Combinatorial interpretations for invariants of smooth curves on toric surfaces

(joint work with Wouter Castryck)

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- k algebraically closed field of $\mathrm{char}(k) = 0$ (e.g. $k = \mathbb{C}$)
- $\mathbb{T}^2 = (k^*)^2$ two-dimensional torus
- $f \in k[x^{\pm 1}, y^{\pm 1}]$ irreducible Laurent polynomial
- $U(f) \subset \mathbb{T}^2$ curve defined by f
- $\Delta = \Delta(f)$ the Newton polygon of f (i.e. the convex hull of the exponent vectors that appear in f with a non-zero coefficient)

Definition

f is non-degenerate with respect to its Newton polygon if for every face $\tau \subset \Delta(f)$ (including $\Delta(f)$ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$.

Definition

An algebraic curve C/k is called Δ-non-degenerate if it is birationally equivalent to $U(f)$ for some Δ-non-degenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

Question

Which geometric properties/birational invariants of C are encoded in the combinatorics of the Newton polygon Δ?

Definition

A toric surface is an algebraic surface that contains $\mathbb{T}^2$ as a Zariski open dense subset, such that the self-action of $\mathbb{T}^2$ extends to an action on the whole surface. (First examples: $\mathbb{A}^2, \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \ldots$)

Example

Let Δ be a two-dimensional lattice polygon and consider the map

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^N : (x, y) \mapsto \left(x^i y^j\right)_{(i,j)\in\Delta\cap\mathbb{Z}^2} \quad (\text{where } N = \sharp(\Delta\cap\mathbb{Z}^2) - 1))$$

Then $\mathrm{Tor}(\Delta) = \overline{\varphi_\Delta(\mathbb{T}^2)}$ is a projective toric surface.

Remark

If f is Δ-non-degenerate, then $C = \overline{\varphi_\Delta(U(f))} \subset \mathrm{Tor}(\Delta)$ is a smooth hyperplane section.

Definition

For lattice polygons $\Delta, \Delta' \subset \mathbb{R}^2$, we say that Δ is equivalent to Δ' (notation: $\Delta \cong \Delta'$) if Δ' is obtained from Δ through a unimodular transformation, i.e. through a transformation of the form

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} i \\ j \end{pmatrix} \mapsto A \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad A \in \mathrm{GL}_2(\mathbb{Z}),\ a_1, a_2 \in \mathbb{Z}.$$

Remark

If a Laurent polynomial

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}(x,y)^{(i,j)}$$

is Δ-non-degenerate and α is a unimodular transformation, then

$$f^\alpha = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}(x,y)^{\alpha(i,j)}$$

is $\alpha(\Delta)$-non-degenerate, and $U(f) \cong U(f^\alpha)$.

Corollary

Each combinatorial interpretation in terms of the Newton polygon of a birational invariant of a non-degenerate curve should be invariant under unimodular transformations.

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Let C be a smooth projective curve.

- A divisor $D = \sum_{P \in C} D(P) \cdot P$ is a finite $\mathbb{Z}$-linear combination of points on C.
- The divisor D is effective iff $D(P) \geq 0$ for each $P \in C$. Notation: $D \geq 0$.
- The degree of D is $\deg(D) = \sum_{P \in C} D(P) \in \mathbb{Z}$.
- The divisor of a rational function $f \in k(C)^*$ is defined as $\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) \cdot P = (f)_0 - (f)_\infty$. Note that $\deg(\mathrm{div}(f)) = 0$.
- Two divisors D, D' on C are linear equivalent iff $D' - D = \mathrm{div}(f)$ for some $f \in k(C)$.

- The Riemann-Roch space of D is

$$\mathcal{L}(D) = \{f \in k(C)^* \mid div(f) + D \geq 0\} \cup \{0\}.$$

Note: this is a linear subspace of $k(C)$ and we denote its dimension by $\ell(D)$.

- Riemann-Roch Theorem: $\exists$ canonical divisor K_C, $\forall$ D :

$$\ell(D) - \ell(K_C - D) = \deg(D) - g + 1,$$

where $g = g(C)$ is the genus of C.
Note that $\ell(K_C) = g$ and $\deg(K_C) = 2g - 2$.

- Corollary : $\begin{cases} \ell(D) = 0 & \text{if } \deg(D) < 0 \\ \ell(D) = \deg(D) - g + 1 & \text{if } \deg(D) > 2g - 2 \end{cases}$

- If $V \subset \mathcal{L}(D)$ is linear, then $\{D + div(f) \mid f \in V\}$ is a linear system on C of degree $d = \deg(D)$ and rank $r = \dim(V) - 1$. Notation: g^r_d.
- A linear system is called complete iff $V = \mathcal{L}(D)$.
- A linear pencil is a linear system of rank 1.
- The gonality $\gamma(C)$ of C is the minimal degree of a linear pencil on C. Equivalently, it is the minimal degree of a non-constant rational map from C to $\mathbb{P}^1$.
- Remark : gonality 1 = rational, gonality 2 = hyperelliptic, gonality 3 = trigonal, etc.
- Facts on gonality :
  - Gonality pencils are always complete.
  - Brill-Noether inequality : $\gamma(C) \leq \left\lfloor \frac{g(C)+3}{2} \right\rfloor$.

- A canonical map for a non-rational curve C is a map of the form
$$C \to \mathbb{P}^{g-1} : P \mapsto (f_1(P) : \ldots : f_g(P)),$$
where $f_1, \ldots, f_g$ are generators of $\mathcal{L}(K_C)$.
- Facts:
  - If C is hyperelliptic, then the canonical image is a rational normal curve of degree $g-1$.
  - If not, then the canonical map is in fact an embedding. The image $C_{can} \subset \mathbb{P}^{g-1}$ is called a canonical model of C.
- Geometric version of Riemann-Roch: If D is an effective divisor on a canonical model C_{can}, then
$$\dim\langle D\rangle = \deg(D) - \dim \mathcal{L}(D).$$

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Theorem (H. Baker 1893, Khovanskii 1977)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the (geometric) genus g of $U(f)$ equals $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$.
More precisely, let $C = \overline{\varphi_\Delta(U(f))}$ be the smooth projective model of $U(f)$ as before. Then there exists a canonical divisor K_C on C such that the Riemann-Roch space $\mathcal{L}(K_C)$ is generated by the functions $\{x^i y^j\}_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2}$.
Hereby, the lattice polygon $\Delta^{(1)}$ is the convex hull of the interior latice points of Δ.

Corollary

- *The curve $U(f)$ is rational iff $\Delta^{(1)} = \emptyset$.*
- *The curve $U(f)$ is hyperelliptic iff $\Delta^{(1)}$ is one-dimensional.*
- *If $\Delta^{(1)}$ is two-dimensional, the map*

$$\varphi_{\Delta^{(1)}} : U(f) \to \mathbb{P}^{g-1} : (x, y) \mapsto \left(x^i y^j\right)_{(i,j)\in\Delta^{(1)}\cap\mathbb{Z}^2}$$

gives rise to a canonical model

$$C_{can} = \overline{\varphi_{\Delta^{(1)}}(U(f))} \subset Tor(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

Remark

The inclusion $C_{can} \subset \mathrm{Tor}(\Delta^{(1)})$ is not a hyperplane section!

Theorem (C.-C.)

Given a lattice polygon Δ with $\Delta^{(1)}$ two-dimensional and f non-degenerate w.r.t. Δ, there is a concrete way to write down generators of the canonical ideal $\mathcal{I}(C_{can})$.

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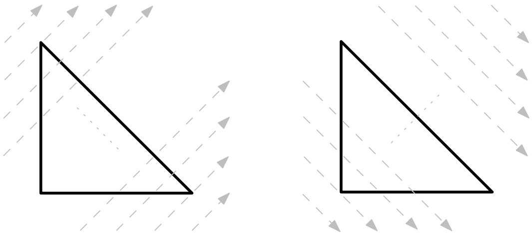
Definition

- A lattice direction is just a primitive element of $v = (a, b) \in \mathbb{Z}^2$.
- For a non-empty lattice polygon Δ and a lattice direction $v = (a, b)$, the width of Δ with respect to v is the minimal d for which there exists an $m \in \mathbb{Z}$ such that Δ is contained in the strip
$$m \leq aY - bX \leq m + d.$$
Note that $w(\Delta, v) = w(\Delta, -v)$.
- The lattice width of Δ is
$$\operatorname{lw}(\Delta) = \min_v w(\Delta, v).$$

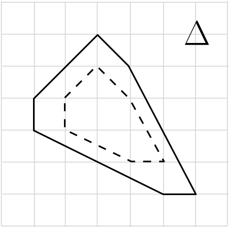
Example

The width of $d\Sigma$ with respect to $(1,1)$ is $2d$, while its width with respect to $(1,-1)$ is d. Here, $\Sigma = \text{conv}\{(0,0),(1,0),(0,1)\}$.



The polygon $d\Sigma$ has lattice width $\text{lw}(d\Sigma) = d$ and there are precisely three lattice directions computing this.

Example



Also in this case, there are three lattice width directions.

Definition

Let Δ, f, $C \subset \mathrm{Tor}(\Delta)$, $v = (a, b)$ be as before.
Then the rational map $U(f) \to \mathbb{T}^1 : (x, y) \mapsto x^a y^b$ extends to a degree $w(\Delta, v)$ morphism $C \to \mathbb{P}^1$. Let g_v be the corresponding base-point free pencil. A pencil on C that arises as g_v for some lattice direction v is called combinatorial.

Remark

- Note that $g_v = g_{-v}$.
- The correspondence between pairs $\pm v$ of lattice directions and combinatorial pencils is usually 1-to-1, but there are counterexamples.

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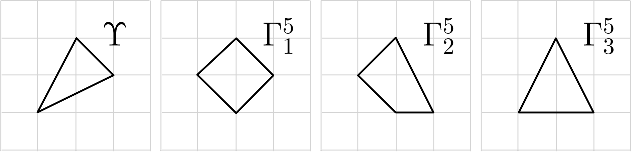
Theorem (R. Kawaguchi, C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Suppose that $\Delta^{(1)}$ is not equivalent to any of the following:

$$\emptyset, \quad (d-3)\Sigma \text{ (for some integer } d \geq 3), \quad \Upsilon, \quad 2\Upsilon, \quad \Gamma_1^5, \quad \Gamma_2^5, \quad \Gamma_3^5.$$

Then every gonality pencil on (the smooth projective model of) $U(f)$ is combinatorial.



Remark

- If $\Delta^{(1)} = \emptyset$ then $U(f)$ is rational, hence of gonality 1.
- If $\Delta^{(1)} \cong (d-3)\Sigma$ then $U(f)$ is birationally equivalent to a smooth projective plane curve of degree d, hence of gonality $d-1$.
- If $\Delta^{(1)} \cong \Upsilon$ then $U(f)$ is a non-hyperelliptic curve of genus 4, hence of gonality 3.
- If $\Delta^{(1)} \cong 2\Upsilon$ then $U(f)$ is birationally equivalent to a smooth intersection of two cubics in $\mathbb{P}^3$, hence of gonality 6.
- If $\Delta^{(1)} \cong \Gamma_i^5$ $(i = 1, 2, 3)$ then $U(f)$ is a non-hyperelliptic, non-trigonal curve of genus 5, hence of gonality 4.

Corollary

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate. Then the gonality $\gamma(U(f))$ of $U(f)$ equals $lw(\Delta^{(1)}) + 2$, unless $\Delta^{(1)} \cong \Upsilon$ (i.e. $\Delta \cong 2\Upsilon$), in which case it equals 3.

Corollary

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate.

- *If $\Delta^{(1)} = \emptyset$ then there is a unique gonality pencil.*
- *If $\Delta^{(1)} \cong \Upsilon$ then the number of gonality pencils is at most 2.*
- *If $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 3$, or if $\Delta^{(1)} \cong 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5$, then there are infinitely many gonality pencils.*
- *In all other cases the number of gonality pencils equals the number of lattice width directions. In particular, the number of gonality pencils is at most 4, and the bound is met iff $\Delta^{(1)} \cong d\Gamma_1^5$ for some $d \geq 2$.*

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Definition

If $\Delta \neq \emptyset$, then its lattice size $\mathrm{ls}(\Delta)$ is defined as the minimal integer $d \geq 0$ such that Δ is equivalent to a lattice polygon that is contained in $d\Sigma$ (set $\mathrm{ls}(\emptyset) = -2$).

Theorem (C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the minimal degree of a (possibly singular) projective plane curve that is birationally equivalent to $U(f)$ is bounded by $\mathrm{ls}(\Delta^{(1)}) + 3$. If $\Delta^{(1)} \cong (d-1)\Upsilon$ for a certain integer $d \geq 2$ (i.e. $\Delta \cong d\Upsilon$), then it is moreover bounded by $3d - 1$.

Definition

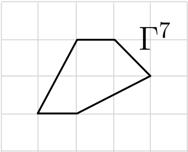
By a near-gonal pencil on a smooth projective curve C/k, we mean a base-point free $g^1_{\gamma(C)+1}$ (note that such pencils need not exist).

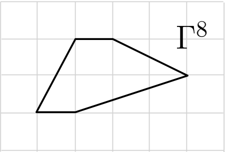
Theorem (C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate and let γ be the gonality of $U(f)$. Suppose that

$$ls(\Delta^{(1)}) \geq lw(\Delta^{(1)}) + 2 \qquad (\bigstar)$$

and that $\Delta^{(1)} \not\cong 2\Upsilon, 3\Upsilon, \Gamma^7, \Gamma^8$. Then every base-point free $g^1_{\gamma+1}$ on (the smooth projective model of) $U(f)$ is combinatorial.





Remark

- If condition ($\bigstar$) fails, then $U(f)$ is birationally equivalent with a (possibly singular) plane curve of degree $\gamma + 1$ or $\gamma + 2$, so $U(f)$ has infinitely many base-point free $g^1_{\gamma+1}$'s.
  This is also the case if $\Delta^{(1)} \cong 2\Upsilon$ (with $\gamma = 6$) or $\Delta^{(1)} \cong \Gamma^7$ (with $\gamma = 4$).
- If $\Delta^{(1)} \cong 3\Upsilon$, then there exists a base-point free g^1_9, but no combinatorial one.
- If $\Delta^{(1)} \cong \Gamma^8$, there are no combinatorial g^1_5's, but there are instances of curves $U(f)$ with a base-point free g^1_5.

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- Clifford index (Kawaguchi, C.-C.)
- Clifford dimension (Kawaguchi, C.-C.)
- Scrollar invariants associated to a combinatorial pencil (C.-C.)
- Completeness of a combinatorial pencil (C.-C.)
- Schreyer's tetragonal invariants (C.-C.)

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Theorem (C.-C.)

If a smooth projective curve C/k carries a point P having a Weierstrass semi-group of embedding dimension 2 (i.e. of the form $a\mathbb{N} + b\mathbb{N}$ for coprime integers $a, b \geq 2$), then this semi-group does not depend on the choice of P.

Theorem (C.-C.)

If a smooth projective curve C/k of gonality $\gamma > 2$ is contained in a Hirzebruch surface $\mathcal{H}_n$, then n is an invariant of C.

- Is the Newton polytope intrinsic for non-degenerate curves? More precise formulation: If a curve C/k is both Δ-non-degenerate and Δ'-non-degenerate, does it follow that $\Delta^{(1)} \cong \Delta'^{(1)}$?
- Is it possible to prove (or disprove) Green's canonical conjecture for non-degenerate curves?

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Thanks!