event, we get $N = n+k+1$ and X is also of minimal degree $k+2$, so X is either a cone over the Veronese surface or a rational normal scroll (see [10]), i.e. we are either in case (ii) or (iii). $\square$

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