Plane curves with 3 or 4 total inflection points

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Abstract. *— In this article, we will study plane curves of a certain degree d with 3 or 4 total inflection points. In particular, we will study their image in the moduli spaces. Also a result on curves with 5 total inflection points is included.*

MSC. *— 14H10, 14H50, 14N20.*

0 Notation and introduction

We will first fix some notations. Let $\mathbb{P}^2$ be the projective plane over some algebraically closed field k and let $\mathcal{P}^2$ be the incidence relation in $(\mathbb{P}^2)^* \times \mathbb{P}^2$, i.e. $\mathcal{P}^2 = \{(L, P) \mid P \in L\}$.

If d and e are nonzero natural numbers, we denote by $V_{d,e} \subset (\mathcal{P}^2)^e$ the set of elements $(\mathcal{L}, \mathcal{P}) = ((L_1, P_1), \ldots, (L_e, P_e))$ with $P_i \notin L_j$ for all $i \neq j$ (hence also $L_i \neq L_j$ for $i \neq j$) and such that there exists a plane curve Γ (not necessarily irreducible) of degree d, not containing one of the lines L_i, with intersection number $i(\Gamma.L_i, P_i) = d$. We say that in this case the pairs (L_i, P_i) are total inflection points of Γ. We write $\overline{V_{d,e}}$ to denote the closure of $V_{d,e}$ in $(\mathcal{P}^2)^e$.

If $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$, denote by $V(\mathcal{L}, \mathcal{P}) \subset \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ the set $\{s \mid dP_1 + \ldots + dP_e \subset Z(s)\}$, whereby dP_i is the subscheme of $\mathbb{P}^2$ corresponding to the divisor dP_i on L_i. So the associated linear system $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ consists of curves Γ of degree d having (L_i, P_i) as total inflection point for all i. If $1 \leq f \leq e$, we will mostly write $(\mathcal{L}_f, \mathcal{P}_f)$ to denote the element $((L_1, P_1), \ldots, (L_f, P_f)) \in V_{d,f}$ (unless stated otherwise).

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We write $\mathcal{V}_{d,e}$ to denote the union of the spaces of curves $\mathbb{P}(V(\mathcal{L},\mathcal{P}))$ with $(\mathcal{L},\mathcal{P}) \in V_{d,e}$. We denote the set of points corresponding to smooth plane curves of $\mathcal{V}_{d,e}$ by $\mathcal{V}^\circ_{d,e}$. Let $m_{d,e} : \mathcal{V}^\circ_{d,e} \to M_{(d-1)(d-2)/2}$ be the moduli map and denote its image by $M(V_{d,e})$.

In [7], the case $d = 4$ (i.e. quartic curves) has been studied intensively. The main tool used there is the so-called λ-invariant, which is nothing else than a cross ratio of four points (see also [4]). In [2], the cases $e = 1, 2$ have been handled and also the cases $e = 3, 4$ for some special configurations of the lines L_i and the points P_i. In [3], a few general results are proven on curves with total inflection points.

In Section 1, we will consider the case $e = 3$ (so three total inflection points). We will give a full description of the components of $V_{d,3}$ for $d \geq 2$ and $M(V_{d,3})$ for $d \geq 5$, prove that they are rational and compute their dimensions (see Theorem 1.1 and Theorem 1.5). In Section 2, we consider the case $e = 4$. Again we find a full list of the components of $V_{d,4}$ for $d \geq 3$ and $M(V_{d,4})$ for $d \geq 6$. We will prove that all components of $V_{d,4}$ (Theorem 2.2) and almost all components of $M(V_{d,4})$ (Theorem 2.11) are rational. In Section 3, we will prove a result on the case $e = 5$ (Theorem 3.2).

We recall a proposition proven in [3], which we will use several times during this article.

Proposition 0.1. *Let $(\mathcal{L},\mathcal{P}) \in V_{d,e}$.*

(a) $\dim(V(\mathcal{L},\mathcal{P})) = \binom{d-e+2}{2} + 1$, *where* $\binom{n}{2}$ *is defined to be* 0 *if* $n < 2$.

(b) Let L be a line in $\mathbb{P}^2$ with $P_i \notin L$ for $1 \leq i \leq e$. Let $V_L(\mathcal{L},\mathcal{P})$ be the image of the restriction map $V(\mathcal{L},\mathcal{P}) \to \Gamma(L, \mathcal{O}_L(d))$. Let $P_{i0} = L_i \cap L$ for $1 \leq i \leq e$. If $d \geq e$, $\dim(V_L(\mathcal{L},\mathcal{P})) = d - e + 2$ *and $\mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$ is a linear system g^{d-e+1}_d on L containing $P_{10} + \ldots + P_{e0} + g^{d-e}_{d-e}$. If $d < e$,* $\dim(V_L(\mathcal{L},\mathcal{P})) = 1$.

(c) Under the assumptions of (b), for $P \in L$ with $P \neq P_{i0}$ for all $1 \leq i \leq e$ one has $((\mathcal{L},\mathcal{P}),(L,P)) \in V_{d,e+1}$ if and only if $dP \in \mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$.

The following lemma is well-known. Since we cannot find a good reference, we include a proof for sake of completeness.

Lemma 0.2. *Let C_1 and C_2 be smooth plane curves of degree $d \geq 4$. Then C_1 and C_2 are isomorphic if and only if they are projectively equivalent; i.e. there exists an automorphism ϕ of $\mathbb{P}^2$ such that $\phi(C_1) = C_2$.*

Proof. In case C_1 and C_2 are isomorphic curves, both are defined by a linear system g^2_d on the same curve C. Let g_i be the linear system on C defining C_i

($i \in \{0,1\}$). In order to prove the existence of ϕ, it is enough to prove that $g_1 = g_2$. Take $D \in g_2$ general, then $D = P_1 + \ldots + P_d$ with $P_i \neq P_j$ for $i \neq j$. We identify C with C_1, hence it is enough to prove that $D = C_1.L$ for a line L in $\mathbb{P}^2$. Let L be the line connecting P_1 and P_2 and assume $P_d \notin L$. From the Adjunction Formula, it follows that the canonical linear system on C_1 is defined by intersections of C_1 with plane curves of degree $d-3$. Using $d-3$ lines in $\mathbb{P}^2$, it is possible to find canonical divisors K_i on C such that $K_i \cap D = P_1 + \ldots + P_i$ for $0 \leq i \leq d-3$. On the other hand, using L and $d-4$ suited lines in $\mathbb{P}^2$, we can find a canonical divisor K_{d-2} containing $P_1 + \ldots + P_{d-2}$ but not P_d. This proves that D imposes at least $d-1$ conditions on K_C, hence $h^0(K_C - D) \leq h^0(K_C) - (d-1) = g - d + 1$. From Riemann-Roch, it follows that $h^0(D) = \deg(D) - g + 1 + h^0(K_C - D) \leq (d - g + 1) + (g - d + 1) = 2$. This contradicts $D \in g_2$. □

1 The case $e = 3$

Let $(\mathcal{P}^2)^{3,0} \subset (\mathcal{P}^2)^3$ be the set of points $(\mathcal{L},\mathcal{P}) = ((L_1,P_1),(L_2,P_2),(L_3,P_3))$ with $P_i \notin L_j$ for $i \neq j$ and let $(\mathcal{P}^2)^{3,0,2} \subset (\mathcal{P}^2)^3$ be the subspace of $(\mathcal{P}^2)^{3,0}$ of elements $(\mathcal{L},\mathcal{P})$ such that the lines L_1, L_2 and L_3 have a common point S. Let $(\mathcal{P}^2)^{3,0,1} = (\mathcal{P}^2)^{3,0} \setminus (\mathcal{P}^2)^{3,0,2}$.

For $(\mathcal{L},\mathcal{P}) \in (\mathcal{P}^2)^{3,0}$, we write g_d^{d-1} to denote the linear system $\mathbb{P}(V_{L_3}(\mathcal{L}_2,\mathcal{P}_2))$ on L_3. Let $P_{1,2,3} = \langle P_1, P_2 \rangle \cap L_3$. Since $d\langle P_1,P_2\rangle \in \mathbb{P}(V(\mathcal{L}_2,\mathcal{P}_2))$, we find $dP_{1,2,3} \in g_d^{d-1}$, hence

$$g_d^{d-1} = \langle P_{1,3} + P_{2,3} + g_{d-2}^{d-2}, dP_{1,2,3} \rangle.$$

Moreover, $(\mathcal{L},\mathcal{P}) \in (\mathcal{P}^2)^{3,0,2}$ is equivalent to $P_{1,3} = P_{2,3}$ and $(\mathcal{L},\mathcal{P}) \in V_{d,3}$ is equivalent to $dP_3 \in g_d^{d-1}$.

We continue in case $(\mathcal{L},\mathcal{P}) \in (\mathcal{P}^2)^{3,0,1}$. We can choose coordinates $(X_1 : X_2 : X_2)$ on $\mathbb{P}^2$. Let R_1, R_2 and R_3 be the coordinate axes. There exists a coordinate transformation ϕ on $\mathbb{P}^2$ such that $\phi(L_i) = R_i : X_i = 0$ for $i \in \{1,2,3\}$, $\phi(P_1) = Q_1 := (0:1:1)$ and $\phi(P_2) = Q_2 := (1:0:1)$, hence $\phi(P_3) = Q_3 := (1:-t:0)$ for some $t \in k$ (note that P_3 is not contained in L_1), $\phi(P_{1,3}) = Q_{1,3} := (0:1:0)$, $\phi(P_{2,3}) = Q_{2,3} := (1:0:0)$ and $\phi(P_{1,2,3}) = Q_{1,2,3} := (1:-1:0)$. On R_3 we can take $(X_1 : X_2)$ as local coordinates, so on R_3 we have $Q_{1,3} = (0:1)$, $Q_{2,3} = (1:0)$, $Q_{1,2,3} = (1:-1)$ and $Q_3 = (1:-t)$. Identify g_d^d on R_3 with the projective space $\mathbb{P}(k[X_1,X_2]_d)$ of homogeneous forms of degree d and use homogeneous coordinates $(a_d : a_{d-1} : \ldots : a_0)$ for $\langle a_d X_1^d + a_{d-1}X_1^{d-1}X_2 + \ldots + a_0 X_2^d \rangle$. The subspace $Q_{1,3} + Q_{2,3} + g_{d-2}^{d-2} \subset g_d^d$ is defined by the linear equations $a_0 = a_d = 0$, while $dQ_{1,2,3}$ corresponds to $\langle (X_1 + X_2)^d \rangle$. It follows that $g_d^{d-1} = \langle Q_{1,3} + Q_{2,3} + g_{d-2}^{d-2}, dQ_{1,2,3} \rangle \subset g_d^d$ has as equation $a_0 = a_d$. The point $Q_3 = (1:-t) \in L_3$ satisfies $dQ_3 \in g_d^{d-1}$ if and only if the form $(tX_1 + X_2)^d$ satisfies the equation, i.e. if and only if $t^d = 1$.

The point $Q_{1,2,3} = (1 : -1)$ on R_3 is a solution of this equation. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ does not divide d, there are exactly d solutions of this equation. If $\mathrm{char}(k) = p > 0$ divides d, we can write $d = d'.p^c$ with d' not divisible by p. In this case, the condition becomes $t^{d'} = 1$, hence there are exactly d' solutions. In case $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ does not divide d, we also write $d' = d$.

Denote $((R_1, Q_1), (R_2, Q_2), (R_3, Q_3))$ by $(\mathcal{R}, \mathcal{Q}(t))$ if $Q_3 = (1 : -t : 0)$. By using 0.1, we see that $(\mathcal{R}, \mathcal{Q}(t)) \in V_{d,3}$ if and only if $t^{d'} = 1$, hence if and only if $t \in \mu_{d'} = \{1, \omega, \ldots, \omega^{d'-1}\}$ whereby ω is a d'-th root of unity in k (for example, if $k = \mathbb{C}$ we can take $\omega = e^{2\pi i/d'}$).

We get that if $(\mathcal{L}, \mathcal{P}) \in V_{d,3} \cap (\mathcal{P}^2)^{3,0,1}$, there exists a coordinate transformation ϕ and an element $t \in \mu_{d'}$ such that $\phi(\mathcal{L}, \mathcal{P}) = (\mathcal{R}, \mathcal{Q}(t))$, hence $(\mathcal{L}, \mathcal{P}) = \phi^{-1}(\mathcal{R}, \mathcal{Q}(t))$. So we can conclude that

$$V_{d,3} \cap (\mathcal{P}^2)^{3,0,1} = \bigcup_{t \in \mu_{d'}} \mathrm{Aut}(\mathbb{P}^2).(\mathcal{R}, \mathcal{Q}(t))$$

and thus it is the union of d' smooth components of dimension 8. For an element $t \in \mu_{d'}$, we write $V_t \subset V_{d,e}$ to denote the component containing $\mathrm{Aut}(\mathbb{P}^2).(\mathcal{R}, \mathcal{Q}(t))$.

Now we study the case where $(\mathcal{L}, \mathcal{P}) \in (\mathcal{P}^2)^{3,0,2}$. We can choose coordinates $(X_1 : X_2)$ on L_3 such that $P_{1,3} = P_{2,3} = (1 : 0)$ and $P_{1,2,3} = (0 : 1)$. As before, we identify g^d_d on L_3 with $\mathbb{P}(k[X_1, X_2]_d)$ and we use coordinates $(a_d : a_{d-1} : \ldots : a_0)$ on g^d_d. The linear subspace $2P_{1,3} + g^{d-2}_{d-2} \subset g^d_d$ has linear equations $a_d = a_{d-1} = 0$. Since $dP_{1,2,3} \in g^d_d$ has coordinate $(1 : 0 : \ldots : 0)$, the equation of $g^{d-1}_d = \langle 2P_{1,3} + g^{d-2}_{d-2}, dP_{1,2,3} \rangle \subset g^d_d$ is given by $a_{d-1} = 0$. Since $P_3 \neq P_{1,3} = P_{2,3}$ (P_3 is not contained in L_1 or L_2), we may assume $P_3 = (a : 1)$ on L_3. Hence $dP_3 \in g^{d-1}_d$ if and only if $(X_1 - aX_2)^d$ satisfies the equation, i.e. if and only if $da = 0$. The solution $a = 0$ corresponds to $P_{1,2,3}$ and if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ does not divide d, there is no other solution. In this case, $(\mathcal{L}, \mathcal{P})$ belongs to the closure V_1 of $\mathrm{Aut}(\mathbb{P}^2).(\mathcal{R}, \mathcal{Q}(1))$. However, if $\mathrm{char}(k) = p > 0$ divides d, all points P_3 on L_3 satisfy $dP_3 \in g^{d-1}_d$, so $(\mathcal{P}^2)^{3,0,2}$ is a component of $V_{d,3}$. Notice that

$$\mathrm{Aut}(\mathbb{P}^2).((R'_1, Q'_1), (R'_2, Q'_2), (R'_3, Q'_3))$$

is a dense subset of $(\mathcal{P}^2)^{3,0,2}$, with $R'_1 : X_1 = 0$, $R'_2 : X_2 = 0$, $R'_3 : X_1 = X_2$, $Q'_1 = (0 : 1 : 0)$, $Q'_2 = (1 : 0 : 0)$ and $Q'_3 = (1 : 1 : 1)$ (so $S = (0 : 0 : 1)$).

Since $\mathrm{Aut}(\mathbb{P}^2)$ is rational, we have proven the following theorem.

Theorem 1.1. *In case $\mathrm{char}(k) = 0$, let $d' = d$; in case $\mathrm{char}(k) = p > 0$, write $d = p^c d'$ with $c \geq 0$ and $p \nmid d'$. There are exactly d' components of $V_{d,3}$ intersecting $(\mathcal{P}^2)^{3,0,1}$. Each of them has dimension 8, the intersection with $(\mathcal{P}^2)^{3,0,1}$ is smooth and exactly one of them is not contained in $(\mathcal{P}^2)^{3,0,1}$. In case $c \geq 1$, $(\mathcal{P}^2)^{3,0,2}$ is another 8-dimensional component of $V_{d,3}$. Each component of $V_{d,3}$ is rational.*

We have the following generalization for the case where $\mathrm{char}(k) = p > 0$ is divisible by d.

Proposition 1.2. *Assume char$(k) = p > 0$ and $d = p^c d'$ with $c \geq 1$ and $p \nmid d'$. Let $e \leq p^c + 1$ and let $L_1, \ldots, L_e$ be lines through a common point S and let $P_i \in L_i \setminus \{S\}$ for $1 \leq i \leq e$. Then $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$.*

Proof. We may assume $e \geq 3$. Let $L = \langle P_1, P_2 \rangle$ and assume $2 \leq x \leq e$ with $P_i \in L$ for $1 \leq i \leq x$ and $P_i \notin L$ otherwise. Write $f = e - x$. In case $f = 0$, we have $dL \in \mathbb{P}(V(\mathcal{L}, \mathcal{P}))$, hence $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$. Let $f > 0$ and assume the claim holds for $f - 1$ (instead of f). Let $Q_{e-f+1} = L_{e-f+1} \cap L$. Let $(\mathcal{L}', \mathcal{P}') \in (\mathcal{P}^2)^{e,0}$ (resp. $(\mathcal{L}'', \mathcal{P}'') \in (\mathcal{P}^2)^{e-1,0}$) be obtained from $(\mathcal{L}, \mathcal{P})$ by replacing (L_{e-f+1}, P_{e-f+1}) by (L_{e-f+1}, Q_{e-f+1}) (resp. by omitting (L_{e-f+1}, P_{e-f+1})). Both $(\mathcal{L}', \mathcal{P}')$ and $(\mathcal{L}'', \mathcal{P}'')$ correspond to $f - 1$ instead of f, hence $(\mathcal{L}', \mathcal{P}') \in V_{d,e}$ and $(\mathcal{L}'', \mathcal{P}'') \in V_{d,e-1}$.

In order to proof the claim, it is enough to show $dP_{e-f+1} \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$. There exists a $\Gamma \in V(\mathcal{L}', \mathcal{P}')$ such that $L_{e-f+1} \not\subset \Gamma$ and $L_{e-f+1} \cap \Gamma = dQ_{e-f+1}$. Since $V(\mathcal{L}', \mathcal{P}') \subset V(\mathcal{L}'', \mathcal{P}'')$ it follows that $dQ_{e-f+1} \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$, hence

$$\mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}'')) = \langle (e-1)S + g^{d-e+1}_{d-e+1}, dQ_{e-f+1} \rangle.$$

Choose coordinates $(x : y)$ on L_{e-f+1} such that $S = (1 : 0)$ and $Q_{e-f+1} = (0 : 1)$. Use coordinates $(a_d : \ldots : a_0)$ on g^d_d on L_{e-f+1} as before. The linear system $(e-1)S + g^{d-e+1}_{d-e+1}$ has equations $a_d = \ldots = a_{d-e+2} = 0$ and $dQ_{e-f+1} = (1 : 0 : \ldots : 0)$. Hence $\mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$ has equations $a_{d-1} = \ldots = a_{d-e+2} = 0$. For $P = (\alpha : \beta) \in L_{e-f+1}$ one has $dP \in \mathbb{P}(V_{L_{e-f+1}}(\mathcal{L}'', \mathcal{P}''))$ if and only if the form $(\beta x - \alpha y)^d$ satisfies those equations. This is equivalent to $\beta^{d-i}\alpha^i \binom{d}{i} = 0$ for $1 \leq i \leq e - 2$. In case $\alpha\beta \neq 0$, all those conditions are satisfied if and only if $e - 2 \leq p^c - 1$, hence $e \leq p^c + 1$. $\square$

Lemma 1.3. *Assume $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$ with $e \leq d$. Then a general element of $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ is a smooth plane curve.*

Proof. The claim follows immediately from [3, Prop. 2.1] by taking $z = 0$. $\square$

For each $t \in \mu_{d'}$, we write $\mathcal{V}_t$ to denote the union of spaces $\mathbb{P}(V(\mathcal{L}, \mathcal{P}))$ for all $(\mathcal{L}, \mathcal{P}) \in V_t$. We denote the set of points corresponding to smooth plane curves of $\mathcal{V}_t$ by $\mathcal{V}^\circ_t$. Since $\mathcal{V}^\circ_t \subset \mathcal{V}^\circ_{d,3}$, we can consider the image of $\mathcal{V}^\circ_t$ under $m_{d,3} : \mathcal{V}^\circ_{d,3} \to M_{(d-1)(d-2)/2}$. We denote this image by $M(V_t)$.

Now we restrict to the case char$(k) = 0$.

Lemma 1.4. *If $d \geq 5$ and V_t is component of $V_{d,3}$, then a general element of $\mathcal{V}^\circ_t$ has exactly 3 total inflection points.*

Proof. Since $V_{d,3}$ has codimension 1 inside $(\mathcal{P}^2)^3$ (see Theorem 1.1 or [3, Ex. 3.4]), we find $V_{d,4}$ has codimension at least 2 inside $(\mathcal{P}^2)^4$, hence $\dim(V_{d,4}) \leq 10$ (in fact, we will show in Section 2 that equality holds). Indeed, fixing $(\mathcal{L}, \mathcal{P}) \in V_{d,3}$, the set

$$S = \{(L, P) \in \mathcal{P}^2 \mid ((\mathcal{L}, \mathcal{P}), (L, P)) \in V_{d,4}\}$$

is at most 2-dimensional, since $S \subset \{(L,P) \in \mathcal{P}^2 \,|\, ((\mathcal{L}_2,\mathcal{P}_2),(L,P)) \in V_{d,3}\}$.

We obtain $\dim(\mathcal{V}_t) = 8 + \binom{d-1}{2}$ and $\dim(\mathcal{V}_{d,4}) \leq 10 + \binom{d-2}{2}$, hence $\dim(\mathcal{V}_t) > \dim(\mathcal{V}_{d,4})$ if $d > 4$. $\square$

Since $V_t = \overline{\mathrm{Aut}(\mathbb{P}^2).(\mathcal{R},\mathcal{Q}(t))}$ for each $t \in \mu_{d'}$, the set $m_{d,3}(\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ)$ is a dense open subset of $M(V_t)$, where again the $\circ$-sign indicates the smooth curves. For a general curve $C \in \mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ$, Lemma 0.2 implies that $[C] = [C'] \in M_{(d-1)(d-2)/2}$ for some $C' \in \mathbb{P}(V(\mathcal{R},\mathcal{Q}(t')))^\circ$ with $t' \in \mu_{d'}$ if and only if $\phi(C) = C'$ for some $\phi \in \mathrm{Aut}(\mathbb{P}^2)$. Lemma 1.4 implies that the lines R_1, R_2 and R_3 are permutated by the map ϕ.

We write $A = (a_{ij})_{i,j=1}^3$ to denote a matrix corresponding to ϕ and $Q_3(t) = (1 : -t : 0)$ for some $t \in \mu_{d'}$.

First we consider the case where $\phi((R_1,R_2,R_3)) = (R_2,R_1,R_3)$. In this case, the matrix becomes

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

We also need $\phi(Q_1) = Q_2$, hence $a_{12} = a_{33}$, and $\phi(Q_2) = Q_1$, hence $a_{21} = a_{33}$. So ϕ corresponds to the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So $\phi(Q_3(t)) = Q_3(t')$ implies $(-t : 1 : 0) = (1 : -t' : 0)$, hence $t' = 1/t$. Note that moreover the equality $t = t'$ holds if and only if $t = 1$ or d is even and $t = -1$. We obtain $M(V_t) = M(V_{1/t})$.

In case $\phi((R_1,R_2,R_3)) = (R_3,R_2,R_1)$, the matrix becomes $\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$. We also need $\phi(Q_2) = Q_2$ and $\phi(Q_3(t)) = Q_1$, hence $a_{13} = a_{31} = -t.a_{22}$. So $\phi(Q_1) = Q_3(t')$ implies $(-t : 1 : 0) = (1 : -t' : 0)$ and thus again $t' = 1/t$.

As above, the case $\phi((R_1,R_2,R_3)) = (R_1,R_3,R_2)$ implies $t' = 1/t$.

In case $\phi((R_1,R_2,R_3)) = (R_2,R_3,R_1)$, the matrix becomes $\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$. In this case we need $\phi(Q_1) = Q_2$ hence $a_{13} = a_{32}$ and $\phi(Q_3(t)) = Q_1$ hence $a_{21} = -t.a_{32}$. Since also $\phi(Q_2) = Q_3(t')$, we get $(1 : -t : 0) = (1 : -t' : 0)$ and thus $t = t'$. It follows that ϕ acts on $\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ$.

In case $\phi((R_1,R_2,R_3)) = (R_3,R_1,R_2)$, we find analogously as the above case that $t = t'$ and that ϕ acts on $\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ$.

In case $t \neq 1$ and $t \neq -1$ if d is even, we find a subgroup $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Aut}(\mathbb{P}^2)$ acting on $\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ$ such that $\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ/(\mathbb{Z}/3\mathbb{Z})$ is birationally equivalent to $M(V_t)$. Since $\mathbb{P}(V(\mathcal{R},\mathcal{Q}(t)))^\circ$ is rational and $\mathbb{Z}/3\mathbb{Z}$ is Abelian, it follows that $M(V_t)$ is also rational (using a result of E. Fischer, see [5]). This extends the result of Casnati and Del Centina (they consider the components $M(V_1)$ and $M(V_{-1})$ in case d is even; see [2, Theorem B]) showing that all components of $M(V_{d,3})$ are rational. All the above results are summarized in the following theorem.

Theorem 1.5. *If $char(k) = 0$ and $d \geq 5$, the set $M(V_{d,3})$ has $1+\frac{d-1}{2}$ components if d is odd and $2+\frac{d-2}{2}$ components if d is even. Moreover, each component is rational.*

2 The case $e = 4$

In this section we will for simplicity assume that $k = \mathbb{C}$.

Assume that $(\mathcal{L}, \mathcal{P})$ is an element of $V_{d,4}$, with no three of the lines L_1, L_2, L_3 and L_4 are concurrent. Since $(\mathcal{L}_3, \mathcal{P}_3) = ((L_1, P_1), (L_2, P_2), (L_3, P_3))$ is an element of $V_{d,3} \cap (\mathcal{P}^2)^{3,0,1}$, there exists an element $t \in \mu_d$ such that $(\mathcal{L}_3, \mathcal{P}_3) \in V_t$. If we consider coordinates $(X_1 : X_2 : X_3)$ on $\mathbb{P}^2$, there exists a coordinate transformation ϕ such that $\phi((\mathcal{L}_3, \mathcal{P}_3)) = (\mathcal{R}, \mathcal{Q}(t))$. Assume that $R_4 := \phi(L_4) : X_3 = AX_1 + BX_2$ (L_4 does not contain $P_{1,2}$). We can use $(X_1 : X_2)$ as local coordinates on R_4. Identify the linear system g^d_d on R_4 with $\mathbb{P}(k[X_1, X_2]_d)$ and use homogeneous coordinates $(a_d : \ldots : a_0)$ for $\langle a_d X_1^d + a_{d-1} X_1^{d-1} X_2 + \ldots + a_0 X_2^d \rangle$. Denote $R_i \cap R_j$ by $Q_{i,j}$ if $1 \leq i < j \leq 4$.

Since $Q_{1,4} = (0 : 1)$, $Q_{2,4} = (1 : 0)$ and $Q_{3,4} = (-B : A)$ on R_4, the linear system $Q_{1,4} + Q_{2,4} + Q_{3,4} + g^{d-3}_{d-3} \subset g^d_d$ is defined by the following equations:

$$\begin{cases} l_1(a_d, \ldots, a_0) := a_0 = 0, \\ l_2(a_d, \ldots, a_0) := a_d = 0, \\ l_3(a_d, \ldots, a_0) := a_d(-B)^d + a_{d-1}(-B)^{d-1}A + \ldots + a_0 A^d = 0. \end{cases}$$

If we define $f_t \in k[X_1, X_2, X_3]_d$ to be

$$(X_3 - X_1)^d + (X_3 - X_2)^d + (-1)^d (tX_1 + X_2)^d - (-1)^d X_1^d - (-1)^d X_2^d - X_3^d,$$

it is easy to see that the curve $\mathcal{C}$ in $\mathbb{P}^2$ with equation $f_t(X_1 : X_2 : X_3) = 0$ is contained in $\mathbb{P}(V(\mathcal{R}, \mathcal{Q}(t)))$, hence the divisor $\mathcal{C} \cap L_4$ is an element of $g^{d-2}_d = \mathbb{P}(V_{R_4}(\mathcal{R}, \mathcal{Q}(t)))$. The divisor $\mathcal{C} \cap R_4$ is defined by

$$\widetilde{a}_d X_1^d + \ldots + \widetilde{a}_0 X_2^d := f_t(X_1, X_2, AX_1 + BX_2) = 0,$$

so we have the following equalities

$$\begin{cases} l_1(\widetilde{a}_d, \ldots, \widetilde{a}_0) = \widetilde{a}_0 = (B-1)^d, \\ l_2(\widetilde{a}_d, \ldots, \widetilde{a}_0) = \widetilde{a}_d = (A-1)^d, \\ l_3(\widetilde{a}_d, \ldots, \widetilde{a}_0) = f_t(-B, A, 0) = (-1)^d (A - tB)^d. \end{cases}$$

Since $g_d^{d-2} = \langle Q_{1,4} + Q_{2,4} + Q_{3,4} + g_{d-3}^{d-3}, \mathcal{C} \cap R_4 \rangle$, we find that g_d^{d-2} is defined by

$$\begin{cases} l_2(\widetilde{a}_d, \ldots, \widetilde{a}_0).l_1(a_d, \ldots, a_0) = l_1(\widetilde{a}_d, \ldots, \widetilde{a}_0).l_2(a_d, \ldots, a_0), \\ l_3(\widetilde{a}_d, \ldots, \widetilde{a}_0).l_2(a_d, \ldots, a_0) = l_2(\widetilde{a}_d, \ldots, \widetilde{a}_0).l_3(a_d, \ldots, a_0). \end{cases}$$

Assume that the local coordinates of Q_4 are $(\alpha : \beta)$. Proposition 0.1 implies that $dQ_4 = \langle (\beta X_1 - \alpha X_2)^d \rangle \in g_d^{d-2}$, hence

$$\begin{cases} (A-1)^d(-\alpha)^d = (B-1)^d \beta^d \\ (-1)^d (A - tB)^d.\beta^d = (A-1)^d(-\beta B - \alpha A)^d \end{cases}$$

so there exist t' and t'' in μ_d such that

$$\begin{cases} -t'(A-1)\alpha = (B-1)\beta \\ t''.(A-tB)\beta = t(A-1)(A\alpha + B\beta) \end{cases}$$

hence

$$t(t'-1)AB = (t't'' - t)A + tt'(1-t'')B.$$

Proposition 2.1. *Under the above assumptions, we have:*

i) $((L_1, P_1), (L_2, P_2), (L_3, P_3)) \in V_t$

ii) $((L_1, P_1), (L_2, P_2), (L_4, P_4)) \in V_{t'}$

iii) $((L_2, P_2), (L_3, P_3), (L_4, P_4)) \in V_{t''}$

iv) $((L_1, P_1), (L_3, P_3), (L_4, P_4)) \in V_{t't''/t}$

Proof. For i, we don't have to prove anything. We will only prove statement iii (ii and iv are analogously). It is enough to prove that

$$((R_2, Q_2), (R_3, Q_3), (R_4, Q_4)) \in V_{t''}.$$

Consider a coordinate transformation φ on $\mathbb{P}^2$ such that $\varphi(R_2) : X_1 = 0$, $\varphi(R_3) : X_2 = 0$, $\varphi(R_4) : X_3 = 0$, $\varphi(Q_2) = (0:1:1)$ and $\varphi(Q_3) = (1:0:1)$, hence

$$\begin{pmatrix} 0 & \frac{-Bt+A}{t} & 0 \\ 0 & 0 & 1-A \\ -A & -B & 1 \end{pmatrix}$$

is a matrix corresponding to φ. So the image of $Q_4 = (\alpha, \beta, A\alpha + B\beta)$ under φ is equal to $(\beta\frac{-Bt+A}{t} : (1-A)(A\alpha + B\beta) : 0) = (1 : -t'' : 0)$. $\square$

We can rewrite the equation $t(t'-1)AB = (t't''-t)A + tt'(1-t'')B$ as

$$(t'-1)AB = (\frac{t't''}{t} - 1)A + t'(1-t'')B.$$

If t', t'' and $t't''/t$ are fixed and not equal to 1, we have a smooth conic of lines L_4 and for each line L_4 on this conic, there is only one point P_4 such that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$. So we get a 9-dimensional component of $V_{d,4}$.

If $t \neq 1$ and exactly one of the numbers t', t'' or $t't''/t$ is equal to 1, it is easy to see that L_4 moves in a pencil of lines through a fixed point ($A \neq 0$ and $B \neq 0$ since otherwise L_4 contains $P_{1,2}$ respectively $P_{1,3}$). Once L_4 is fixed, we have only one choice for P_4 so that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$. Again, this gives us a 9-dimensional component of $V_{d,4}$.

If at least two of the numbers t, t', t'' or $t't''/t$ are equal to 1, the points P_1, P_2, P_3, P_4 are collinear and so $t = t' = t'' = t't''/t = 1$. In this case, we get no condition on the line L_4 and for each line L_4 we have one point P_4 such that $(\mathcal{L}, \mathcal{P}) \in V_{d,4}$, in particular P_4 is the point on L_4 collinear with P_1, P_2 and P_3. Hence this case gives rise to a 10-dimensional component of $V_{d,4}$.

Let ν_d be the set of elements (t, t', t'') with $t, t', t'' \in \mu_d$ and no 2 or 3 of the elements t, t', t'' and $t't''/t$ equal to 1. It is easy to see that $(t, t', t'') \in \nu_d$ if $(t, t', t'') \in (\mu_d)^3$ is not of the form $(a, 1, 1)$, $(1, a, 1)$, $(1, 1, a)$, $(a, a, 1)$, $(a, 1, a)$ or $(1, a, 1/a)$ with $a \neq 1$, hence $|\nu_d| = d^3 - 6(d-1)$. We denote for $(t, t', t'') \in \nu_d$, the corresponding component of $V_{d,4}$ by $V_{t,t',t''}$.

Notice that $V_{t,t',t''}$ is 9-dimensional for each $(t, t', t'') \in \nu_d \setminus \{(1, 1, 1)\}$ and that $V_{1,1,1}$ is 10-dimensional. All these components are rational, since they are birationally equivalent to $\mathrm{Aut}(\mathbb{P}^2) \times \mathbb{P}^1$ or $\mathrm{Aut}(\mathbb{P}^2) \times (\mathbb{P}^1)^2$.

Now assume that $(\mathcal{L}, \mathcal{P})$ is an element of $V_{d,4}$, with

$$(\mathcal{L}_3, \mathcal{P}_3) = ((L_1, P_1), (L_2, P_2), (L_3, P_3)) \in (\mathcal{P}^2)^{3,0,2},$$

hence L_1, L_2 and L_3 are concurrent. From the case $e = 3$, it follows that P_1, P_2 and P_3 are collinear (since $(\mathcal{L}_3, \mathcal{P}_3) \in V_{d,3} \cap (\mathcal{P}^2)^{3,0,2}$) and $(\mathcal{L}_3, \mathcal{P}_3) \in V_1$.

If L_4 contains the intersection point $S = L_1 \cap L_2 \cap L_3$, we have that all lines L_1, L_2, L_3 and L_4 are concurrent and the points P_1, P_2, P_3 and P_4 have to be collinear, hence $(\mathcal{L}, \mathcal{P}) \in V_{1,1,1}$.

If L_4 does not contain the point S, there exist elements t' and t'' in μ_d such that $((L_1, P_1), (L_2, P_2), (L_4, P_4)) \in V_{t'}$ and $((L_2, P_2), (L_3, P_3), (L_4, P_4)) \in V_{t''}$, hence $(\mathcal{L}, \mathcal{P}) \in V_{1,t',t''}$.

The above results are summarized in the following theorem.

Theorem 2.2. *For $k = \mathbb{C}$, the set $V_{d,4}$ has $d^3 - 6(d-1) - 1$ components of dimension 9 and one component of dimension 10. Each component is rational.*

For all $(t,t',t'') \in \nu_d$, denote by $\mathcal{V}_{t,t',t''}$ the union of the spaces of curves $\mathbb{P}(V(\mathcal{L},\mathcal{P}))$ with $(\mathcal{L},\mathcal{P}) \in V_{t,t',t''}$ and by $\mathcal{V}^\circ_{t,t',t''}$ the subset of points corresponding to smooth curves. Since $\mathcal{V}^\circ_{t,t',t''} \subset \mathcal{V}^\circ_{d,4}$, we can consider the image of $\mathcal{V}^\circ_{t,t',t''}$ under the moduli map $m_{d,4} : \mathcal{V}^\circ_{t,t',t''} \to M_{(d-1)(d-2)/2}$. Denote this image by $M(V_{t,t',t''})$.

If $(\mathcal{L}_3,\mathcal{P}_3) \in V_t$ general, denote by $\mathcal{V}_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3) \subset \mathcal{V}_{t,t',t''}$ the union of spaces of curves $\mathbb{P}(V((\mathcal{L}_3,\mathcal{P}_3),(L_4,P_4)))$ with $((\mathcal{L}_3,\mathcal{P}_3),(L_4,P_4)) \in V_{t,t',t''}$ and by $\mathcal{V}_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3)^\circ$ the points corresponding to smooth curves. Let $M(V_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3))$ be the image of $\mathcal{V}_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3)^\circ$ under the moduli map $m_{d,4}$. Since for a general element $(\mathcal{L},\mathcal{P}) \in V_{t,t',t''}$, there exists a $\phi \in \mathrm{Aut}(\mathbb{P}^2)$ and an element $(L_4,P_4) \in \mathcal{P}^2$ such that $(\mathcal{L},\mathcal{P}) = \phi((\mathcal{L}_3,\mathcal{P}_3),(L_4,P_4))$, we have that $M(V_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3))$ is an open dense subset of $M(V_{t,t',t''})$. So if we want to prove that $M(V_{t,t',t''})$ is rational, it is enough to show that $M(V_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3))$ is rational for a general element $(\mathcal{L}_3,\mathcal{P}_3) \in V_t$.

Proposition 2.3. *If $d \geq 6$ and $(t,t',t'') \in \nu_d \setminus \{(1,1,1)\}$, then a general curve in $\mathcal{V}^\circ_{t,t',t''}$ has exactly 4 total inflection points. In particular, if $(\mathcal{L},\mathcal{P}) \in V_{t,t',t''}$ general, a general curve of $\mathbb{P}(V(\mathcal{L},\mathcal{P}))^\circ$ has exactly 4 total inflection points.*

Proof. Suppose $d \geq 6$ and that a general curve in $\mathcal{V}^\circ_{t,t',t''}$ has more than 4 inflection points. Then there exists a closed subset $W \subset V_{d,5}$ such that the map $p : W \to V_{d,4} : (\mathcal{L},\mathcal{P}) \mapsto (\mathcal{L}_4,\mathcal{P}_4)$ has as image $V_{t,t,t''}$ and such that the union $\mathcal{W}$ of spaces of curves $\mathbb{P}(V(\mathcal{L},\mathcal{P}))$ with $(\mathcal{L},\mathcal{P}) \in W$ is equal to $\mathcal{V}_{t,t',t''}$. We know that $\dim \mathcal{W} = \dim W + \binom{d-3}{2}$ and $\dim \mathcal{V}_{t,t',t''} = 9 + \binom{d-2}{2}$, hence $\dim W = 9 + \binom{d-2}{2} - \binom{d-3}{2} = 6 + d \geq 12$. Since $\dim(V_{t,t',t''}) = 9$, the dimension of a general fibre of p is equal to $d-3 \geq 3$. On the other hand, such a general fiber can be seen as a subset of $\mathcal{P}^2$ and hence its dimension is at most 3. So it follows that $d = 6$ and a general fiber of p is equal to $\mathcal{P}^2$. If we define the map q to be the projection $W \to V_{d,3} : (\mathcal{L},\mathcal{P}) \mapsto ((L_1,P_1),(L_2,P_2),(L_5,P_5))$, we get that $p(W) = (\mathcal{P}^2)^3$, hence $V_{d,3} = (\mathcal{P}^2)^3$, a contradiction. □

Proposition 2.4. *Assume $d \geq 6$ and $(t,t',t'') \in \nu_d \setminus \{(1,1,1)\}$. If $(\mathcal{L}_3,\mathcal{P}_3)$ is a general element of V_t, the space $\mathcal{V}_{t,t',t''}(\mathcal{L}_3,\mathcal{P}_3)$ is rational.*

Proof. There exists a smooth rational curve $\Gamma \subset (\mathbb{P}^2)^*$ (if $1 \notin \{t', t'', t't''/t\}$, Γ is a conic; if $t \neq 1$ and only one of the numbers t', t'' and $t't''/t$ is equal to 1, Γ is a line), such that for $L_4 \in \Gamma$ general one finds a unique point $P_4 \in L_4$ such that $((\mathcal{L}_3,\mathcal{P}_3),(L_4,P_4)) \in V_{t,t',t''}$.

This defines a curve $\widetilde{\Gamma} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$ together with a projection $\widetilde{\Gamma} \to \Gamma : (L_4,P_4) \mapsto L_4$ that is generically injective, hence $\widetilde{\Gamma} \to \Gamma$ is a birational equivalence. Since Γ is smooth, the normalization map of $\widetilde{\Gamma}$ defines an inverse morphism $\Gamma \to \widetilde{\Gamma}$, so $\widetilde{\Gamma}$ is isomorphic to Γ and $\widetilde{\Gamma}$ is rational and smooth.

On $\widetilde{\Gamma} \times \mathbb{P}^2$, consider the closed subscheme $\mathcal{D}$ flat over $\widetilde{\Gamma}$ with the fiber over (L_4,P_4) being $dP_4 \subset L_4$.

Let $p_1 : \widetilde{\Gamma} \times \mathbb{P}^2 \to \widetilde{\Gamma}$ and $p_2 : \widetilde{\Gamma} \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projections and consider

$$\Phi : (p_2^*(\mathcal{O}_{\mathbb{P}^2}(d))) \to p_2^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_{\mathcal{D}}$$

and

$$p_{1*}(\Phi) : p_{1*}(p_2^*(\mathcal{O}_{\mathbb{P}^2}(d))) = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \times \widetilde{\Gamma} \to p_{1*}(p_2^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_{\mathcal{D}}) = E_{\mathcal{D}}$$

whereby $E_{\mathcal{D}}$ is locally free of rank d. Let $\widetilde{\Phi}$ be the map $p_{1*}(\Phi)$ restricted to $V(\mathcal{L}_3, \mathcal{P}_3) \times \widetilde{\Gamma}$ and consider the short exact sequence

$$0 \to \mathrm{Ker}(\widetilde{\Phi}) \to V(\mathcal{L}_3, \mathcal{P}_3) \times \widetilde{\Gamma} \to \mathrm{Im}(\widetilde{\Phi}) \to 0.$$

$\mathrm{Ker}(\widetilde{\Phi})$ and $\mathrm{Im}(\widetilde{\Phi})$ are vector bundles over $\widetilde{\Gamma}$ since they are torsion free and $\widetilde{\Gamma}$ is a smooth curve. For $\widetilde{x} \in \widetilde{\Gamma}$, one has $\mathrm{Tor}_1(\mathrm{Im}(\widetilde{\Phi}), k(\widetilde{x})) = 0$, hence we have an exact sequence

$$0 \to \mathrm{Ker}(\widetilde{\Phi}) \otimes k(\widetilde{x}) \to V(\mathcal{L}_3, \mathcal{P}_3) \to \mathrm{Im}(\widetilde{\Phi}) \otimes k(\widetilde{x}) \to 0$$

with $\dim[\mathrm{Im}(\widetilde{\Phi}) \otimes k(\widetilde{x})] = d - 2$ by construction. From $\mathrm{Im}(\widetilde{\Phi}) \subset E_{\mathcal{D}}$, we obtain the following commutative diagram

$$\begin{tikzcd}
V(\mathcal{L}_3, \mathcal{P}_3) \arrow[r] \arrow[dr, "\widetilde{\Phi}(\widetilde{x})"'] & \mathrm{Im}(\widetilde{\Phi}) \otimes k(\widetilde{x}) \arrow[d, "u"'] \\
& E_{\mathcal{D}} \otimes k(\widetilde{x})
\end{tikzcd}$$

By definition of $\widetilde{\Gamma}$ one has $\mathrm{rank}(\widetilde{\Phi}(\widetilde{x})) = d - 2$, hence u is injective. This shows $\mathrm{Ker}(\widetilde{\Phi}) \otimes k(\widetilde{x}) = \mathrm{Ker}(\widetilde{\Phi}(\widetilde{x}))$, hence $\mathrm{Ker}(\widetilde{\Phi}) \otimes k(\widetilde{x}) = V_d((\mathcal{L}_3, \mathcal{P}_3), (L_4, P_4))$. Consider the projection of $\mathbb{P}(\mathrm{Ker}(\widetilde{\Phi})) \subset \mathbb{P}(V(\mathcal{L}_3, \mathcal{P}_3)) \times \widetilde{\Gamma}$ on $\mathbb{P}(V(\mathcal{L}_3, \mathcal{P}_3))$. Since $\mathbb{P}(\mathrm{Ker}(\widetilde{\Phi}))$ is rational and this projection is generically injective (see Prop. 2.3), we find that the image of the projection is rational. □

Now let $\mathcal{S}_4$ be the symmetric group of order 4, where we denote $\sigma \in \mathcal{S}_4$ by $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$. Define $\theta : \mathcal{S}_4 \times \nu_d \to \nu_d$ as the map that maps $(\sigma, (t, t', t''))$ to (t_0, t_1, t_2) if the component consisting of elements

$$(\mathcal{L}_\sigma, \mathcal{P}_\sigma) := ((L_{\sigma(1)}, P_{\sigma(1)}), \ldots, (L_{\sigma(4)}, P_{\sigma(4)})) \in V_{d,4}$$

with $(\mathcal{L}, \mathcal{P}) \in V_{t,t',t''}$ is equal to V_{t_0,t_1,t_2}.

Proposition 2.5. *If $d \geq 6$ and $(t, t', t''), (t_0, t_1, t_2) \in \nu_d \setminus \{(1, 1, 1)\}$, we have $M(V_{t,t',t''}) = M(V_{t_0,t_1,t_2})$ if and only if $\theta(\sigma, (t, t', t'')) = (t_0, t_1, t_2)$ for some element $\sigma \in \mathcal{S}_4$.*

Proof. The image of a curve contained in $\mathcal{V}^\circ_{t,t',t''}$ (resp. $\mathcal{V}^\circ_{t_0,t_1,t_2}$) under an automorphism $\phi \in \mathrm{Aut}(\mathbb{P}^2)$ remains in $\mathcal{V}^\circ_{t,t',t''}$ (resp. $\mathcal{V}^\circ_{t_0,t_1,t_2}$), hence $M(V_{t,t',t''}) = M(V_{t_0,t_1,t_2})$ if and only if $\mathcal{V}^\circ_{t,t',t''} = \mathcal{V}^\circ_{t_0,t_1,t_2}$. Since $d \geq 6$, a general curve contained in $\mathcal{V}^\circ_{t,t',t''}$ or $\mathcal{V}^\circ_{t_0,t_1,t_2}$ has exactly 4 total inflection points. Thus if $\mathcal{V}^\circ_{t,t',t''} = \mathcal{V}^\circ_{t_0,t_1,t_2}$, the total inflection points just are ordered in a different way and so there has to be a permutation $\sigma \in \mathcal{S}_4$ such that $V_{t_0,t_1,t_2} = \{(\mathcal{L}_\sigma, \mathcal{P}_\sigma) \,|\, (\mathcal{L}, \mathcal{P} \in V_{t,t',t''}))\}$. □

Write $O[t,t',t'']$ to denote $\theta(\mathcal{S}_4 \times \{(t,t',t'')\})$ and $S[t,t',t'']$ to denote

$$\{\sigma \in \mathcal{S}_4 \,|\, \theta(\sigma,(t,t',t'')) = (t,t',t'')\}$$

for $(t,t',t'') \in \nu_d$.

Proposition 2.6. *For each $(t,t',t'') \in \nu_d$, we have that $S[t,t',t'']$ is a subgroup of $\mathcal{S}_4$ and $|S[t,t',t'']|.|O[t,t',t'']| = 24 = |\mathcal{S}_4|$. Moreover, the sets $O[t,t',t'']$ with $(t,t',t'') \in \nu_d$ form a partition of ν_d.*

Proof. Note that θ is a left group action of $\mathcal{S}_4$ on ν_d, $S[t,t',t'']$ is the stabilizer of (t,t',t'') and $O[t,t',t'']$ is the orbit of (t,t',t''). The statement of the proposition now follows from classical theorems on group actions. □

Consider the following table.

$\sigma \in \mathcal{S}_4$	$\theta(\sigma,(t,t',t''))$	$T \in \nu_d$ with $\theta(\sigma,T) = T$
$(1,2,3,4)$	(t,t',t'')	$T = (t,t',t'') \in \nu_d$
$(1,3,2,4)$	$(\frac{1}{t}, \frac{t't''}{t}, \frac{1}{t''})$	$T = (1,1,1)$ or $T = (-1,a,-1)$ with $a \in \mu_d \setminus \{1\}$ and d even
$(1,2,4,3)$	$(t', t, \frac{1}{t''})$	$T = (1,1,1)$ or $T = (a,a,-1)$ with $a \in \mu_d \setminus \{1\}$ and d even
$(1,4,2,3)$	$(\frac{1}{t'}, \frac{t}{t't''}, t'')$	$T = (a, \frac{1}{a}, a^3)$ with $a \in \mu_d$
$(1,3,4,2)$	$(\frac{t't''}{t}, \frac{1}{t}, t'')$	$T = (a, \frac{1}{a}, a^3)$ with $a \in \mu_d$
$(1,4,3,2)$	$(\frac{t}{t't''}, \frac{1}{t'}, \frac{1}{t''})$	$T = (1,1,1)$ or $T = (a,-1,-1)$ with $a \in \mu_d \setminus \{1\}$ and d even
		continued on the next page

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$\sigma \in \mathcal{S}_4$	$\theta(\sigma,(t,t',t''))$	$T \in \nu_d$ with $\theta(\sigma,T)=T$
$(2,1,3,4)$	$(\frac{1}{t},\frac{1}{t'},\frac{t't''}{t})$	$T=(1,1,1)$ or $T=(-1,-1,a)$ with $a \in \mu_d \setminus \{1\}$ and d even
$(2,1,4,3)$	$(\frac{1}{t'},\frac{1}{t},\frac{t}{t't''})$	$T=(a,\frac{1}{a},a)$ with $a \in \mu_d$ or $T=(a,\frac{1}{a},-a)$ with $a \in \mu_d \setminus \{1,-1\}$ and d even
$(2,3,1,4)$	$(t,t'',\frac{t}{t't''})$	$T=(a^3,a,a)$ with $a \in \mu_d$
$(2,3,4,1)$	$(t'',t,\frac{t't''}{t})$	$T=(a,a,a)$ with $a \in \mu_d$
$(2,4,1,3)$	$(t',\frac{1}{t''},\frac{t't''}{t})$	$T=(a,a,\frac{1}{a})$ with $a \in \mu_d$
$(2,4,3,1)$	$(\frac{1}{t''},t',\frac{t}{t't''})$	$T=(a,a^3,\frac{1}{a})$ with $a \in \mu_d$
$(3,1,2,4)$	$(t,\frac{t}{t't''},t')$	$T=(a^3,a,a)$ with $a \in \mu_d$
$(3,1,4,2)$	$(\frac{t}{t't''},t,\frac{1}{t'})$	$T=(a,a,\frac{1}{a})$ with $a \in \mu_d$
$(3,2,1,4)$	$(\frac{1}{t},\frac{1}{t''},\frac{1}{t'})$	$T=(1,1,1)$ or $T=(-1,a,\frac{1}{a})$ with $a \in \mu_d \setminus \{1\}$ and d even
$(3,2,4,1)$	$(\frac{1}{t''},\frac{1}{t},t')$	$T=(\frac{1}{a},a,a)$ with $a \in \mu_d$
$(3,4,1,2)$	$(\frac{t't''}{t},t'',t')$	$T=(a,a,a)$ with $a \in \mu_d$ or $T=(-a,a,a)$ with $a \in \mu_d \setminus \{1,-1\}$ and d even
$(3,4,2,1)$	$(t'',\frac{t't''}{t},\frac{1}{t'})$	$T=(a,\frac{1}{a},a)$ with $a \in \mu_d$
$(4,1,2,3)$	$(t',\frac{t't''}{t},\frac{1}{t'})$	$T=(a,a,a)$ with $a \in \mu_d$
$(4,1,3,2)$	$(\frac{t't''}{t},t',\frac{1}{t})$	$T=(a,a^3,\frac{1}{a})$ with $a \in \mu_d$
$(4,2,1,3)$	$(\frac{1}{t'},t'',\frac{1}{t})$	$T=(\frac{1}{a},a,a)$ with $a \in \mu_d$
$(4,2,3,1)$	$(t'',\frac{1}{t'},t)$	$T=(1,1,1)$ or $T=(a,-1,a)$ with $a \in \mu_d \setminus \{1\}$ and d even
$(4,3,1,2)$	$(\frac{t}{t't''},\frac{1}{t''},t)$	$T=(a,\frac{1}{a},a)$ with $a \in \mu_d$
$(4,3,2,1)$	$(\frac{1}{t''},\frac{t}{t't''},\frac{1}{t})$	$T=(a,a,\frac{1}{a})$ with $a \in \mu_d$ or $T=(a,-a,\frac{1}{a})$ with $a \in \mu_d \setminus \{1,-1\}$ and d even

In the third columns, all $T \in \nu_d$ are listed such that $\theta(\sigma,T)=T$. These

$T \in \nu_d$ are computed in the following way. For example, if $\sigma = (1,3,2,4)$ and $T = (t, t', t'') \in \nu_d \subset (\mu_d)^3$ such that $\theta(\sigma, T) = T$, we have $t = \frac{1}{t}$, $\frac{t't''}{t} = t'$ and $\frac{1}{t''} = t''$, hence $t^2 = (t'')^2 = \frac{t}{t''} = 1$. If $t = t'' = 1$, we must have $t' = 1$ since $(t, t', t'') \in \nu_d$. In the other case, we have $t = t'' = -1$ and $t' \neq 1$ since $(t, t', t'') \in \nu_d$.

Proposition 2.7. *Let $(t, t', t'') \in \nu_d$. The set $O[t, t', t'']$ has less than 24 elements if and only if (t, t', t'') is of the form*

$$(a^3, a, a), (a, \frac{1}{a}, a^3), (a, a^3, \frac{1}{a}), (a, \frac{1}{a}, \frac{1}{a}), (a, a, a), (a, \frac{1}{a}, a) \ \textit{or}\ (a, a, \frac{1}{a})$$

or d is even and (t, t', t'') is of the form

$$(-1,-1,a), (-1,a,-1), (a,-1,-1), (-1,a,\frac{1}{a}), (a,-1,a), (a,a,-1),$$
$$(-a,a,a), (a,\frac{1}{a},-a) \ \textit{or}\ (a,-a,\frac{1}{a}).$$

Moreover, then $O[t, t', t'']$ is of the form $\{(1,1,1)\}$, $\{(-1,-1,-1)\}$ (d even), $\{(-i,i,i),(i,-i,-i)\}$ ($d \in 4\mathbb{Z}$),

$$A(a) := \Big\{ (a^3,a,a), (a,\frac{1}{a},a^3), (a,a^3,\frac{1}{a}), (\frac{1}{a},a,a),$$
$$(\frac{1}{a^3},\frac{1}{a},\frac{1}{a}), (\frac{1}{a},a,\frac{1}{a^3}), (\frac{1}{a},\frac{1}{a^3},a), (a,\frac{1}{a},\frac{1}{a}) \Big\}$$

for some $a \in \{\omega^j \,|\, 0 < j < d/2\} \setminus \{i\}$,

$$B(a) := \Big\{ (a,a,a), (a,\frac{1}{a},a), (a,a,\frac{1}{a}), (\frac{1}{a},\frac{1}{a},\frac{1}{a}), (\frac{1}{a},a,\frac{1}{a}), (\frac{1}{a},\frac{1}{a},a) \Big\}$$

for some $a \in \{\omega^j \,|\, 0 < j < d/2\}$,

$$C(a) := \Big\{ (-1,-1,a), (-1,a,-1), (a,-1,-1), (-1,a,\frac{1}{a}), (a,-1,a), (a,a,-1),$$
$$(-1,-1,\frac{1}{a}), (-1,\frac{1}{a},-1), (\frac{1}{a},-1,-1), (-1,\frac{1}{a},a), (\frac{1}{a},-1,\frac{1}{a}), (\frac{1}{a},\frac{1}{a},-1) \Big\}$$

for some $a \in \{\omega^j \,|\, 0 < j < d/2\}$ (d even) or

$$D(a) := \Big\{ (-a,a,a), (a,\frac{1}{a},-a), (a,-a,\frac{1}{a}), (a,-a,-a), (-a,-\frac{1}{a},a), (-a,a,-\frac{1}{a}),$$
$$(-\frac{1}{a},\frac{1}{a},\frac{1}{a}), (\frac{1}{a},a,-\frac{1}{a}), (\frac{1}{a},-\frac{1}{a},a), (\frac{1}{a},-\frac{1}{a},-\frac{1}{a}), (-\frac{1}{a},-a,\frac{1}{a}), (-\frac{1}{a},\frac{1}{a},-a) \Big\}$$

for some $a \in \{\omega^j \,|\, 0 < j < d/4\}$ (d even).

Proof. Since $|S[t,t',t'']|.|O[t,t',t'']| = |\mathcal{S}_4|$, we have $|O[t,t',t'']| < 24$ if and only if $\{Id\} \subsetneq S[t,t',t'']$, hence if and only if there exists a $\alpha \in \mathcal{S}_4 \setminus \{Id\}$ such that $\theta(\alpha,(t,t',t'')) = (t,t',t'')$. These $(t,t',t'') \in \nu_d$ can be found in the third column of the table above. For such a $(t,t',t'') \in \nu_d$ (with $|O[t,t',t'']| < 24$), we can compute $O[t,t',t'']$ by using the second column of the table. We use hereby the fact that the sets $O[t,t',t'']$ form a partition of ν_d. □

If $(t,t',t'') \in \nu_d$ does not appear in the list given in Proposition 2.7, we have $|O[t,t',t'']| = 24$. Hence, in case d is odd, the number of components of $M(V_{d,4})$ is equal to

$$1 + 2.\frac{d-1}{2} + \frac{d^3 - 6(d-1) - 1 - (8+6).\frac{d-1}{2}}{24} = \frac{d^3 + 11d + 12}{24}.$$

Analogously, we can see that if d is odd, $M(V_{d,4})$ has $\frac{d^3+20d}{24}$ components.

The following proposition gives us a full list of the subgroups $S[t,t',t''] \subset \mathcal{S}_4$ for all $(t,t',t'') \in \nu_d$ (up to isomorphism).

Proposition 2.8. *Let $(t,t',t'') \in \nu_d$.*

- *if $(t,t',t'') = (1,1,1)$ or $(-1,-1,-1)$ (d even): $S[t,t',t''] = \mathcal{S}_4$,*
- *if $d \in 4\mathbb{Z}$, $(t,t',t'') = (-i,i,i)$ or $(i,-i,-i)$: $S[t,t',t''] = \mathcal{A}_4$,*
- *if $(t,t',t'') \in A(a)$ for some $a \in \{\omega^j \,|\, 0 < j < d/2\} \setminus \{i\}$: $S[t,t',t''] \cong \mathbb{Z}_3$,*
- *if $(t,t',t'') \in B(a)$ for some $a \in \{\omega^j \,|\, 0 < j < d/2\}$: $S[t,t',t''] \cong \mathbb{Z}_4$,*
- *if $(t,t',t'') \in C(a)$ for some $a \in \{\omega^j \,|\, 0 < j < d/2\}$: $S[t,t',t''] \cong \mathbb{Z}_2$,*
- *if $(t,t',t'') \in D(a)$ for some $a \in \{\omega^j \,|\, 0 < j < d/4\}$: $S[t,t',t''] \cong \mathbb{Z}_2$,*
- *in the other cases: $S(t,t',t'') = \{Id\} \subset \mathcal{S}_4$.*

Proof. To compute $S[t,t',t'']$ for an element $(t,t',t'') \in \nu_d$, we only have to write down all the elements $\sigma \in S_4$ with $\theta(\sigma,(t,t',t'')) = (t,t',t'')$ (this can be done using the table). The number of elements $|S[t,t',t'']|$ is equal to $24/|O[t,t',t'']|$. □

Proposition 2.9. *If $d \geq 6$ and*

$$(t,t',t'') \in \nu_d \setminus \{(1,1,1),(-1,-1,-1),(-i,i,i),(i,-i,-i)\},$$

the component $M(V_{t,t',t''}) \subset M(V_{d,4})$ is rational.

Proof. Let $(\mathcal{L}_3, \mathcal{P}_3) \in V_t$ be general. It is enough to prove that $M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)$ is rational. We will first prove that

$$M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)) \cong \frac{\mathcal{V}_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ}{S[t,t',t'']}.$$

Assume that C and C' are general smooth curves in $\mathcal{V}_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ$ with $C \in \mathbb{P}(V((\mathcal{L}, \mathcal{P})))$ and $C' \in \mathbb{P}(V(\mathcal{L}', \mathcal{P}'))$. We have that $[C] = [C'] \in M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3))$ if and only if there exists an automorphism ϕ of $\mathbb{P}^2$ such that $\phi(C) = C'$. If the latter happens, since C and C' have exactly 4 total inflection points, the automorphism ϕ changes the order of the total inflection points, hence there exists a permutation $\sigma \in \mathcal{S}_4$ such that $\phi(\mathcal{L}, \mathcal{P}) = (\mathcal{L}'_\sigma, \mathcal{P}'_\sigma)$. Thus we get that $(\mathcal{L}', \mathcal{P}')$ and $(\mathcal{L}'_\alpha, \mathcal{P}'_\alpha)$ are contained in $V_{t,t',t''}$, so $\sigma \in S[t,t',t'']$. On the other side, if $C \in \mathbb{P}(V(\mathcal{L}, \mathcal{P})) \subset \mathcal{V}_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ$ and $\sigma \in S[t,t',t'']$, the curve C also belongs to $\mathbb{P}(V(\mathcal{L}_\alpha, \mathcal{P}_\alpha))$. Since there exists an automorphism φ of $\mathbb{P}^2$ such that $\varphi(\mathcal{L}_\alpha, \mathcal{P}_\alpha)$ is of the form $((\mathcal{L}_3, \mathcal{P}_3), (L'_4, P'_4))$, we have $C' = \varphi(C) \in \mathcal{V}_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3)^\circ$.

By Proposition 2.8, we have that $S[t,t',t''] \subset \mathcal{S}_4$ is Abelian. By using a result due to E. Fischer (see [5]) and Proposition 2.4, we conclude that $M(V_{t,t',t''}(\mathcal{L}_3, \mathcal{P}_3))$ is rational. $\square$

Remark 2.10. From [2, Theorem C] follows that $M(V_{1,1,1})$ is rational. We cannot use the above arguments in order to prove that $M(V_{t,t',t''})$ is rational for

$$(t,t',t'') \in \{(1,1,1), (-1,-1,-1), (-i,i,i), (i,-i,-i)\},$$

since in each of these cases, the group $S[t,t',t'']$ is not Abelian.

The results on the components of $M(V_{d,4})$ are summarized in the following theorem.

Theorem 2.11. *Assume $k = \mathbb{C}$ and $d \geq 6$. If d is odd, $M(V_{d,4})$ has $\frac{d^3+11d+12}{24}$ components and each of these components is rational. If d is even, $M(V_{d,4})$ has $\frac{d^3+20d}{24}$ components and at most τ of them are not rational, whereby $\tau = 3$ if $4|d$ and $\tau = 1$ if $d \equiv 2 \mod 4$.*

3 A result for the case $e = 5$

We will first give a new proof of the following result of A.M. Vermeulen (see [7, Prop. 2.12]).

Proposition 3.1. *Let $(\mathcal{L}, \mathcal{P})$ be an element of $(\mathcal{P}^2)^e$ such that no 3 lines are concurrent. If for all $2 \leq i < j \leq e$, we have $((L_1, P_1), (L_i, P_i), (L_j, P_j)) \in V_{d,3}$, then $(\mathcal{L}, \mathcal{P}) \in V_{d,e}$.*

Proof. It is easy to see that it is enough to prove the following claim: assume $(\mathcal{L},\mathcal{P}) = ((L_1,P_1),\dots,(L_e,P_e)) \in V_{d,e}$ and $(L,P) \in \mathcal{P}^2$ such no 3 of the lines $L_1,\dots,L_e,L$ are concurrent. If moreover $((L_1,P_1),(L_i,P_i),(L,P)) \in V_{d,3}$ for all $i \in \{2,\dots,e\}$, we have $((\mathcal{L},\mathcal{P}),(L,P)) \in V_{d,e+1}$.

We will first consider the case where $e \leq d+1$. Let $\Gamma \in \mathbb{P}(V(\mathcal{L},\mathcal{P}))$ be a curve not containing one of the lines L_i. We have

$$\mathbb{P}(V_L(\mathcal{L},\mathcal{P})) = \langle P_{1,0} + \ldots + P_{e,0} + g^{d-e}_{d-e}, \Gamma.L\rangle$$

($\{\Gamma.L\}$ in case $e = d+1$) with $P_{i,0} = L_i \cap L$. Since $\Gamma \in \mathbb{P}(V((L_1,P_1),(L_i,P_i)))$ and $((L_1,P_1),(L_i,P_i),(L,P)) \in V_{d,3}$ for all $i \in \{2,\dots,e\}$, one has

$$dP \in \mathbb{P}(V((L_1,P_1),(P_i,L_i))) = \langle P_{1,0} + P_{i,0} + g^{d-2}_{d-2}, \Gamma.L\rangle.$$

Of course, we have

$$\mathbb{P}(V_L(\mathcal{L},\mathcal{P})) \subset \bigcap_{i=2}^{e} \mathbb{P}(V_L((L_1,P_1),(L_i,P_i)))$$

and $\dim(\mathbb{P}(V_L(\mathcal{L},\mathcal{P}))) = d-e+1$.

For $2 \leq i \leq e$, we have $P_{1,0}+\ldots+P_{i,0}+g^{d-i}_{d-i} \subset \bigcap_{j=2}^{i} \mathbb{P}(V_L((L_1,P_1),(L_j,P_j)))$. Take $F \in g^{d-i}_{d-i}$ with $P_{i+1,0} \notin F$, then $P_{1,0}+\ldots+P_{i,0}+F \notin P_{1,0}+P_{i+1,0}+g^{d-2}_{d-2}$. If $P_{1,0}+\ldots+P_{i,0}+F \in \mathbb{P}(V_L((L_1,P_1),(L_{i+1},P_{i+1})))$, then for some $G \in g^{d-2}_{d-2}$, we have $P_{1,0}+\ldots+P_{i,0}+F \in \langle P_{1,0}+P_{i+1,0}+G, \Gamma.L\rangle$, hence

$$\Gamma.L \in \langle P_{1,0}+P_{i+1,0}+G, P_{1,0}+\ldots+P_{i,0}+F\rangle,$$

so $P_{1,0} \in \Gamma \cap L$. This implies $L_1 \subset \Gamma$ since $P_{1,0} \in \Gamma \cap L_1$, so we get a contradiction and $P_{1,0}+\ldots+P_{i,0}+F \notin \mathbb{P}(V_L((L_1,P_1),(L_{i+1},P_{i+1})))$ and so

$$\dim\left\{\bigcap_{j=2}^{i+1} \mathbb{P}(V_L((L_1,P_1),(L_j,P_j)))\right\} < \dim\left\{\bigcap_{j=2}^{i} \mathbb{P}(V_L((L_1,P_1),(L_j,P_j)))\right\}.$$

This proves $\dim\left\{\bigcap_{j=2}^{e} \mathbb{P}(V_L((L_1,P_1),(L_j,P_j)))\right\} = d-e+1$, hence

$$\mathbb{P}(V_L(\mathcal{L},\mathcal{P})) = \bigcap_{j=2}^{e} \mathbb{P}(V_L((L_1,P_1),(L_j,P_j)))$$

and so $dP \in \mathbb{P}(V_L(\mathcal{L},\mathcal{P}))$, hence $((\mathcal{L},\mathcal{P}),(L,P)) \in V_{d,e+1}$.

Now assume $e \geq d+2$. We have $\mathbb{P}(V((L_1,P_1),\dots,(L_{d+1},P_{d+1}))) = \{\Gamma\}$. Since $(\mathcal{L},\mathcal{P}) \in V_{d,e}$, we have $dP_i = \Gamma.L_i$ for all $i \in \{1,\dots,e\}$. The previous part of this proof implies $((L_1,P_1),\dots,(L_{d+1},P_{d+1}),(L,P)) \in V_{d,d+2}$, hence $dP \in \Gamma.L$. So we find $\Gamma \in \mathbb{P}(V((\mathcal{L},\mathcal{P}),(L,P)))$, i.e. $((\mathcal{L},\mathcal{P}),(L,P)) \in V_{d,e+1}$. □

Theorem 3.2. *Assume that $V \subset V_{d,5}$ a component is of dimension 10 such that for a general $(\mathcal{L}, \mathcal{P}) \in V$, no three of the points $P_1, \ldots, P_5$ are collinear. Then d is even and $(\mathcal{L}, \mathcal{P}) \in V_{2,5}$.*

Proof. Assume that

$$((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_4, P_4)) \in V_{t_1, t_1', t_1''}$$

and

$$((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_5, P_5)) \in V_{t_2, t_2', t_2''}.$$

Note that $\dim(V_{t_1,t_1',t_1''}) = \dim(V_{t_2,t_2',t_2''}) = 9$ (we even have that none of the numbers $t_1, t_1', t_1'', t_2, t_2'$ or t_2'' is equal to 1) and that $t_1 = t_2$. Since $\dim(V_{t_1}) = 8$ and $\dim(V) = 10$, for general elements $((\mathcal{L}_3, \mathcal{P}_3), (L_4', P_4')) \in V_{t_1,t_1',t_1''}$ and $((\mathcal{L}_3, \mathcal{P}_3), (L_5', P_5')) \in V_{t_2,t_2',t_2''}$ we have $((\mathcal{L}_3, \mathcal{P}_3), (L_4', P_4'), (L_5', P_5')) \in V$.

One needs $((L_1, P_1), (L_2, P_2), (L_4, P_4), (L_5, P_5)) \in V_{t_3,t_3',t_3''}$ general for some $t_3, t_3', t_3'' \in \mu_d$ (we omitted the accents in (L_4', P_4') and (L_5', P_5') for notational reasons). Using the base $\{(L_1, P_1), (L_2, P_2), (L_3, P_3)\}$, let $(X_1 : X_2 : X_3)$ be the coordinates of $\mathbb{P}^2$, and by using $\{(L_1, P_1), (L_2, P_2), (L_4, P_4)\}$, let them be $(X_1' : X_2' : X_3')$. Assume L_4 has as equation $X_3 = AX_1 + BX_2$. It is easy to see that

$$\begin{pmatrix} 1-A & 0 & 0 \\ 0 & 1-B & 0 \\ -A & -B & 1 \end{pmatrix}$$

is a matrix corresponding to the coordinate transformation from the coordinates $(X_1 : X_2 : X_3)$ to $(X_1' : X_2' : X_3')$. The equation $X_3' = A'X_1' + B'X_2'$ of L_5 becomes $X_3 = [A'(1-A) + A]X_1 + [B'(1-B) + B]X_2$. Hence $(\widetilde{A}, \widetilde{B}) = (A'(1-A) + A, B'(1-B) + B)$ is a general solution of

$$(t_2' - 1)\widetilde{A}\widetilde{B} = (\frac{t_2' t_2''}{t_2} - 1)\widetilde{A} + t_2'(1 - t_2'')\widetilde{B}. \tag{1}$$

This equation holds for general (A', B') satisfying

$$(t_3' - 1)A'B' = (\frac{t_3' t_3''}{t_3} - 1)A' + t_3'(1 - t_3'')B'. \tag{2}$$

Since (2) is an equation without constant term, the constant term in (1) has to be equal to zero, so we get

$$(t_2' - 1)AB = (\frac{t_2' t_2''}{t_2} - 1)A + t_2'(1 - t_2'')B. \tag{3}$$

This equation should be satisfied for general (A, B) satisfying

$$(t_1' - 1)AB = (\frac{t_1' t_1''}{t_1} - 1)A + t_1'(1 - t_1'')B, \tag{4}$$

hence (3) and (4) have to define the same curve. If we take $A = 1$ we get that

$$B = \frac{\frac{t_1' t_1''}{t_1} - 1}{t_1' t_1'' - 1} = \frac{\frac{t_2' t_2''}{t_2} - 1}{t_2' t_2'' - 1}.$$

Since $t_1 = t_2$, we get $t_1' t_1'' = t_2' t_2''$, hence the coefficients of A in (3) and (4) are equal. So we get that the coefficients of AB in (3) and (4) are also equal, so $t_1' = t_2'$ and thus $t_2' = t_2''$. We can conclude that $((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_4, P_4))$ and $((L_1, P_1), (L_2, P_2), (L_3, P_3), (L_5, P_5))$ belong to the same component of $V_{d,4}$.

Now let $(\mathcal{L}^{(j)}, \mathcal{P}^{(j)}) = ((\mathcal{L}_3, \mathcal{P}_3), (L_j, P_j))$ be general elements of $V_{t_1, t_1', t_1''}$ for each $j = 4, \ldots, m$. We see that $((\mathcal{L}_3, \mathcal{P}_3), (L_i, P_i), (L_j, P_j)) \in V$ for each $i, j \in \{4, \ldots, m\}$ with $i \neq j$. Hence we get that $((L_1, P_1), (L_i, P_i), (L_j, P_j)) \in V_{d,3}$, so $(\overline{\mathcal{L}}, \overline{\mathcal{P}}) = ((L_1, P_1), (L_2, P_2), \ldots, (L_m, P_m)) \in V_{d,m}$. Let $\Gamma \in \mathbb{P}(V(\overline{\mathcal{L}}, \overline{\mathcal{P}}))$ with $L_i \not\subset \Gamma$ for all i and let $n_1\Gamma_1 + \ldots + n_s\Gamma_s$ be its decomposition into irreducible curves. Write d_i to denote the degree of Γ_i

Assume $s \geq 2$. Since

$$d = i(\Gamma.L_j, P_j) = \sum_{i=1}^{s} n_i.i(\Gamma_i.L_j, P_j) \leq \sum_{i=1}^{s} n_i.d_i = d,$$

we get that (L_j, P_j) is also a total inflection point of Γ_i for all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, m\}$. A fortiori, the points $P_1, \ldots, P_m$ are contained in $\cap_{i=1}^{s} \Gamma_i$, thus m is bounded. We get a contradiction, so $s = 1$ and $\Gamma = n_1\Gamma_1$.

If $d_1 \geq 3$, then the number of total inflection points of Γ_1 is bounded by d, hence $d_1 \leq 2$. Since $d_1 = 1$ is excluded, we find $d_1 = 2$ and $(\mathcal{L}, \mathcal{P}) \in V_{2,5}$. $\square$

Remark 3.3. A point $(\mathcal{L}, \mathcal{P}) \in (\mathcal{P}^2)^5$ is contained in $V_{2,5}$ if and only if there exists a smooth conic $C \subset \mathbb{P}^2$ through the five points $P_1, \ldots, P_5$ such that L_i is the tangent line to C at P_i for all i. It is clear that $V_{2,5}$ is 10-dimensional (see also [3, Ex. 3.5]).

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