A Tropical Proof of the Brill-Noether Theorem

(joint work with J. Draisma, S. Payne and E. Robeva)

Filip Cools

K.U.Leuven (Belgium)

June 15th, 2010

Classical Brill-Noether Theory (BNT)

Fix integers $g, r, d \geq 0$, and denote the Brill-Noether number by $\rho(g, r, d) := g - (r+1)(g - d + r)$.

(a) If $\rho(g, r, d) \geq 0$, then every smooth curve $X/\mathbb{C}$ of genus g has a divisor D of degree $\leq d$ and rank r.

(b) If $\rho(g, r, d) < 0$, then on a general smooth curve $X/\mathbb{C}$ of genus g, there is no divisor D of degree $\leq d$ and rank r.

BNT for metric graphs ?

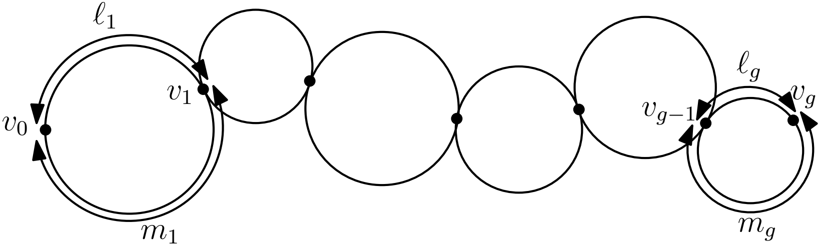
In the paper "Specialization of linear systems from curves to graphs", M. Baker proves part (a) of BNT for metric graphs, i.e.

> *If* $\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0$*, then every metric graph* Γ *of genus* g *has a divisor* D *of degree* $\leq d$ *and rank* r*.*

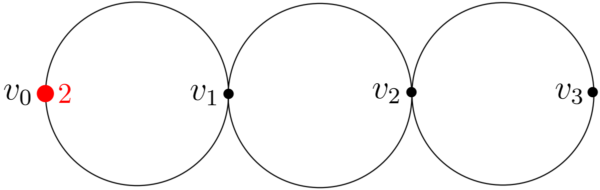
He also conjectures that part (b) of BNT is true under the following form.

> *If* $\rho(g, r, d) < 0$*, then there exists a metric graph* Γ *of genus* g *for which there is no divisor* D *of degree* $\leq d$ *and rank* r*.*

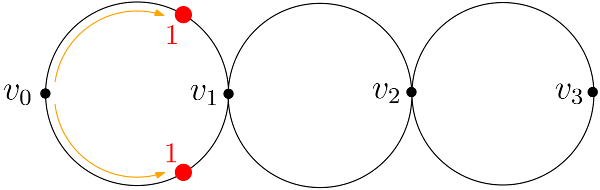
Consider the metric graph Γ of genus g which is a chain of g loops, where the lengths of the two edges of the ith loop are equal to ℓ_i and m_i.



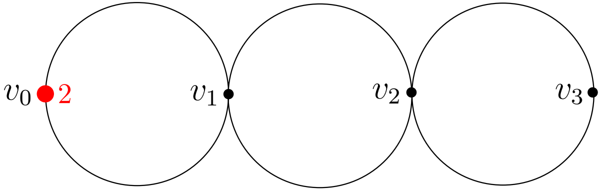
Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



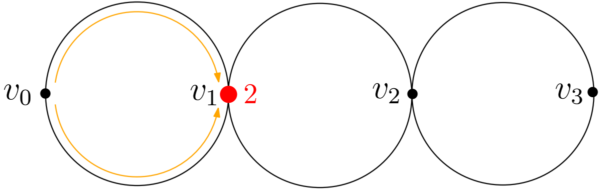
is equivalent with



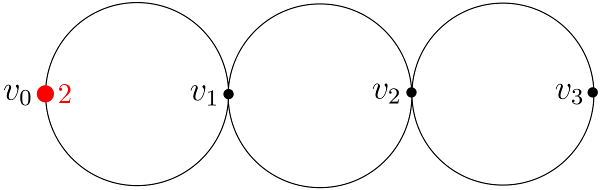
Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



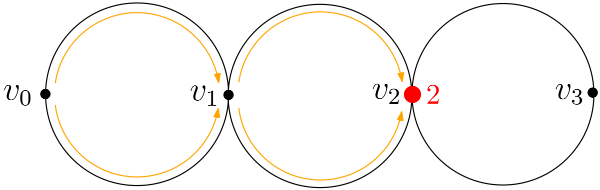
is equivalent with



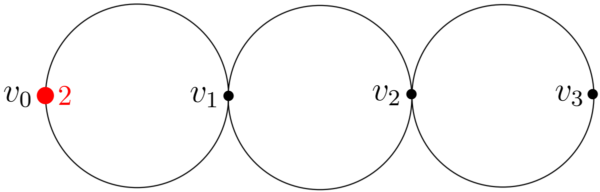
Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



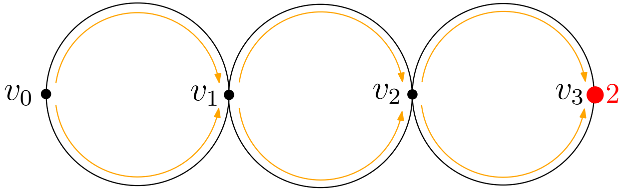
is equivalent with



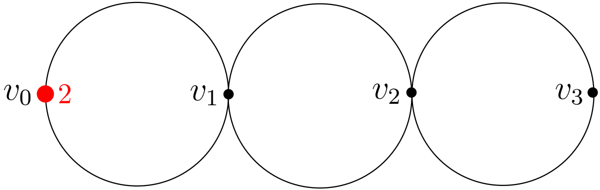
Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



is equivalent with



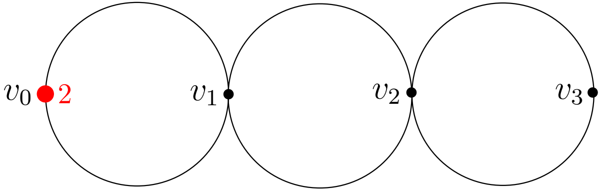
Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



The divisor $D = 2v_0$ has rank 1. So for $g = 3$, $d = 2$ and $r = 1$, there does exists a divisor on Γ of rank r and degree $\leq d$, but $\rho = g - (r+1)(g - d + r) = -1$.

Conclusion: not a good example for part (b) of BNT for metric graphs.

Example: $g = 3$ and $\ell_i = m_i = 1$ for all i



The divisor $D = 2v_0$ has rank 1. So for $g = 3$, $d = 2$ and $r = 1$, there does exists a divisor on Γ of rank r and degree $\leq d$, but $\rho = g - (r+1)(g - d + r) = -1$.
Conclusion: not a good example for part (b) of BNT for metric graphs.

Idea

Assume that the edge lengths ℓ_i and m_i are generic positive numbers.

Precise definition: The metric graph Γ is *generic* if none of the ratios ℓ_i/m_i is equal to the ratio of two positive integers whose sum is less or equal to $2g-2$.

Example: $\ell_i = 2g-2$ and $m_i = 1$ for all i.

Theorem

Assume Γ is generic.

(a) If $\rho(g, r, d) < 0$, there is no divisor D on Γ with degree $\leq d$ and rank r. (i.e. part (b) of BNT for metric graphs)

(b) If $\rho(g, r, d) \geq 0$, then the dimension of $W_d^r(\Gamma)$ is equal to $\min\{\rho(g, r, d), g\}$.

(c) If $\rho(g, r, d) = 0$, there are exactly

$$\lambda = g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}$$

linear systems on Γ of rank r and degree d.

Theorem

Assume Γ is generic.

(a) If $\rho(g, r, d) < 0$, there is no divisor D on Γ with degree $\leq d$ and rank r. (i.e. part (b) of BNT for metric graphs)

(b) If $\rho(g, r, d) \geq 0$, then the dimension of $W_d^r(\Gamma)$ is equal to $\min\{\rho(g, r, d), g\}$.

(c) If $\rho(g, r, d) = 0$, there are exactly

$$\lambda = g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}$$

linear systems on Γ of rank r and degree d.

Theorem

Assume Γ is generic.

(a) If $\rho(g, r, d) < 0$, there is no divisor D on Γ with degree $\leq d$ and rank r. (i.e. part (b) of BNT for metric graphs)

(b) If $\rho(g, r, d) \geq 0$, then the dimension of $W_d^r(\Gamma)$ is equal to $\min\{\rho(g, r, d), g\}$.

(c) If $\rho(g, r, d) = 0$, there are exactly

$$\lambda = g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}$$

linear systems on Γ of rank r and degree d.

Proposition

Assume Γ is generic, $r = 1$ and $g = 2d - 2$ (so $\rho(g, r, d) = 0$). Then there is a bijection between $W^1_d(\Gamma)$ and

$$\left\{ \begin{array}{l} \text{lattice paths } p = (p_0, \ldots, p_g) \text{ in } \mathbb{Z} \text{ satisfying } p_0 = p_g = 1, \\ p_i \geq 1 \text{ and } p_i - p_{i-1} = \pm 1 \text{ for all } i \in \{1, \ldots, g\} \end{array} \right\}$$

as follows. If $p = (p_0, \ldots, p_g)$ is such a path, let D_p be the divisor on Γ with

- one chip in v_0,
- one (extra) chip on the unique point w_i of the ith loop satisfying $p_{i-1}v_{i-1} + w_i \sim p_i v_i$ if $p_i - p_{i-1} = 1$,
- no (extra) chips on the ith loop if $p_i - p_{i-1} = -1$.

Note that D_p is v_0-reduced. The bijection maps p to the linear system $|D_p|$.

Proposition

Assume Γ is generic, $r = 1$ and $g = 2d - 2$ (so $\rho(g, r, d) = 0$). Then there is a bijection between $W^1_d(\Gamma)$ and

$$\left\{ \begin{array}{l} \text{lattice paths } p = (p_0, \ldots, p_g) \text{ in } \mathbb{Z} \text{ satisfying } p_0 = p_g = 1, \\ p_i \geq 1 \text{ and } p_i - p_{i-1} = \pm 1 \text{ for all } i \in \{1, \ldots, g\} \end{array} \right\}$$

as follows. If $p = (p_0, \ldots, p_g)$ is such a path, let D_p be the divisor on Γ with

- one chip in v_0,
- one (extra) chip on the unique point w_i of the ith loop satisfying $p_{i-1}v_{i-1} + w_i \sim p_i v_i$ if $p_i - p_{i-1} = 1$,
- no (extra) chips on the ith loop if $p_i - p_{i-1} = -1$.

Note that D_p is v_0-reduced. The bijection maps p to the linear system $|D_p|$.

Proposition

Assume Γ is generic, $r = 1$ and $g = 2d - 2$ (so $\rho(g, r, d) = 0$). Then there is a bijection between $W^1_d(\Gamma)$ and

$$\left\{ \begin{array}{l} \text{lattice paths } p = (p_0, \dots, p_g) \text{ in } \mathbb{Z} \text{ satisfying } p_0 = p_g = 1, \\ p_i \geq 1 \text{ and } p_i - p_{i-1} = \pm 1 \text{ for all } i \in \{1, \dots, g\} \end{array} \right\}$$

as follows. If $p = (p_0, \dots, p_g)$ is such a path, let D_p be the divisor on Γ with

- one chip in v_0,
- one (extra) chip on the unique point w_i of the ith loop satisfying $p_{i-1}v_{i-1} + w_i \sim p_i v_i$ if $p_i - p_{i-1} = 1$,
- no (extra) chips on the ith loop if $p_i - p_{i-1} = -1$.

Note that D_p is v_0-reduced. The bijection maps p to the linear system $|D_p|$.

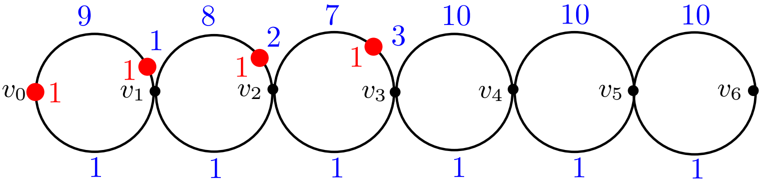
Example: $r = 1,\ g = 6,\ d = 4$

In this case, there are precisely five lattice paths that satisfy the conditions:

- $(1, 2, 3, 4, 3, 2, 1)$
- $(1, 2, 3, 2, 3, 2, 1)$
- $(1, 2, 3, 2, 1, 2, 1)$
- $(1, 2, 1, 2, 3, 2, 1)$
- $(1, 2, 1, 2, 1, 2, 1)$

Example: $r = 1$, $g = 6$, $d = 4$, $p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to

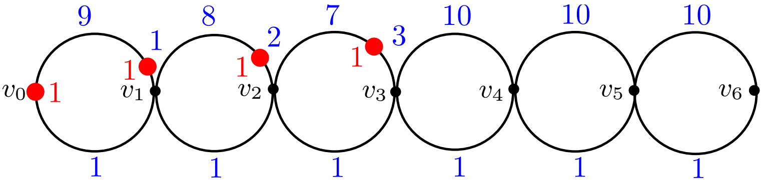


since we have e.g.

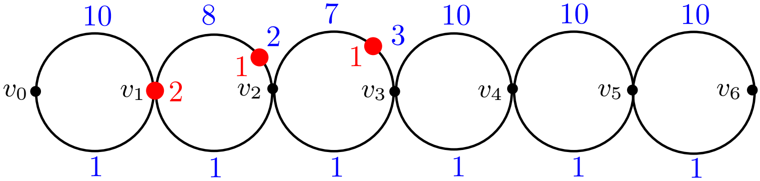
- one chip at v_0,
- one chip at the unique point w_2 of the 2nd loop satisfying $p_1 v_1 + w_2 = 2v_1 + w_2 \sim p_2 v_2 = 3v_2$ since $p_2 - p_1 = 1$,
- no chip at the 4th loop since $p_4 - p_3 = -1$.

Example: $r = 1,\ g = 6,\ d = 4,\ p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to

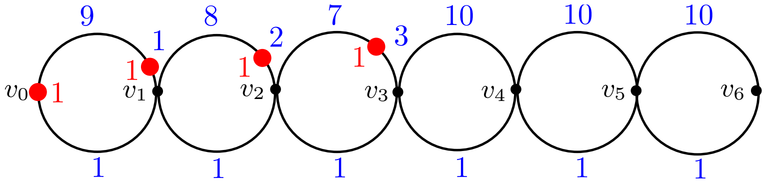


and is equivalent with

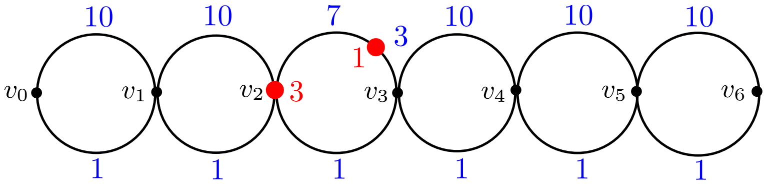


Example: $r = 1,\ g = 6,\ d = 4,\ p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to

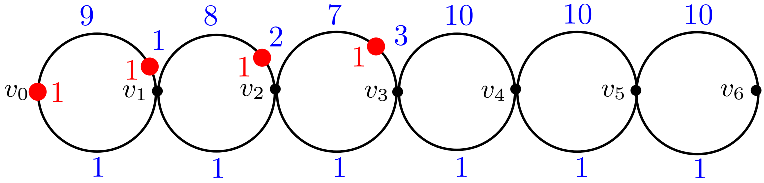


and is equivalent with

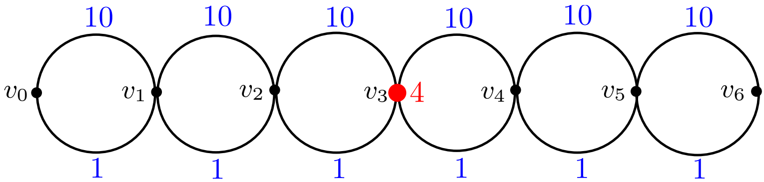


Example: $r = 1,\ g = 6,\ d = 4,\ p = (1,2,3,4,3,2,1)$

The divisor D_p is equal to

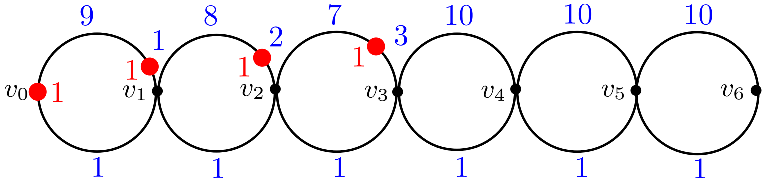


and is equivalent with

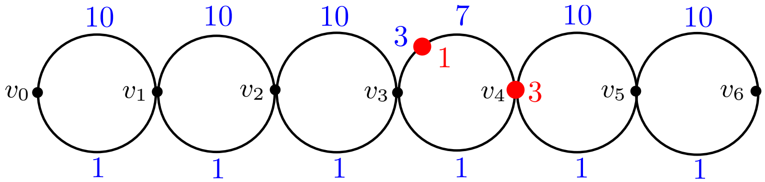


Example: $r = 1,\ g = 6,\ d = 4,\ p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to

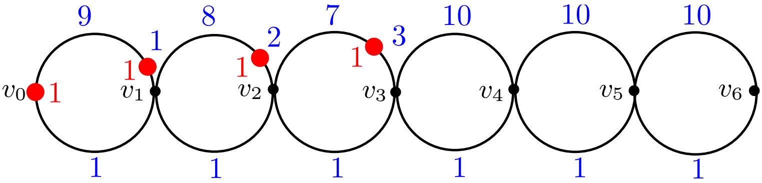


and is equivalent with

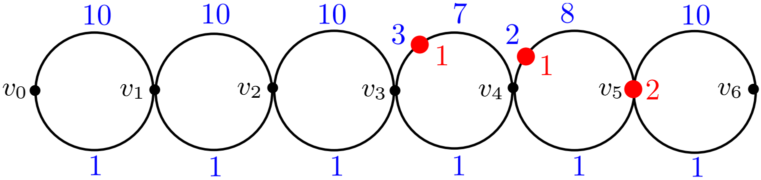


Example: $r = 1,\ g = 6,\ d = 4,\ p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to

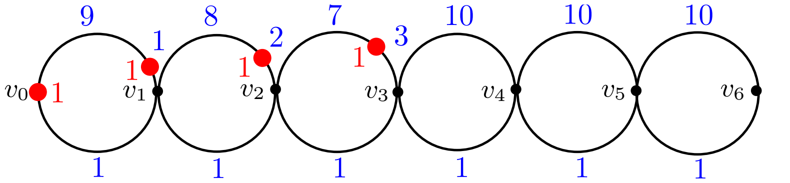


and is equivalent with

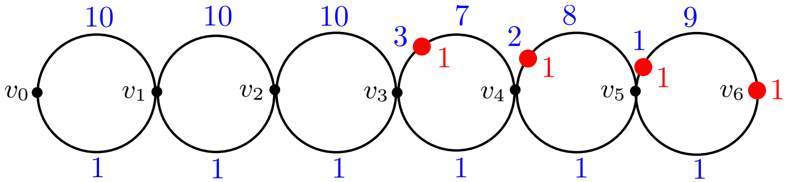


Example: $r = 1,\ g = 6,\ d = 4,\ p = (1, 2, 3, 4, 3, 2, 1)$

The divisor D_p is equal to



and is equivalent with



Corollary of the theorem

Using Baker's Specialization Lemma, we get a new (tropical) characteristic-free proof of the BNT for curves.

Main result of the recent preprint "Linear pencils on graphs and on real curves" (joint work with M. Coppens)

There exists a smooth complex curve X (resp. X') of genus $g = 2d - 2$ defined over $\mathbb{R}$ having exactly $\lambda = \frac{1}{d}\binom{2d-2}{d-1}$ linear pencils of degree d such that all of them (resp. exactly $\lambda' = \binom{d-1}{\lceil \frac{d-1}{2} \rceil}$ of them) are real.

d	2	3	4	5	6	7	8	9	10
g	2	4	6	8	10	12	14	16	18
λ	1	2	5	14	42	132	429	1430	4862
λ'	1	2	3	6	10	20	35	70	126

Note that $\lim_{d \to \infty} \frac{\lambda'}{\lambda} = 0$.