Newton polygons and curve gonalities

Filip Cools
joint work with Wouter Castryck

K.U. Leuven

Tropical Geometry workshop (Edinburgh)
April 5, 2012

Introduction

- $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$: irreducible Laurent polynomial
- $\Delta(f)$: its Newton polygon
  - i.e. if
    $$f = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij} x^i y^j,$$
    then
    $$\Delta(f) = \mathrm{Conv}\{(i,j) \in \mathbb{Z}^2 \mid c_{ij} \neq 0\} \subset \mathbb{R}^2$$
- $C(f)$: curve in $\mathbb{T}^2_{\mathbb{C}} = (\mathbb{C} \setminus \{0\})^2$ defined by f

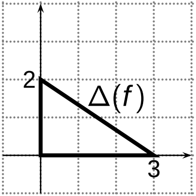
Theorem

(Baker, 1893) The (geometric) genus of $C(f)$ is bounded by the number of $\mathbb{Z}^2$-points in the interior of $\Delta(f)$.

(Khovanskii, 1977) Generically, this bound is attained.

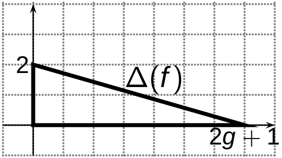
Examples

- $f = y^2 - x^3 - Ax - B$ with $B \neq 0$



$\#(\Delta^\circ \cap \mathbb{Z}^2) = 1$
the genus of $C(f)$ is equal to one
iff $4A^3 + 27B^2 \neq 0$

- $f = y^2 - h(x)$ with $\deg h = 2g + 1$ and $h(0) \neq 0$



$\#(\Delta^\circ \cap \mathbb{Z}^2) = g$
the genus of $C(f)$ is equal to g iff
$h(x)$ has no multiple roots

Central question of this talk

Question

Does there exist a similar combinatorial interpretation for the gonality?

- gonality = minimal degree of a non-constant rational map to $\mathbb{P}^1_{\mathbb{C}}$
- hyperelliptic = gonality 2 (by definition)

Central question of this talk

- A lattice polygon is the convex hull in $\mathbb{R}^2$ of a finite number of $\mathbb{Z}^2$-points (also called lattice points).
- The genus of a two-dimensional lattice polygon Δ is the (geometric) genus of the curve defined by a generic Laurent polynomial f with $\Delta(f) = \Delta$.
- Notation: $g(\Delta)$. By the foregoing: $g(\Delta) = \#(\Delta^\circ \cap \mathbb{Z}^2)$.
- The gonality of a two-dimensional lattice polygon Δ is the gonality of the curve defined by a generic Laurent polynomial f with $\Delta(f) = \Delta$.
- Notation: $\gamma(\Delta)$. Well-defined by a semi-continuity argument.

Question (reformulated)

Does there exist a purely combinatorial interpretation for $\gamma(\Delta)$?

Some terminology and easy facts

- A $\mathbb{Z}$-affine transformation is a map
$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y)A + b$$
with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$.
- Two lattice polygons Δ and Δ' are equivalent if there is a $\mathbb{Z}$-affine transformation φ such that $\varphi(\Delta) = \Delta'$. (Notation: $\Delta \equiv \Delta'$)
- A $\mathbb{Z}$-affine transformation φ acts on $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ as
$$f = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij}(x, y)^{(i,j)} \quad \mapsto \quad \varphi(f) = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij}(x, y)^{\varphi(i,j)}.$$
- $\Delta(\varphi(f)) = \varphi(\Delta(f))$ and $C(f) \cong C(\varphi(f))$.

The lattice width as an upper bound

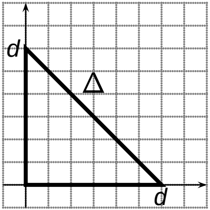
- The lattice width of a non-empty lattice polygon Δ is the minimal d for which there is a $\mathbb{Z}$-affine transformation φ such that

$$\varphi(\Delta) \subset \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq d\}.$$

- Notation: $\mathrm{lw}(\Delta)$.
- Convention: $\mathrm{lw}(\emptyset) = -1$.
- Easy fact: $\gamma(\Delta) \leq \mathrm{lw}(\Delta)$.
  - Let f be a generic Laurent polynomial with $\Delta(f) = \Delta$.
  - Let φ be a $\mathbb{Z}$-affine transformation realizing $\mathrm{lw}(\Delta)$.
  - $C(f) \cong C(\varphi(f))$, so it suffices to deal with $C(\varphi(f))$.
  - Then $C(\varphi(f)) \to \mathbb{A}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}} : (x, y) \mapsto x$ is of degree at most d.

Sharp?

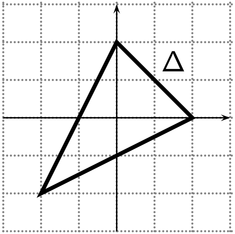
- Counterexample 1



- $\gamma(\Delta) = d - 1$ (Namba, 1979: gonality of smooth plane curves)
- $\text{lw}(\Delta) = d$, since every edge contains $d + 1$ lattice points

Sharp?

- Counterexample 2



- $\gamma(\Delta) \leq 3$ (by Brill-Noether Theorem, curves of genus 4 are at most 3-gonal)
- $\text{lw}(\Delta) = 4$, because the interior polygon contains an interior $\mathbb{Z}^2$-point itself

The interior polygon

- Let Δ be a two-dimensional lattice polygon. The convex hull of the interior lattice points is called the interior polygon of Δ.
  Notation: $\Delta^{(1)}$

Theorem (–, Lubbes & Schicho)

$\mathrm{lw}(\Delta^{(1)}) = \mathrm{lw}(\Delta) - 2$, *unless* $\Delta \equiv \mathrm{Conv}\{(0,0),(d,0),(0,d)\}$ *for* $d \geq 2$, *in which case* $\mathrm{lw}(\Delta) = d$ *and* $\mathrm{lw}(\Delta^{(1)}) = d - 3$.

- Thus in fact $\gamma(\Delta) \leq \mathrm{lw}(\Delta^{(1)}) + 2$. This rules out Counterexample 1 as an exceptional case. Counterexample 2 is more fundamental.
- Algorithm for computing $\mathrm{lw}(\Delta)$.

Conjecture

$\gamma(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 2$, *unless* $\Delta \equiv \mathrm{Conv}\{(2,0),(0,2),(-2,-2)\}$, *in which case* $\gamma(\Delta) = 3$.

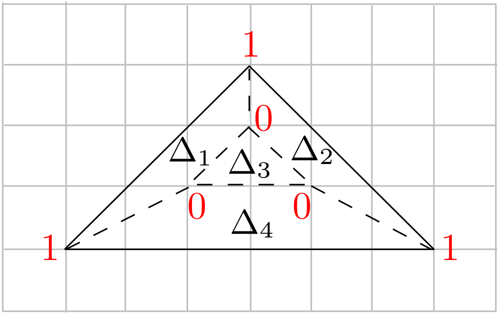
The metric graph $\Gamma(h)$

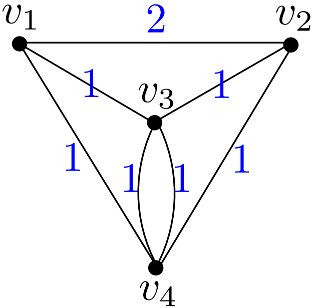
- Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon.
- Let $\Delta_1, \ldots, \Delta_r \subset \Delta$ be a regular subdivision.
- Let $h : \Delta \to \mathbb{R}$ be an upper-convex piece-wise linear function such that its restrictions to $\Delta_1, \ldots, \Delta_r$ are linear. Assume that $h(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$.
- Definition metric graph $\Gamma(h)$:
  - vertices $v_1, \ldots, v_r$
  - number of edges between v_i and v_j is the integral length of $\Delta_i \cap \Delta_j$
  - length of an edge between v_i and v_j is the greatest common divisor of the 2×2-minors of

$$\begin{pmatrix} a_{i1} & a_{i2} & 1 \\ a_{j1} & a_{j2} & 1 \end{pmatrix},$$

  where $(a_{k1}, a_{k2}, 1)$ is a primitive normal vector to the graph of $h|_{\Delta_k}$.

Example





Note that the edge (v_1, v_2) has length equal to 2 since the corresponding 2×3-matrix is

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Lower bound for $\gamma(\Delta)$

- Given a metric graph Γ, denote by $\gamma(\Gamma)$ the gonality of Γ, i.e.

$$\gamma(\Gamma) = \min\{d \mid \exists D \in Div_d(\Gamma) : r_{BN}(D) \geq 1\}.$$

Theorem

If $h : \Delta \to \mathbb{R}$ gives rise to a regular subdivision (as above), then

$$\gamma(\Gamma(h)) \leq \gamma(\Delta).$$

- Idea of proof:
  - let $\mathrm{Tor}(\widetilde{\Delta})$ be the toric threefold corresponding to h
  - consider the toric degeneration of $Tor(\Delta)$ to $\cup_{i=1}^{r} Tor(\Delta_i)$
  - view $C(f)$ as a generic hyperplane section of the toric surface $Tor(\Delta)$ and let it degenerate
  - use Baker's Specialization Lemma (might need to blow-up some boundary $\mathbb{T}^1$'s at the bottom of $\widetilde{\Delta}$!)

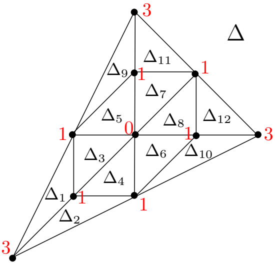
Lower bound = gonality?

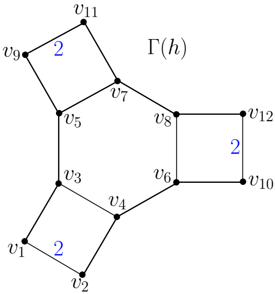
- We expect that it is always possible to obtain equality:

Conjecture

There always exists a height function $h : \Delta \to \mathbb{R}$ such that $\gamma(\Gamma(h)) = \gamma(\Delta)$.

- Example: our Counterexample 2.





$\gamma(\Delta) = \gamma(\Gamma(h)) = 3.$

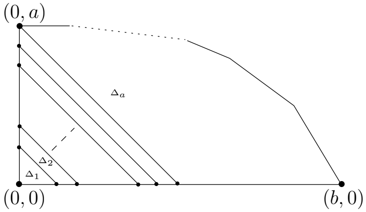
Lower bound = upper bound?

- A proof of the following combinatorial statement would solve it all:

Conjecture

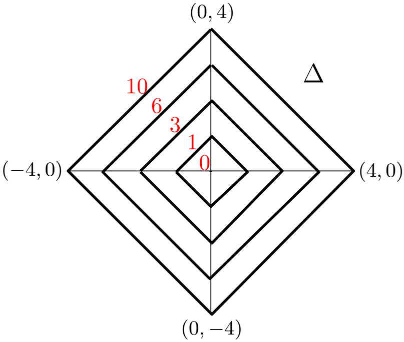
There always exists a height function $h : \Delta \to \mathbb{R}$ *such that* $\gamma(\Gamma(h)) = \mathrm{lw}(\Delta^{(1)}) + 2$, *except if* $\Delta \equiv \mathrm{Conv}\{(2,0),(0,2),(-2,-2)\}$.

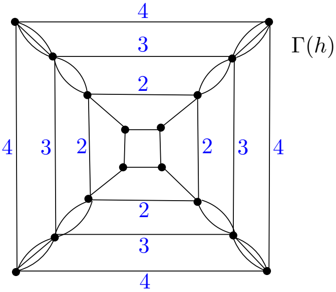
- Example of a lattice polygon Δ for which we can prove the above conjecture:



Lower bound = upper bound?

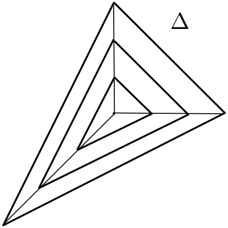
- A specific guess for the height function h: the “union skin” subdivision of Δ
- Example: $\gamma(\Delta) = \gamma(\Gamma(h)) = \mathrm{lw}(\Delta) = 8$

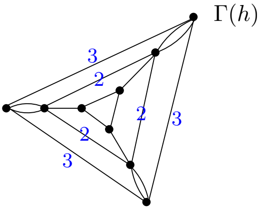




Other invariants: Clifford index and dimension

- Question : does the metric graph $\Gamma(h)$ (corresponding to the union skin subdivision of Δ) have the same Clifford index and dimension as the generic curve $C(f)$?
- Example: $\Delta = \text{Conv}\{(3,0),(0,3),(-3,-3)\}$
  - $C(f)$ has a g^3_9 since it is the intersection of two cubics in $\mathbb{P}^3$. The Clifford index is $9 - 2.3 = 3 < 6 - 2.1 = 4$ and the Clifford dimension is 3.
  - $\Gamma(h)$ also has a g^3_9.





- Thanks for listening!