Newton polygons and curve gonalities

Filip Cools
joint work with Wouter Castryck

K.U. Leuven

Tropical geometry and computational biology (Saarbrücken)
September 29, 2011

Introduction

- $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$: irreducible Laurent polynomial
- $\Delta(f)$: its Newton polygon
  - i.e. if

$$f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} x^i y^j,$$

then

$$\Delta(f) = \text{Conv}\{(i,j) \in \mathbb{Z}^2 \mid c_{ij} \neq 0\} \subset \mathbb{R}^2$$

- $C(f)$: curve in $\mathbb{T}^2_{\mathbb{C}} = (\mathbb{C} \setminus \{0\})^2$ defined by f

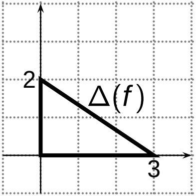
Theorem

(Baker, 1893) The (geometric) genus of $C(f)$ is bounded by the number of $\mathbb{Z}^2$-points in the interior of $\Delta(f)$.

(Khovanskii, 1977) Generically, this bound is attained.

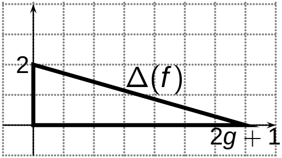
Examples

- $f = y^2 - x^3 - Ax - B$ with $B \neq 0$



$\#(\Delta^\circ \cap \mathbb{Z}^2) = 1$
the genus of $C(f)$ is equal to one iff $4A^3 + 27B^2 \neq 0$

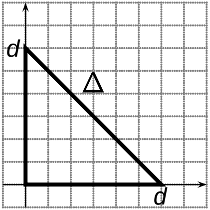
- $f = y^2 - h(x)$ with $\deg h = 2g + 1$ and $h(0) \neq 0$



$\#(\Delta^\circ \cap \mathbb{Z}^2) = g$
the genus of $C(f)$ is equal to g iff $h(x)$ has no multiple roots

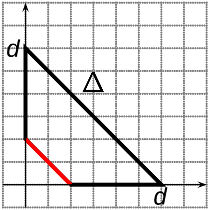
Examples

- f polynomial of degree d: then $\Delta(f)$ is contained in



$$\#(\Delta^\circ \cap \mathbb{Z}^2) = \frac{(d-1)(d-2)}{2}$$

If $C(f)$ has a singularity at (x_0, y_0), then $\Delta(f(x + x_0, y + y_0))$ is contained in



$$\#(\Delta^\circ \cap \mathbb{Z}^2) = \frac{(d-1)(d-2)}{2} - 1$$

Central question of this talk

Question

Does there exist a similar combinatorial interpretation for the gonality?

- gonality = minimal degree of a non-constant rational map to $\mathbb{P}^1_{\mathbb{C}}$
- hyperelliptic = gonality 2 (by definition)

Central question of this talk

- A lattice polygon is the convex hull in $\mathbb{R}^2$ of a finite number of $\mathbb{Z}^2$-points (also called lattice points).
- The genus of a two-dimensional lattice polygon Δ is the (geometric) genus of the curve defined by a generic Laurent polynomial f with $\Delta(f) = \Delta$.
- Notation: $g(\Delta)$. By the foregoing: $g(\Delta) = \#(\Delta^\circ \cap \mathbb{Z}^2)$.
- The gonality of a two-dimensional lattice polygon Δ is the gonality of the curve defined by a generic Laurent polynomial f with $\Delta(f) = \Delta$.
- Notation: $\gamma(\Delta)$. Well-defined by a semi-continuity argument.

Question (reformulated)

Does there exist a purely combinatorial interpretation for $\gamma(\Delta)$?

Some terminology and easy facts

- A $\mathbb{Z}$-affine transformation is a map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y)A + b$$

  with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$.

- Two lattice polygons Δ and Δ' are equivalent if there is a $\mathbb{Z}$-affine transformation φ such that $\varphi(\Delta) = \Delta'$. (Notation: $\Delta \equiv \Delta'$)

- A $\mathbb{Z}$-affine transformation φ acts on $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ as

$$f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} (x, y)^{(i,j)} \quad \mapsto \quad \varphi(f) = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} (x, y)^{\varphi(i,j)}.$$

- $\Delta(\varphi(f)) = \varphi(\Delta(f))$ and $C(f) \cong C(\varphi(f))$.

The lattice width as an upper bound

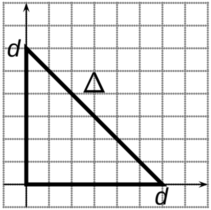
- The lattice width of a non-empty lattice polygon Δ is the minimal d for which there is a $\mathbb{Z}$-affine transformation φ such that

$$\varphi(\Delta) \subset \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq d\}.$$

- Notation: $\mathrm{lw}(\Delta)$.
- Convention: $\mathrm{lw}(\emptyset) = -1$.
- Easy fact: $\gamma(\Delta) \leq \mathrm{lw}(\Delta)$.
  - Let f be a generic Laurent polynomial with $\Delta(f) = \Delta$.
  - Let φ be a $\mathbb{Z}$-affine transformation realizing $\mathrm{lw}(\Delta)$.
  - $C(f) \cong C(\varphi(f))$, so it suffices to deal with $C(\varphi(f))$.
  - Then $C(\varphi(f)) \to \mathbb{A}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}} : (x,y) \mapsto x$ is of degree at most d.

Sharp?

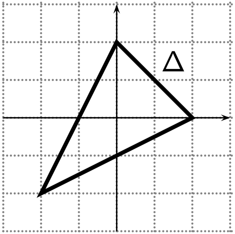
- Counterexample 1



- $\gamma(\Delta) = d - 1$ (Namba, 1979: gonality of smooth plane curves)
- $\mathrm{lw}(\Delta) = d$, since every edge contains $d + 1$ lattice points

Sharp?

- Counterexample 2



- $\gamma(\Delta) \leq 3$ (by Brill-Noether Theorem, curves of genus 4 are at most 3-gonal)
- $\text{lw}(\Delta) = 4$, because the interior polygon contains an interior $\mathbb{Z}^2$-point itself

The interior polygon

- Let Δ be a two-dimensional lattice polygon. The convex hull of the interior lattice points is called the interior polygon of Δ.
  Notation: $\Delta^{(1)}$

Theorem (–, Lubbes & Schicho, 2010)

$\mathrm{lw}(\Delta^{(1)}) = \mathrm{lw}(\Delta) - 2$, *unless* $\Delta \equiv \mathrm{Conv}\{(0,0),(d,0),(0,d)\}$ *for* $d \geq 2$, *in which case* $\mathrm{lw}(\Delta) = d$ *and* $\mathrm{lw}(\Delta^{(1)}) = d - 3$.

- Thus in fact $\gamma(\Delta) \leq \mathrm{lw}(\Delta^{(1)}) + 2$. This rules out Counterexample 1 as an exceptional case. Counterexample 2 is more fundamental.
- Algorithm for computing $\mathrm{lw}(\Delta)$.

Conjecture

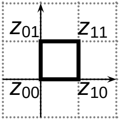
$\gamma(\Delta) = \mathrm{lw}(\Delta^{(1)}) + 2$, *unless* $\Delta \equiv \mathrm{Conv}\{(2,0),(0,2),(-2,-2)\}$, *in which case* $\gamma(\Delta) = 3$.

Toric surfaces

- To each $(i,j) \in \Delta \cap \mathbb{Z}^2$ we associate a formal variable z_{ij}.
- The toric surface

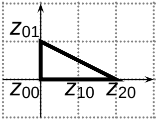
$$\mathrm{Tor}(\Delta) \subset \mathbb{P}_{\mathbb{C}}^{\#(\Delta\cap\mathbb{Z}^2)-1} = \mathrm{Proj}\,\mathbb{C}[z_{ij}]$$

  is defined by all homogeneous binomial relations that are 'induced by the combinatorics of Δ'.
- Example: $\Delta = \mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\}$.



$\mathrm{Tor}(\Delta) : z_{10}z_{01} = z_{00}z_{11}$
(hyperboloid)

- Example: $\Delta = \mathrm{Conv}\{(0,0),(2,0),(0,1)\}$.



$\mathrm{Tor}(\Delta) : z_{20}z_{00} = z_{10}^2$
(cone)

Toric surfaces

- Alternatively, Tor(Δ) is the Zariski closure of the image of

$$\mathbb{T}^2_{\mathbb{C}} \hookrightarrow \mathbb{P}^{\#(\Delta\cap\mathbb{Z}^2)-1}_{\mathbb{C}} : (x,y) \mapsto (x^i y^j)_{(i,j)\in\Delta\cap\mathbb{Z}^2}.$$

- Under this map, the curve $C(f) \subset \mathbb{T}^2_{\mathbb{C}}$ with

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{ij} x^i y^j \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

  maps to the hyperplane section of Tor(Δ) defined by

$$\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{ij} z_{ij} = 0.$$

- $g(\Delta)$ = ‘sectional genus’ of Tor(Δ).
- $\gamma(\Delta)$ = ‘sectional gonality’ of Tor(Δ).

Proof of conjecture for $\text{lw}(\Delta^{(1)}) = -1$

- $\gamma(\Delta) \leq \text{lw}(\Delta^{(1)}) + 2 = 1 \leq \gamma(\Delta)$

Proof of conjecture for $\mathrm{lw}(\Delta^{(1)}) = 0$

- $\gamma(\Delta) \leq \mathrm{lw}(\Delta^{(1)}) + 2 = 2$.
- On the other hand $\gamma(\Delta) \geq 2$ since $g(\Delta) \geq 1$.

Proof of conjecture for $\mathrm{lw}(\Delta^{(1)}) = 1$

- $\gamma(\Delta) \leq \mathrm{lw}(\Delta^{(1)}) + 2 = 3$.
- By a refined version of Khovanskii's Theorem, the curve $C(f)$ is canonically embedded by

$$\pi : C(f) \to \mathbb{P}_{\mathbb{C}}^{g(\Delta)-1} : (x, y) \mapsto (x^i y^j)_{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2}$$

  if f is non-degenerate with respect to Δ (generic condition).
- $\mathrm{lw}(\Delta^{(1)}) = 1 \Rightarrow \Delta^{(1)}$ is two-dimensional
  $\Rightarrow$ assume that $\{(0,0), (1,0), (0,1)\} \subset \Delta^{(1)}$
  $\Rightarrow \mathbb{C}(\pi(C(f))) = \mathbb{C}(C(f)) = \mathrm{Frac}(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]/(f))$
  $\Rightarrow g(\pi(C(f))) = g(C(f)) = g(\Delta) > 1$
  $\Rightarrow \gamma(\Delta) \geq 3$

Proof of conjecture for $\mathrm{lw}(\Delta^{(1)}) = 2$

- $\gamma(\Delta) \leq \mathrm{lw}(\Delta^{(1)}) + 2 = 4$.
- Analogously as in the above case, we get $\gamma(\Delta) \geq 3$ (and $\pi(C(f)) \subset \mathrm{Tor}(\Delta^{(1)})$).
- Suppose $\gamma(\Delta) = 3$ and $\#(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$.
  - Since $\#(\partial\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$, $\mathrm{Tor}(\Delta^{(1)})$ is generated by quadrics (Koelman, 1993).
  - Since $\gamma(\Delta) = 3$, the intersection of all quadrics containing $\pi(C(f))$ is a surface of sectional genus 0 (Petri, 1923).
  - Hence the sectional genus of $\mathrm{Tor}(\Delta^{(1)})$ is zero, i.e. $g(\Delta^{(1)}) = 0$.
  - $\Delta^{(1)(1)} = \emptyset \Rightarrow \mathrm{lw}(\Delta^{(1)(1)}) = -1 \Rightarrow \mathrm{lw}(\Delta^{(1)}) = 1$ or $\Delta^{(1)} \equiv \mathrm{Conv}\{(0,0),(2,0),(0,2)\}$: contradiction.
- Suppose $\gamma(\Delta) = 3$ and $\#(\partial\Delta^{(1)} \cap \mathbb{Z}^2) = 3$.
  - $g(\Delta^{(1)}) = 0$: contradiction as above.
  - $g(\Delta^{(1)}) > 0 \Rightarrow \Delta^{(1)} \equiv \mathrm{Conv}\{(-1,-1),(1,0),(0,1)\}$: Counterexample 2

Proof of conjecture for $\mathrm{lw}(\Delta^{(1)}) > 2$

- **?**
- One naturally bumps into Green's canonical conjecture (a generalization of Petri's theorem).
- But:
  - Green's conjecture is unproven.
  - Even if it were proven, we require a better understanding of the Betti table of $\pi(C(f))$ in terms of $\Delta(f)$.

Linear systems on curves

- Let $C/\mathbb{C}$ be a non-singular algebraic curve.
- A divisor on C is an element of the free abelian group generated by C:

$$\mathrm{Div}(C) = \left\{ \sum_{P \in C} n_P \cdot P \,\middle|\, n_P \in \mathbb{Z}, n_P = 0 \text{ for all but finitely many } P \right\}.$$

- The degree of a divisor is $\sum n_P$. It is called effective if all $n_P \geq 0$.
- To $g \in \mathbb{C}(C)$, one can associate a divisor $\mathrm{div}(g) = \sum_P \mathrm{ord}_P(g) \cdot P$. It has degree 0.
- Two divisors D and D' are called equivalent if $D' - D = \mathrm{div}(g)$ for some $g \in \mathbb{C}(C)$.

Linear systems on curves

- The complete linear system $|D|$ is the set of all effective divisors that are equivalent to D.
- A complete linear system can be given the structure of a projective space, by identifying $E \in |D|$ with the function g for which $D + \operatorname{div} g = E$ (well-defined up to a scalar).
- The rank $r(|D|)$ is the dimension of this projective space.
- Alternatively, $r(|D|) = \max\{ k \mid \forall E \in \operatorname{Div}_+^k C : |D - E| \neq \emptyset\}$.
- Gonality = minimal d for which C has a complete linear system $|D|$ of degree d and rank one.

Linear systems on metric graphs

- Let Γ be a metric graph.
- A divisor on Γ? An element $D = \sum_{P\in\Gamma} n_P \cdot P$ of the free abelian group generated by the points of Γ.
- The degree of a divisor is $\sum n_P$. It is called effective if all $n_P \geq 0$.
- Rational functions on Γ? Continuous map $g : \Gamma \to \mathbb{R}$ such that the restriction of g to an edge is piecewise-linear with only finitely many pieces and integer slopes.
- Divisor associated to a rational function $g : \Gamma \to \mathbb{R}$?

$$\mathrm{div}(g) = \sum_{P\in\Gamma} \mathrm{ord}_P(g) \cdot P,$$

  where $\mathrm{ord}_P(g) \in \mathbb{Z}$ is the sum of the incoming slopes of g at P. Note that it has degree 0.
- Two divisors D and D' are called equivalent if $D' - D = \mathrm{div}(g)$ for some rational function g on Γ.

Linear systems on metric graphs

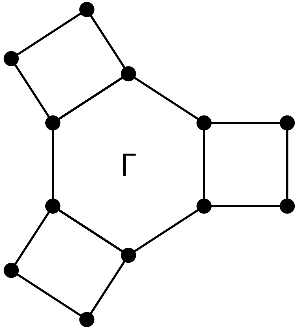
- Complete linear system $|D|$? The set of all effective divisors that are equivalent to D.
- Rank of $|D|$?

$$r(|D|) = \max\{\, k \mid \forall E \in \mathrm{Div}_+^k \Gamma : |D - E| \neq \emptyset\}.$$

- The gonality of Γ is the minimal d for which Γ carries a positive rank complete linear system of divisors of degree d.
- Notation: $\gamma(\Gamma)$.

Linear systems on graphs

- Example : $\gamma(\Gamma) = 3$



Specializing linear systems from curves to graphs

- Let $\mathfrak{X}$ be a flat and proper scheme over $\mathbb{C}[[t]]$ such that
  - $X = \mathfrak{X} \otimes \mathbb{C}((t))$ is a smooth curve,
  - $X_{\mathbb{C}} = \mathfrak{X} \otimes \mathbb{C}$ decomposes into a union of smooth curves that intersect each other transversally.
- Then $\mathfrak{X}$ is called a strongly semi-stable arithmetic surface.
- To $\mathfrak{X}$, one associates a metric graph, by identifying each component of $X_{\mathbb{C}}$ with a vertex and each intersection point with an edge, and by taking all edge lengts equal to one.
- Notation: $\Gamma(\mathfrak{X})$.

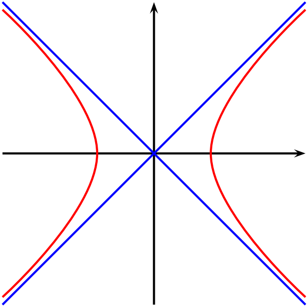
Theorem (Baker, 2008)

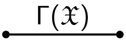
There is a natural degree-preserving way of specializing a divisor D on X to a divisor $\rho(D)$ on $\Gamma(\mathfrak{X})$, such that $r(|\rho(D)|) \geq r(|D|)$.

- Corollary: $\gamma(\Gamma(\mathfrak{X})) \leq \gamma(X)$.

Specializing linear systems from curves to graphs

- Example: $\mathfrak{X} : x^2 - y^2 + tz^2$ over $\mathbb{C}[[t]]$





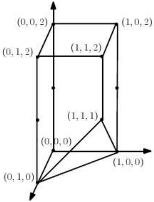
- $X : x^2 - y^2 + tz^2$ over $\mathbb{C}((t))$ (hyperbola)
- $X_{\mathbb{C}} : x^2 - y^2 = (x+y)(x-y)$ over $\mathbb{C}$ (two lines)

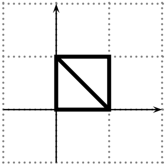
Toric degenerations

- Consider

$$f \in \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{ij}(t)x^i y^j \quad \in \mathbb{C}[t][x^{\pm 1}, y^{\pm 1}]$$

  and suppose that it is 'sufficiently generic'.
- Let $\widetilde{\Delta}$ be its 3D Newton polytope ($f \in \mathbb{C}[t, x^{\pm 1}, y^{\pm 1}]$).
- Then the lower facets of $\widetilde{\Delta}$ induce a subdivision $\{\Delta_i\}_i$ of Δ.
- Example: $f = (1+t^2) + (1+t^2)x + (1+t^2)y + (t+t^2)xy$.





Toric degenerations

- Now $f \in \mathbb{C}[t, x^{\pm 1}, y^{\pm 1}]$ corresponds to a hyperplane section $\widetilde{H}$ of the toric threefold $\mathrm{Tor}(\widetilde{\Delta})$.
- There is a natural morphism $p : \mathrm{Tor}(\widetilde{\Delta}) \to \mathbb{P}^1_{\mathbb{C}}$ such that for all $t_0 \in \mathbb{A}^1_{\mathbb{C}}$ one has
  - if $t_0 \neq 0$ then $p^{-1}(t_0) \cong \mathrm{Tor}(\Delta)$,
  - $p^{-1}(0) \cong \bigcup_i \mathrm{Tor}(\Delta_i)$.
- When restricted to $\left(\mathrm{Tor}(\widetilde{\Delta}) \setminus p^{-1}\{\infty\}\right) \cap \widetilde{H}$ this yields a strongly semi-stable arithmetic surface $\mathfrak{X}$ over $\mathbb{C}[[t]]$.
  - X is a hyperplane section of $\mathrm{Tor}(\Delta)$ over $\mathbb{C}((t))$.
  - $X_{\mathbb{C}}$ decomposes into a union of hyperplane sections of the $\mathrm{Tor}(\Delta_i)$ over $\mathbb{C}$.
  - Two such components will intersect each other transversally in $\#(\Delta_i \cap \Delta_j \cap \mathbb{Z}^2) - 1$ distinct points.

Toric degenerations

- Thus: $\Gamma(\mathfrak{X})$ is fully determined by the combinatorics of the subdivision.
- Namely: each Δ_i corresponds to a vertex v_i, and there are $\#(\Delta_i \cap \Delta_j \cap \mathbb{Z}^2) - 1$ edges between v_i and v_j.
- Notation: $\Gamma(\{\Delta_i\}_i)$.
- By Baker's theorem: $\gamma(\Gamma(\{\Delta_i\}_i)) \leq \gamma(X)$.
- A semi-continuity argument and $\mathbb{C}\{\{t\}\} \cong \mathbb{C}$ implies:

Theorem

For every regular subdivision $\{\Delta_i\}_i$ of a two-dimensional lattice polygon Δ, one has $\gamma(\Gamma(\{\Delta_i\}_i)) \leq \gamma(\Delta)$.

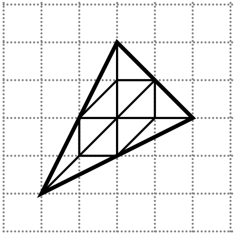
Toric degenerations

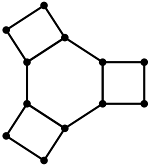
- We expect that it is always possible to obtain equality:

Conjecture

There always exists a regular subdivision $\{\Delta_i\}_i$ *such that* $\gamma(\Gamma(\{\Delta_i\}_i)) = \gamma(\Delta)$.

- Example: our Counterexample 2.





$\gamma(\Delta) = \gamma(\Gamma(\{\Delta_i\}_i)) = 3.$

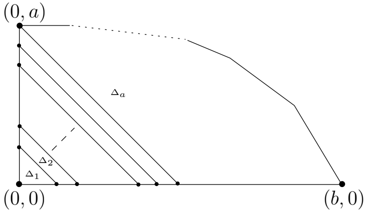
A purely combinatorial conjecture

- A proof of the following combinatorial statement would solve it all:

Conjecture

There always exists a regular subdivision $\{\Delta_i\}_i$ *such that* $\gamma(\Gamma(\{\Delta_i\}_i)) = \mathrm{lw}(\Delta^{(1)}) + 2$, *except if* $\Delta \equiv \mathrm{Conv}\{(2,0),(0,2),(-2,-2)\}$.

- Example of a lattice polygon Δ for which we can prove the above conjecture:



- Thanks for listening!