When Strings Collide: The Gravitational Interaction of Conical Strings

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Abstract

Up to now calculations of the interaction of cosmic strings have neglected gravity. We consider the purely gravitational interactions that occur at large distances, using the conical line singularity for the gravitational field of a string. We construct spaces with multiple intersecting conical strings, that are exactly consistent with General Relativity, and which can be covered in a single Minkowski coordinate patch, using a Regge calculus type construction. We show that after two such strings pass through each other they remain connected by another string, and we derive the branching rules which govern the junction of three strings. These rules apply to conical type strings in any smoothly curved background, whether they are straight or curved, moving or stationary, and they show that, at the junction, the three strings must be as coplanar as is possible in such a space. For these results to be matched onto the short range results of Field Theory calculations, it is suggested that gravitational radiation must be introduced. This would mean that gravitation is not negligible in these interactions.

1 INTRODUCTION

Some interesting recent calculations have confirmed the hypothesis that colliding (or intersecting) cosmic strings almost always break and reconnect, except at very high relative velocities. These calculations were done for both global strings [23,25] and local strings [20,21]. For strings of different winding numbers, however, there remains a connecting string of reduced winding number [14]. In all cases the initial strings were straight. These cosmic strings are believed, from Field Theory calculations, to form following the decay of the false vacuum in the early universe [13,27,28,22], and the tendency for strings to reconnect after colliding is needed to prevent the universe becoming string dominated [13,27,22]. However these reconnection calculations used a Minkowski space background, which means that the curvature of space and the gravitational interactions were ignored. Though a typical value for the deficit angle of a string at rest is only $\sim 10^{-5}$ radians, the apparent value for a rapidly moving string may be anything up to π , and clearly two colliding strings must have a relative velocity. Since the line density of strings is so large and the deficit angle is constant out to all distances (in the case of straight strings), this is a potentially serious omission, from the point of view of General Relativity. Indeed, string reconnections result in strings that are sharply bent, and it has been shown by Unruh et al. [24], and Clarke, Ellis and Vickers [2], that the gravitational field is not negligible in such cases. It is therefore of interest to investigate the gravitational interactions of strings.

This paper performs a calculation complementary to those of Shellard, Matzner, Laguna, Moriarty et al. (SMLM) [23,20,14,21] in that it concentrates on the gravitational fields of the strings, and

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works in the "intermediate" region, where the space is locally flat. For two colliding strings there is an interaction region where the string fields may be quite complicated, and which may be somewhat larger than the effective diameter of isolated strings. Nevertheless, by the time strings are expected to be colliding, this scale is extremely small compared with any astronomical distances (cf. the initial separations used by SMLM) and totally negligible compared with the curvature of space, or the curvature of long strings (i.e. open strings or extremely large loops), which is of the order of the horizon size. Thus we can easily find an intermediate scale which is very small compared with the curvature of the string, and also very large compared with the interaction distance. As was noted by Matzner [20] the remote parts of the two colliding strings cannot initially know a reconnection has occurred, so they keep moving as if it hasn't, at least until the future null cone of the reconnection passes.

The pure deficit angle or conical line singularity is normally used in General Relativity to represent the gravitational field of an "isolated" straight cosmic string. (The assumption of "isolation" is strictly only valid at early times when the particle horizons of forming strings are very small.) The effective equation of state of the strings, with tension equal to line density, requires this type of metric [26,8,10]. This was confirmed by Garfinkle [6] and more recently by Gregory [9] and Laguna-Castillo and Matzner [15,16], who showed that, under reasonable assumptions, the Einstein-Yang-Mills equations for the gravity plus the scalar and gauge fields that are responsible for local strings, produce an asymptotically conical metric at large radii. (The asymptotic metric of global strings is not always well behaved, but neither is their flat space energy density.) Other work, primarily intended to show that the external gravitational field does not uniquely determine the string's interior matter tensor, and that the deficit angle is not simply related to the line density, also support the asymptotically conical description [5,7,4]. Even for gently curved strings, it has been shown that the conical approximation is good on scales much smaller than the radius of curvature [4].

The calculation below gives an exact description of the resulting gravitational field in the intermediate distance range defined above, following a collision of two conical strings, regardless of the details of the string field interactions locally. It will be shown that this description remains valid on the large scale, but it obviously cannot make predictions about the actual results of string collisions, as the small scale string field interactions are not considered. Neverthless, the small and intermediate scales must join up, so the relationship between these results and the small scale interactions derived from Field Theory alone is subsequently discussed.

2 COLLIDING STRINGS AND GRAVITY

2.1 One String Spaces

A space-time with one straight string is commonly "contructed" by drawing a straight line with two planes radiating from it at a small angle δ to each other, "extracting" the semi-infinite wedge shaped region of space between them, and identifying points on the two planes. In Minkowski space, M^4 , the planes extend to timelike 3-planes, and since the intrinsic metric and the extrinsic curvature on the planes are the same, the Darmois matching conditions are satisfied [3,11,1,17], and there is no discontinuity across the identification 3-surface, the curvature of the space being concentrated into the conical singularity on the vertex of the wedge, where the identification surfaces are discontinuous. It is just as easy to make the string move — either by taking out a wedge bounded by moving planes, or else by boosting to a new frame, extracting a stationary wedge of space, and boosting back again. The identification surfaces match smoothly, as before, so a string moving through vacuum leaves no gravitational wake. Only motion perpendicular to a string's length can be detected, as a string is

invariant under a boost along its length. The value of the deficit angle predicted from Field Theory is about 10^{-5} radians, but the apparent deficit angle of a rapidly moving string may be anything up to π (cf. Ref. 28).

The metric for such strings is usually written in cylindrical polar coordinates, and of course has no discontinuous surfaces, but if this space in embedded in M^4 the wedge structure can be recovered. (Obviously there is no preferential direction for drawing the wedge, so the choice is a matter of convenience.) Though the string space is no longer globally invariant under a boost, the original M^4 invariances are preserved locally in any region of space not enclosing the string. Parallel transport of a tetrad of orthogonal vectors round a loop circling a string, whether moving or stationary, results only in a spatial rotation of the tetrad. In a frame in which the string is stationary, the temporal vector is completely unaffected, so we can represent the effect of transport around the string by a 3-d rotation vector parallel to the string. This representation, as well as the view of the string space as M^4 with a wedge extracted, is used extensively below. These conical line singularities were first considered by Marder in 1959 [19], long before the advent of cosmic strings.

2.2 Two String Spaces

Any spacetime which is everywhere locally M^4 , except on a finite number of timelike 2-surfaces, we call almost everywhere Minkowskian (AEM), and we say a string is straight if it appears straight to all nearby observers looking along its length.

Constructing a space with two straight strings, starting from M^4 , merely involves extracting two wedges (stationary or moving) in one go (see Fig. 1a). (If one wedge is taken out, and the identification is made before trying to take out the other, it is not so obvious whether this can be done. Doing both together converts the sliced M^4 space directly to a new AEM space.) Provided the two wedges do not intersect each other, they can have any orientation. To an observer who sees one string behind the other, the front string looks straight, but the rear one will seem to be bent, since each observer perceives the space in terms of Euclidean coordinates constructed radially away from himself (Fig. 1b). This highlights the fact that the relative orientations of distant lines and vectors are path dependent and therefore ambiguous in AEM spaces.

If the two strings are now given an initial velocity, so that they eventually intersect, one may ask what happens at and after the collision. Presumably, the two strings bend each other by their respective deficit angles, but it turns out that is not all. We give a stationary model below, and leave the case of relative motion till later.

2.3 Two Strings After Colliding

This section determines the geometry of a two string space after the strings have collided, using parallel transport round equivalent loops. The same answer may be obtained by a more simple minded argument, that considers the geometry of intersecting wedges. Although the methods are quite independent, the wedge argument is presented first in order to set up coordinates on the space, and also because the interpretation and visualisation of the results is much harder using the parallel transport method alone.

We again start from M^4 , and for simplicity we take out two stationary wedges of space at right angles, so that the extracted wedges intersect, as shown in Fig. 2a, and the deficit angles are δ_1 and δ_2 . (In other words, the wedges of Fig. 1a have been moved into each other.) (To the observer

who sees one string behind the other, the front string appears bent and the rear one appears straight (Fig. 2b), so if he were watching them pass through each other, there would be no change of apparent orientations.)

Once the identifications have been made, the strings obviously have slight bends, or kinks, as shown in Fig. 3, and there is a four-fold identification along a line joining the two bending points. (This line, as well as the two parts of each string, are essentially the same as the "struts" of Regge calculus, which is used for the numerical approximation of curved space-times, though here the curvature of the space really is discontinuous.) If the original Minkowski coordinates are retained, then there are coordinate discontinuities along two semi-infinite surfaces, as shown, which intersect only along the connecting strut.

The total angle round this connecting strut can be calculated using spherical trigonometry. A perpendicular circle round the strut consists of four sections, separated by the four surfaces of discontinuity. Refer back to Fig. 2a, showing the wedges before the identifications were made. In each section, a quarter loop may be constructed as a circular arc in the spacelike plane orthogonal to both wedge surfaces, and calculation of the angle of this arc yields

$$\frac{\pi}{2} - \frac{\alpha}{4} \quad , \tag{1}$$

where

$$\sin\left(\frac{\alpha}{4}\right) = \sin\left(\frac{\delta_1}{2}\right)\sin\left(\frac{\delta_2}{2}\right) \quad , \tag{2}$$

with all four the same. The orientation of the strut is parallel to $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where

$$\cos\theta = \frac{-\sin(\delta_1/2)\cos(\delta_2/2)}{\sqrt{1-\sin^2(\delta_1/2)\sin^2(\delta_2/2)}} , \qquad (3)$$

$$\sin \theta = \frac{\cos(\delta_1/2)}{\sqrt{1 - \sin^2(\delta_1/2)\sin^2(\delta_2/2)}}$$
, (4)

$$\phi = \frac{\pi}{2} - \frac{\delta_2}{2} \quad . \tag{5}$$

with appropriate sign changes for each quadrant. The derivation is given in Appendix B.

This same result is now derived from arguments of parallel transport. Let there be an AEM space that only contains two straight perpendicular non-intersecting strings. Parallel transport around any loop that does not circle a string results in no change. We choose a loop that lies between the strings. The loop may be deformed into any shape without affecting the result of parallel transport, provided it is not made to cross any strings. So, as the two strings are moved together until they intersect and pass through ech other, the loop must be deformed into the shape of a tennis ball seam to avoid being crossed. (This loop is drawn in Fig. 2a for the foregoing choice of coordinates embedded in M^4 , and it still lies entirely in a single connected M^4 coordinate patch.) The loop can again be continuously deformed (to the one shown in Fig. 3) without crossing any strings, so that it touches itself (e.g. at the two surfaces of coordinate discontinuity) at a total of 4 points. Thus, there can still be no rotation of the vector triad around the loop.

Now this loop is equivalent to four closed loops around each side of the two strings, plus one around the connecting strut, so the sum of the rotations due to transport round the separate loops must add to zero. Transporting a spacelike vector triad round the upper part of string 2 results in a rotation by $-\delta_2$ in the y - z plane, which we represent in the figure by a vector parallel to string 2, using the right hand rule. So when this loop is split off, the remaining loop must have a rotation by

 δ_2 , as shown in Fig. 4a. After all four loops have been split off, the fifth loop has a sequence of four rotations (Fig. 4b), and the product of the four rotation matrices in the appropriate order gives the net effect of transport round this fifth loop. Clearly the sense of transport is fixed in relation to the senses for the other four loops, but there is no obvious starting point.

In order to calculate this matrix product, it is now necessary to choose specific coordinates. We take the Minkowski-like coordinates of Fig. 2a, mapped onto the two string space. In these coordinates, the strings remain in the x and z directions even after the collision, so, starting parallel transport from point A on the fifth loop (Fig. 4b), the net rotation matrix is the product of four perpendicular rotation matrices, of magnitudes δ_2 , δ_1 , $-\delta_2$, and $-\delta_1$ (Fig. 4b). It is found (with the help of REDUCE) that the product matrix is a rotation by α , about an axis pointing towards ($\sin \theta \cos \phi$, $\sin \theta \sin \phi$, $\cos \theta$), as calculated in eqs. (2)-(5) above. This direction is exactly the orientation of the connecting strut in the original coordinates, as viewed from the quadrant that loop 5 started and ended in (both x and z positive). This result remains true whichever quadrant the starting point is put in. Thus, despite the ambiguity of the orientation of the strut (that a multiple string space must have), a consistent answer is obtained. The REDUCE code is given in Appendix A.

It has been shown that parallel transport around the strut is associated with a pure spatial rotation, and by the right hand rule, the strut is actually a deficit angle string of strength α . The rotation α is approximately $\delta_1 \delta_2$, and consequently in this case the string is quite weak. Nevertheless it cannot be ignored since we are seeking an exact "Solution" of Einstein's equations.

2.4 The Junction of Three Strings

From the foregoing, it has become clear that an AEM space can have three deficit angle strings joining at a point. The construction in Fig. 5 for stationary strings, shows that there must be zero net rotation of a vector triad around the loop drawn, as it actually encloses no strings and can be contained in a single M^4 coordinate patch. As above, the two end loops around strings 1 and 2 are broken off, and the appropriate rotation vectors are added in. The third loop contains the negative of rotations 1 and 2, which must necessarily generate the rotation due to string 3. In other words, the three rotation matrices, due to δ_1 , δ_2 , and δ_3 , must multiply to give the identity. The product of three matrices for rotations about three arbitrary axes (with orientations θ_1 , ϕ_1 , etc.) is huge, and is impractical to solve directly, even with REDUCE. An approximation to first order in the δ 's gives (see Appendix C)

$$\delta_1 \cos \theta_1 + \delta_2 \cos \theta_2 + \delta_3 \cos \theta_3 = 0 \quad , \tag{6}$$

$$\delta_1 \sin \theta_1 \cos \phi_1 + \delta_2 \sin \theta_2 \cos \phi_2 + \delta_3 \sin \theta_3 \cos \phi_3 = 0 \quad , \tag{7}$$

$$\delta_1 \sin \theta_1 \sin \phi_1 + \delta_2 \sin \theta_2 \sin \phi_2 + \delta_3 \sin \theta_3 \sin \phi_3 = 0 \quad . \tag{8}$$

Thus the x, y, and z components of the rotation vectors must add to zero, and the vectors must obviously be coplanar — to first order. Given only δ_1 , δ_2 , and χ as in Fig. 5, then by the above equations,

$$\delta_3 = \sqrt{\delta_1^2 + \delta_2^2 + 2\delta_1\delta_2\cos\chi} \quad , \tag{9}$$

$$\sin\sigma = \frac{\delta_1 \sin \chi}{\delta_3} , \qquad (10)$$

$$\sin\omega = \frac{\delta_2 \sin\chi}{\delta_3} . \tag{11}$$

The above result was derived without reference to any coordinate system, but it is only approximate, and the ambiguity of the strings' orientations is hidden. (The result here is first order in the δ 's,

whereas the previous result for α is second order.) It is possible to derive exact equations relating the strengths and orientations of the three strings (Appendix D), again using spherical trigonometry. Unfortunately they are quite messy, but they do reduce to the above results to first order. They also give an estimate of the degree of non-coplanarity of the junction, as measured in Minkowski coordinates. The exact expression in the appendix gives to second order:

$$e = \frac{\delta_1 \delta_2 \sin \chi}{\delta_3} \quad , \tag{12}$$

where e is the angle of the third string above or below the plane of the other two, depending which side it is viewed from. Although the junction appears concave to any observer, it is necessarily symmetric and looks the same from both sides, so that it is as coplanar as is possible in such a space, where there are less than 4π radians round the junction point.

Applying this to the two 3-string junctions that appear following the collision of two straight strings (Fig. 6), gives the expected values

$$\omega_1 = \frac{\pi}{2} + \frac{\delta_2}{2} = \sigma_1 \quad , \tag{13}$$

$$\alpha = 2\delta_1 \sin\left(\frac{\delta_2}{2}\right) = \delta_1 \delta_2 \quad , \tag{14}$$

$$\omega_2 = \frac{\pi}{2} + \frac{\delta_1}{2} = \sigma_2 \quad , \tag{15}$$

$$\alpha = 2\delta_2 \sin\left(\frac{\delta_1}{2}\right) = \delta_1 \delta_2 \quad . \tag{16}$$

The fact that the rotation vectors of the strings must be effectively coplanar and sum to zero at a three string junction implies that the tensions in the strings will also be exactly balanced, and there will be no resulting acceleration of the junction point. Thus, in the case of the two strings passing through each other, the weak connecting string is just enough to prevent acceleration now that the two original strings have bent each other.

2.5 Moving Strings in Curved Spaces

In general we expect strings to be moving. However, we can always boost to a frame in which one string is stationary, and then boost along that string so that a given junction point is also stationary, so consequently all the strings at the junction are stationary in that frame. Thus the above formulas apply to the junction of any three conical type strings in a space which is everywhere locally flat.

Using the above methods, the result of the collision of moving strings can be calculated, remembering that the apparent deficit angle of a moving string may be anything up to π . Although the actual values of α , and the amount of bending of each string, may be considerably different from the stationary approximation used here, there must still be a connecting string, whose strength is at least α , and can be of the same order as δ_1 or δ_2 . The degree of "non-coplanarity" may also be increased or decreased greatly for a moving junction, depending on the direction of motion.

For slowly moving strings, the three-string formulas generalise as follows. We represent each string's world-sheet by a pair of orthogonal vectors, one being the timelike four-velocity of the junction point, u^{α} , and the other being the orthogonal spacelike vector lying in the string's world-sheet, n^{α} , with magnitude equal to the appropriate δ value. Equations (6)-(8) may then be written,

$$n_1^{\alpha} + n_2^{\alpha} + n_3^{\alpha} = 0 \tag{17}$$

However, these formulas only apply for small δ 's, and since the deficit angle of a rapidly moving string may be anything up to π , the exact equations of Appendix D would have to be used, so it is probably best to work in the stationary frame.

Moreover we claim that these relations for the junction of three strings apply even in curved spacetimes, where matter is present. It was noted in the introduction that for expected cosmological situations, there exists a scale on which the spacetime is virtually flat and the strings are virtually straight, that is still much larger than the string field interaction scale. On this scale a locally almost everywhere Minkowskian tangent space can be set up about the junction (provided the junction point is not also a curvature singularity in the sense that the Kretschmann scalar diverges as it is approached). An appropriate boost then makes the junction stationary, so the above equations apply in that region also.

2.6 Reconnections

According to Field Theory, the collision of two strings is most likely to result in the strings reconnecting to form two sharp elbows, rather than passing through each other, as shown in Fig. 7a,b. In a strictly AEM space, this is not possible. Starting from two straight strings in AEM space, it is apparent from the foregoing that after reconnection the two must remain linked by a third string (Fig. 7c), which has a strength of the same order as the original pair, as the angle at the elbow is of order 1 in general. Again the tensions balance and there is no acceleration of any of the strings.

Furthermore, there is a causality problem. In order to effect the reconnection, two of the component strings would have to undergo sudden velocity changes. This change can only propagate along the strings at light speed, but a curved or kinked string is not possible in an AEM space. Though it might be possible to arrange a reconnection in a strongly curved space, in general it seems much more likely that the strings would pass through each other, creating a connecting string. This conclusion may be modified if the local tangent space is not AEM, as is discussed below.

2.7 Extension to Robertson-Walker Spaces

Since the Robertson-Walker (RW) metrics are conformably flat, it is evident that the above constructions can be transferred directly to the case of models with uniform spatial curvature, but with straight strings becoming geodesic strings, provided (i) the spatial sections are open, and (ii) the strings are comoving (as non-comoving strings cause wakes in the matter), which obviously means there are no collisions. For k = 0 models no modifications are needed, while for k = -1 models, the wedges of space that are extracted must be bounded by totally geodesic comoving 3-surfaces, so that the intrinsic metric and extrinsic curvature are identical on both, and the Darmois matching conditions [3,11,1,17] are satisfied. For more realistic lumpy models that are approximately RW on some scale, there is no reason why the strings could not move through the matter also, and, provided the density in the matter wake is not divergent close to a string, the locally AEM assumption then holds, and the above results are qualitatively unchanged.

The closed k = +1 models are rather more tricky, as all geodesics either link each other or they intersect. This means that the deficit angle due to any string directly afrects all the others, so although the above construction works locally, it is not easy to make a rigorous global construction. However Fig. 8 shows how two comoving geodesic strings can be combined in a spatial 3-sphere. The closed geodesics are arranged so that each one passes through the centre of the other one and perpendicular to its plane. Fig. 8a shows the arrangement schematically, and 8b shows the embedding of a spatial 2section that has either string as an "equator". Geodesic surfaces radiating from either string necessarily intersect the other one at right angles, and so there is no problem of the strings bending each other when the "wedges" of space are extracted. The metric for this space is

$$ds^{2} = -dt^{2} + R^{2}(t) \{ d\chi^{2} + (1 - \epsilon_{1})^{2} \cos^{2} \chi \, d\psi^{2} + (1 - \epsilon_{2})^{2} \sin^{2} \chi \, d\phi^{2} \} \quad , \tag{18}$$

where $0 \le \psi \le 2\pi$, $0 \le \phi \le 2\pi$, $0 \le \chi \le \pi/2$. The axes $\chi = 0$ and $\chi = \pi/2$ are in fact the two strings, and, due to the closure of the spatial sections t = const., the surfaces $\chi = \text{const.}$ are "cylinders" about both axes, in the sense that they lie at constant distance from them. The deficit angles of the strings are $\delta_i = 2\pi\epsilon_i$.

3 DISCUSSION

We have considered the behaviour of the standard conical type cosmic strings in almost everywhere Minkowskian "solutions" of Einstein's equations and in open Robertson-Walker spaces, and derived results that must also apply in curved spaces that are locally AEM. It has been shown that two straight strings that pass through each other become bent, and must remain connected by another string, which has just the deficit angle required to prevent acceleration of the bending point due to string tension. The connecting string is weak if the collision is slow, but may be quite strong for high speed collisions. Colliding strings cannot reconnect in AEM spaces, and even in strongly curved spaces this seems unlikely. In any case, an extra connecting string must form. It has also been shown that the junction of three strings must be as coplanar as is possible in such spaces, and that the angles and tensions of the strings must satisfy a vector triangle rule, so that the junction point does not accelerate in this case either.

Thus the only possible result of the collision of two straight line conical singularities, for purely gravitational interactions, is that they pass through each other, and they remain connected by another string.

In this scenario, the problem of the universe becoming string dominated remains, since colliding strings cannot reconnect and accelerate away, so loops cannot form and decay and the number of strings cannot decrease.

Since the domain of validity of the two sets of results — those presented above for intermediate scales, and those of Field Theory for small scales — is quite separate, we do not necessarily have any contradiction. On the other hand, we do have to consider how the transition between the two might be achieved, and ultimately prove this is (or isn't) the case.

3.1 Relationship with Field Theory Calculations

A special case of the above result is the collision of two parallel strings. If the winding sense of the string fields round the two strings is opposite, then the two strings may annihilate, according to Field Theory. However, the gravitational field of a string is not altered by its winding sense, so the field of two parallel strings, provided they are well separated, is independent to lowest order, of whether they have the same or opposite winding senses. So in both cases the space has a total deficit angle initially equal (roughly) to the sum of the two separate values. For annihilation to take place, this global curvature must also be removed. (One string could have an excess angle, i.e. $0 \le \phi_1 < 2\pi + \delta$ and $0 \le \phi_2 \le 2\pi - \delta$, but this requires the "anti-string" to have a negative density and a negative tension.) Even in this simple case the two approaches do not seem to agree. In contrast, Moriarty,

Myers and Rebbi's result [21], that head-on collisions of two parallel strings with the same winding sense briefly form a doubly wound string which emits a singly wound pair at 90° to the original motion, presents no problem for GR.

It should be remembered that the results of this paper apply to the intermediate and large scales, that are much larger than the string diameter or interaction range of two strings. Consequently, these results, though exact and independent of the interacting string fields, do not constrain the local results of string collisions, for the very reason that they don't consider them. In other words, the actual result of a string collision is hidden within the future null cone of the collision event, but the net gravitational effect of the matter plus expanding radiation inside the null cone must be the same as the two bent strings plus connecting string described above. The speed of light is relatively slow on astronomical scales, so it would take time for the information that the universe is not string dominated to diffuse out. Though it is tempting to relate the purely gravitational results to those of Laguna and Matzner [14], for strings of different winding number, the connecting strings that result in the two cases occur for quite different reasons. In the former there is no reduction in the amount of string (length times deficit angle) whereas in the latter the total amount of string (length times winding number) is reduced by the "peeling" effect of the collision and partial reconnection. Even when the strengths of the two strings are the same, there is still a connecting string in the gravitational calculation, but not in the Field Theory calculation. Nevertheless, it would be interesting to find out whether Laguna and Matzner's results also have fixed relationships between the strengths of the strings and the angles at which they join.

The most likely effect that might reconcile the two sets of results, ignored by both this approach and those of SMLM, is the emission of gravitational radiation, which may dissipate some of the curvature from the interaction region. It is obvious from the calculations of Shellard [23] and Matzner [20] that the string fields do sometimes oscillate briefly after a collision. In fact Khan [12] has shown that Higgs vortices (strings) cannot interpenetrate unless radiation is emitted. Furthermore, Marder [18] showed that outgoing cylindrical gravitational radiation, carrying positive energy, could decrease the effective deficit angle of the exterior space as it passed. Thus the exterior spacetime as calculated above, would be converted to some other external field as the gravitational radiation passed by, accompanied by radiation due to other fields. If the curvature due to the connecting string is entirely removed, this could be a strong source of gravitational radiation, particularly for high velocity collisions. The emission of this gravitational radiation must be a continuous process. Right after a collision the connecting string would be very short, so only relatively little radiation need be emitted to remove it, but as the reconnected strings move away and the null cone expands, the effective connecting string, as felt by the more distant regions, becomes ever longer, so the radiation needed to eliminate it grows linearly. The radiation would most likely come from the motion of the curved elbows of the reconnected strings.

Given the small scale results of Field Theory calculations — that colliding strings normally break and reconnect — then the resulting reconnected strings are initially bent through large angles in very small distances, and it has been shown by Unruh et al. [24,2] that the gravitational field is not negligible, and the tidal forces (Weyl Tensor) are then comparable to the internal forces of the string. Thus gravitation cannot be ignored in such interactions, nor treated in the weak field limit. Since Matzner's calculations [20] already needed a super-computer, an attempt to calculate such an interaction using the full Einstein-Yang-Mills equations would be a major computing project.

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A Program ALFROT-LSP

REDUCE program to calculate the product, a0, of four rotations — by +delta2 about the z axis, +delta1 about x, - delta2 about z, and -delta1 about x — by multiplying rotation matrices. Also calculates the matrix, ah, for a rotation by -alpha about an axis pointing towards (theta, phi), where alpha, theta and phi are all defined below. The two rotation matrices are exactly the same.

```
matrix aa, rxp, rxm, rzp, rzm, ah;
matrix r1p, r1m, r2p, r2m, a0;
c3**2 := 1-s3**2;
                                                         % Identities.
c4**2 := 1-s4**2;
sa4 := s3*a4;
                       \% Sin(alpha/4) = sin(delta1/2)*sin(delta2/2).
ca4 := sqrt(1-sa4**2);
ca4**2 := 1-sa4**2;
ca2 := 1-2*sa4**2;
                        % Cos(alpha/2) in terms of sin(alpha/4) etc.
sa2 := 2*sa4*ca4;
ca := 1-2*sa2**2;
                          % Cos(alpha) in terms of sin(alpha/2) etc.
sa := 2*sa2*ca2;
aa := mat((ca, 0, -sa)),
                                   % Rotation by -alpha about y axis.
          (0,1,0),
          (aa,0,ca));
cp := c4;
                               % Phi = delta2/2. (Phi = 0 on y axis!)
sp := s4;
                                         % Theta, in terms of delta1.
st := s3*c4/sqrt(1-s3**2*s4**2);
ct := c3/sqrt(1-s3**2*s4**2);
                                     % delta2. (Theta = 0 on y axis!)
rxp := mat((1,0,0),
                                    % Rotation by theta about x axis,
           (0,ct,-st),
           (0,st,ct));
rxm := mat((1,0,0),
                                             % and by -theta about x.
           (0,ct,st),
           (0,-st,ct));
rzp := mat((cp,-sp,0),
                                      % Rotation by phi about z axis.
           (sp,cp,0),
           (0,0,1));
rzm := mat((cp,sp,0),
                                                % and by -phi about z.
           (-sp,cp,0),
           (0,0,1));
```

```
ah := rzp*rxp*aa*rxm*rzm;
                                   % Rotation by -alpha about an axis
                                     % at (theta, phi) to the y axis.
c1 := 1-2*s3**2;
                        % Cos(delta1) in terms of sin(delta1/2) etc.
s1 := 2*s3*c3;
c2 := 1-2*s4**2;
                        % Cos(delta2) in terms of sin(delta2/2) etc.
s2 := 2*s4*c4;
r1p := mat((1,0,0)),
                                  % Rotation by +delta1 about x axis.
           (0,c1,-s1),
           (0,s1,c1));
r1m := mat((1,0,0),
                                  % Rotation by -delta1 about x axis.
           (0,c1,s1),
           (0,-s1,c1));
r2p := mat((c2, -s2, 0)),
                                  % Rotation by +delta2 about z axis.
           (s2, c2, 0),
           (0,0,1));
r2m := mat((c2,s2,0),
                                  % Rotation by -delta2 about z axis.
           (-s2, c2, 0),
           (0,0,1));
a0 := r1p*r2p*r1m*r2m;
                              % Product of the above four rotations,
                                           % in order round the loop.
out aaout;
                                                   % Save the results.
write aa := aa;
write ah := ah;
write a0 := a0;
                              % Compare a0 and ah. They're the same.
write ah-a0 := ah-a0;
shut aaout;
```

end;

B Exact Calculation of α

Here the angle between the wedge planes shown in Fig. 2a is calculated. Fig. 9a shows how these planes intersect a sphere centered on some point where the two planes meet. Planes parallel to the x - y and y - z directions are also shown intersecting the sphere. We wish to find λ . In Fig. 9b the quadrilateral where these 4 great circles intersect is divided into spherical triangles by another great circle. From the upper right triangle one gets

$$\sin a = \frac{\sin(\delta_2/2)}{\sin c} = \cos\left(\frac{\pi}{2} - a\right) \tag{19}$$

$$\sin b = \frac{\sin(\delta_1/2)}{\sin c} = \cos\left(\frac{\pi}{2} - b\right) \tag{20}$$

$$\cos c = \cos\left(\frac{\delta_1}{2}\right)\cos\left(\frac{\delta_2}{2}\right) \tag{21}$$

$$\sin^2 c = \sin^2 \left(\frac{\delta_1}{2}\right) + \sin^2 \left(\frac{\delta_2}{2}\right) - \sin^2 \left(\frac{\delta_1}{2}\right) \sin^2 \left(\frac{\delta_2}{2}\right) \quad , \tag{22}$$

where (22) comes from (21), while from the lower left triangle

$$\cos(\pi - \lambda) = -\cos\left(\frac{\pi}{2} - a\right)\cos\left(\frac{\pi}{2} - b\right) + \sin\left(\frac{\pi}{2} - a\right)\sin\left(\frac{\pi}{2} - b\right) \quad , \tag{23}$$

which then leads to

$$\cos \lambda = \sin\left(\frac{\delta_1}{2}\right) \sin\left(\frac{\delta_2}{2}\right) = \sin\left(\frac{\pi}{2} - \lambda\right) \quad . \tag{24}$$

The angle $\pi/2 - \lambda$ is the deficit for one quarter of the loop, i.e. $\alpha/4$, and the same is derived for each quadrant, so eqs. (1) & (2) follow.

C Program ROT3A-LSP

REDUCE program to find the product of three small rotations about arbitrary axes by calculating and then multiplying the rotation matrices of each one. The three rotations are d1, d2, and d3, but the result is only found to first order in the d's.

```
matrix t1p, t1m, p1p, p1a, d1p, r1,
t2p, t2m, p2p, p2m, d2p, r2,
t3p, t3m, p3p, p3m, d3p, r3,
rt, id3;
ct1**2 := 1-st1**2;
                                                % Trig relationships.
cp1**2 := 1-sp1**2;
                                               % ct1 is cos(theta 1).
ct2**2 := 1-st2**2;
cp2**2 := 1-sp2**2;
                                            % sp2 is sin(phi 2), etc.
ct3**2 := 1-st3**2;
cp3**2 := 1-sp3**2;
dl*d2*d3 := 0;
                                        % Discard higher order terms.
d1*d2 := 0;
d1*d3 := 0;
d2*d3 := 0;
t1p := mat((ct1,0,st1),
                              % Rotation by theta = t1 about y axis.
           (0,1,0),
           (-st1,0,ct1));
t1m := mat((ct1,0,-st1),
                                                   % Rotation by -t1.
           (0,1,0),
           (st1,0,ct1));
p1p := mat((cp1,-sp1,0),
                                 % Rotation by phi = p1 about z axis.
           (sp1,cp1,0),
           (0,0,1));
p1m := mat((cp1,sp1,0),
                                                   % Rotation by -p1.
           (-sp1,cp1,0),
           (0,0,1));
                             % Rotation by d1 (delta1) about z axis.
d1p := mat((1,-d1,0),
           (d1,1,0),
                                             % to lowest order in d1.
           (0,0,1));
r1 := p1p*t1p*d1p*t1m*p1m;
                                          % Matrix for rotation by d1
```

% about theta = t1, phi = p1. t2p := mat((ct2,0,st2), % Same for d2.... (0,1,0),(-st2,0,ct2)); t2m := mat((ct2,0,-st2), (0,1,0),(st2,0,ct2)); p2p := mat((cp2,-sp2,0), (sp2,cp2,0), (0,0,1));p2m := mat((cp2,sp2,0), (-sp2,cp2,0), (0,0,1));d2p := mat((1,-d2,0), (d2,1,0), (0,0,1));r2 := p2p*t2p*d2p*t2m*p2m; t3p := mat((ct3,0,at3), % Same for d3.... (0,1,0),(-st3,0,ct3)); t3m := mat((ct3,0,-st3), (0,1,0),(st3,0,ct3)); p3p := mat((cp3,-sp3,0), (sp3,cp3,0), (0,0,1));p3m := mat((cp3,sp3,0), (-sp3,cp3,0), (0,0,1));d3p := mat((1,-d3,0), (d3,1,0), (0,0,1));r3 := p3p*t3p*d3p*t3m*p3m; id3 := mat((1,0,0), % Identity matrix. (0,1,0),(0,0,1));rt := r3*r2*r1-id3; % When product of the three rotations % is zero, the matrix product produces % the identity, i.e. matrix rt is zero. out rot3aout; write r1 := r1; % These are the expressions that must write r2 := r2;% be set to zero to obtain zero not write r3 := r3;% rotation. Write them to file.

write rt := rt; shut rot3aout; end;

D Exact Results for 3-String Junctions

An exact relationship between the strengths and the orientations of three stationary strings that meet at a point, as in Fig. 5, is derived here.

Suppose we are given two strings, of strengths δ_1 and δ_2 , and the angle between them, χ . The two strings define a plane, and the deficit wedges may be oriented symmetrically about this plane, so that they coincide along a pair of lines, as in the front part of Fig. 2b. Identification across the wedges brings these two lines together, forming the third string. Fig. 10a shows the intesection of these wedges with a sphere centred on the junction point. The angle between the two upper faces of the wedges is slightly less than π , and the same between the two lower faces, so there is a deficit angle around this string, provided

$$\chi < \pi$$
 (25)

If $\chi > \pi$ there is an excess angle, which we disallow, and $\chi = 0$ merely corresponds to one wedge inside the other, which is effectively a single straight string. By the same argument it follows that $\sigma < \pi$ and $\omega < \pi$.

Spherical trigonometry for triangle ABX gives

$$\cos(\pi - \lambda) = -\cos\left(\frac{\delta_1}{2}\right)\cos\left(\frac{\delta_2}{2}\right) + \sin\left(\frac{\delta_1}{2}\right)\sin\left(\frac{\delta_2}{2}\right)\cos(\sigma + \omega) \quad , \tag{26}$$

and since

$$\sigma + \omega + \chi = 2\pi \quad , \tag{27}$$

and δ_3 is defined by

$$\frac{\delta_3}{2} = \pi - \lambda \quad , \tag{28}$$

this becomes

$$\cos\left(\frac{\delta_3}{2}\right) = \sin\left(\frac{\delta_1}{2}\right)\sin\left(\frac{\delta_2}{2}\right)\cos(\chi) - \cos\left(\frac{\delta_1}{2}\right)\cos\left(\frac{\delta_2}{2}\right)$$
(29)

Equation (9) is easily recovered by expanding to second order in the δ 's.

Now considering triangles AXC and BXC separately gives

$$\sin e = \frac{\sin \sigma \sin(\delta_1/2)}{\sin f_1} = \frac{\sin \omega \sin(\delta_2/2)}{\sin f_2}$$
, (30)

$$\cos f_1 = \sin\left(\frac{\delta_1}{2}\right)\cos\sigma \quad , \tag{31}$$

$$\cos f_2 = \sin\left(\frac{\delta_2}{2}\right)\cos\omega \quad , \tag{32}$$

$$\cos\left(\frac{\delta_1}{2}\right) = \sin f_1 \cos e \quad , \tag{33}$$

$$\cos\left(\frac{\delta_2}{2}\right) = \sin f_2 \cos e \quad . \tag{34}$$

Inserting (33) and (34) in (30) and substituting for ω in terms of χ and σ from (27) yields

$$\sin \sigma = \frac{\tan(\delta_2/2) \sin \chi}{\sqrt{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi}} \quad , \tag{35}$$

so that

$$\cos \sigma = \frac{-(\tan(\delta_1/2) + \tan(\delta_2/2)\cos\chi)}{\sqrt{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi}} \quad , \tag{36}$$

and similarly

$$\cos \omega = \frac{-(\tan(\delta_2/2) + \tan(\delta_1/2)\cos\chi)}{\sqrt{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi}} \quad .$$
(37)

Using these in (31) and (32) then leads to

$$\sin f_1 = \sqrt{\frac{\cos^2(\delta_1/2)(\tan(\delta_1/2) + \tan(\delta_2/2)\cos\chi)^2 + \tan^2(\delta_2/2)\sin^2\chi}{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi}} ,$$
(38)

and

$$\sin f_2 = \sqrt{\frac{\cos^2(\delta_2/2)(\tan(\delta_2/2) + \tan(\delta_1/2)\cos\chi)^2 + \tan^2(\delta_1/2)\sin^2\chi}{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi}} \quad .$$
(39)

Finally inserting these results in (33) and (34) gives

$$\sin e = \frac{\tan(\delta_1/2)\tan(\delta_2/2)\sin\chi}{\sqrt{\tan^2(\delta_1/2) + \tan^2(\delta_2/2) + 2\tan(\delta_1/2)\tan(\delta_2/2)\cos\chi + \tan^2(\delta_1/2)\tan^2(\delta_2/2)\sin^2\chi}}$$
(40)

which, to second order becomes

$$e \approx \frac{\delta_1 \delta_2 \sin \chi}{\delta_3}$$
 . (41)

The angle e is a measure of the deviation of the third string from the plane of the first two, and it is of order δ . By symmetry it is clear that the three string junction appears to be slightly concave by the same amount, whichever side it is viewed from.

E Figures



Fig. 1a. The wedge construction for a two string space. This diagram may also be thought of as an embedding of the 2-string space in Euclidean (i.e. Minkowski) space. String 1 is in the background and is parallel to the *x*-axis, string 2 is parallel to the *z*-axis, and the deficit angles are δ_1 and δ_2 respectively. The wedges of "missing space" are shown finite here but actually extend to infinity in three directions. The two faces of each wedge are identified as the same set of points, under the obvious isometry. In general the two strings do not need to be orthogonal or stationary as shown here.



Fig. 1b. A different embedding of the same space, showing it as an observer at X percieves it. Each observer tends to interpret what he sees in terms of Euclidean coordinates constructed away from himself, so Fig. 1a is more natural for an observer at O. This diagram may be obtained from 1a by making a cut through the y - z plane and rotating about string 2, so that the front wedge closes, and a new one opens behind the string. Thus the observer at X sees the front string as straight and the rear one as bent. (He can actually see the bending point twice.)



Fig. 2a. The wedge construction for two strings after they have passed through each other. The wedges of Fig. 1a have been moved into each other, and string 2 is now behind string 1. Here they are perpendicular and stationary, but the diagram is basically the same for moving strings at any angle. The two faces of each wedge identify into each other, so the lines of intersection of the wedges form a four-fold identification. Since this diagram also represents the embedding of the actual space in Minkowski space, the curve shown (with arrows) is a closed loop completely contained in a single Minkowski coordinate patch.



Fig. 2b. How an observer at X perceives the space of Fig. 2a. It is obtained from 2a in the same way as for Figs. 1a and 1b. In this case the front string appears bent and the rear one appears straight. Clearly, as the two strings pass through each other, it seems to the observer at X that string 2 stays straight and string 1 stays bent. In a multiple string space, the apparent orientation of a particular line cannot change unless a string passes across the line of sight.



Fig. 3. A schematic representation of the space containing two strings after they have passed through each other, which in equivalent to Fig. 2a after the identifications have been made. The strings are both bent and the wedges have become semi-infinite surfaces of coordinate discontinuity (in the original M^4 coordinates), which intersect along a finite "strut". The closed loop of Fig. 2a has been deformed into the one shown, which touches these surfaces at eight points but doesn't pass through them.



Fig. 4a. How the sub-loop round the upper part of string 2 is detached, and the apropriate rotations are added into both sides. Parallel transport round string 2 results in a spatial rotation through $-\delta_2$, which is represented in the diagram by a vector. The direction is determined by applying the right hand rule to the sense of transport round the sub-loop. The result of transport around the remaining loop is then the opposite rotation $+\delta_2$, about the same axis.



Fig. 4b. The rotation vector for the five loops after all four sub-loops round strings 1 and 2 have been detached. If parallel transport round the fifth loop is started and ended at A, then the resultant of the four perpendicular rotations is the rotation represented by α , which is parallel to the strut as viewed from that quadrant.



Fig. 5. A schematic representation of the junction of three strings, and the parallel transport loop that is used, together with the appropriate rotation vectors. Though it turns out that the three must be effectively coplanar, this is not assumed in the calculations.



Fig. 6. The angles used to recover the result for two strings after colliding, from the three string junction result.



Fig. 7a,b,c. Reconnection of strings that touch is not allowed in AEM spaces. If two strings come together (a), Field Theory predicts that the strings almost always break and reconnect, forming two elbows that then accelerate away from each other (b). From the results for 3-string junctions, if this occurred, there must be a connecting string joining the two elbows (c), which prevents any acceleration. However there is also a causality problem, due to the sudden change in velocities that is required to effect the reconnection. This implies the strings must become bent, which in not allowed in a space which is everywhere locally flat.



Fig. 8a,b. It is possible to construct two unconnected geodesic strings in an otherwise closed Robertson-Walker model. Because of the closed spatial surfaces, the strings must necessarily loop each other (or intersect twice), and they can be arranged so that each one passes through the centre of the other one (a). The embedding of a 2-dimensional constant time slice is shown in (b). String 1 loops the "equator" and is cut at right angles by the "wedge" due to string 2, which only intersects this surface at the poles. The wedge due to string 1, and the length of string 2 lie in the dimension not shown here. Exactly the same diagram can be drawn with the two strings interchanged.



Fig. 9a. The perpendicular angle between the surfaces of the wedges shown in Fig. 2a is found by taking a unit sphere about a point on the the strut (where the two surfaces intersect each other) and drawing in the intersection of these surfaces with the sphere, as well as the intersection of planes parallel to the x - y and y - z directions. (The axes shown here are all displaced from those of Fig. 2a.) Spherical trigonometry is used to calculate λ .







Fig. 10a. The wedges due to two strings, BO and OA, not necessarily of the same strength, are shown intersecting a unit sphere centred on the junction point O, along the curves BC, BC', CA, and C'A. The wedges have been chosen to be symmetric about the plane defined by AOB, and χ is the angle between the two strings. The angle between the wedges' surfaces ACB (or AC'B) is slightly less than π , and is calculated from spherical trigonometry.



Fig. 10b. The spherical triangles ABC, AXC, and BXC from Fig. 10a are shown enlarged here.