SZEKERES MODELS AND THEIR WORMHOLES

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We investigate quasi-spherical Szekeres models, including the anisotropic generalisation of the Lemaitre-Tolman wormhole topology. We: (a) derive the conditions for physically reasonable models, including a regular origin, maxima and minima, and the absence of shell-crossings; (b)obtain the relations between the local mass dipole, apparent horizon, light propagation rate, and shell crossings; (c) show non-zero dipole requires non-zero density, and cannot compensate for the effects of non-vacuum in any direction, so communication through the neck is still worse than the vacuum case; (d) show that a handle topology cannot be created by identifying hypersurfaces on either side of a wormhole, unless a surface layer is allowed. This impossibility includes the vacuum (Schwarzschild-Kruskal-Szekeres) case.

1. Understanding the Szekeres model

1.1. *Lemaître-Tolman Model*

We start with the simpler and better-known Lemaître-Tolman (LT) metric [1,2], which is spherically-symmetric but inhomogeneous radially:

$$ds^2 = -dt^2 + \frac{R'^2}{1+f}\,dr^2 + R^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

where $f(r)$ is an arbitrary function that determines the curvature of the spatial (constant t) sections, $R(t,r)$ is the areal radius, and $' \equiv \partial/\partial r$. The evolution of R is determined by

$$\dot{R}^2 = \frac{2M}{R} + f \tag{2}$$

where $\dot{} \equiv \partial/\partial t$ and $M(r)$ is an arbitrary function representing the gravitational mass inside the comoving shell at coordinate radius r. Here f acquires a second interpretation as the energy per unit mass of the matter shell with comoving radius r. This is actually a Friedmann equation with parameters that depend on r.

The solution for $R(t,r)$ has three cases — ever-expanding, re-collapsing, and the borderline case, and involves a third arbitrary function, a(r), the time at which the big bang happens locally. For example, the re-collapsing solution, in terms of parameter η is

$$R = \frac{M}{(-f)}(1-\cos\eta)\ , \qquad (\eta - \sin\eta) = \frac{(-f)^{3/2}(t-a)}{M} \tag{3}$$

The density is

$$8\pi\rho = \frac{2M'}{R^2R'} \tag{4}$$

1.2. *Szekeres Model*

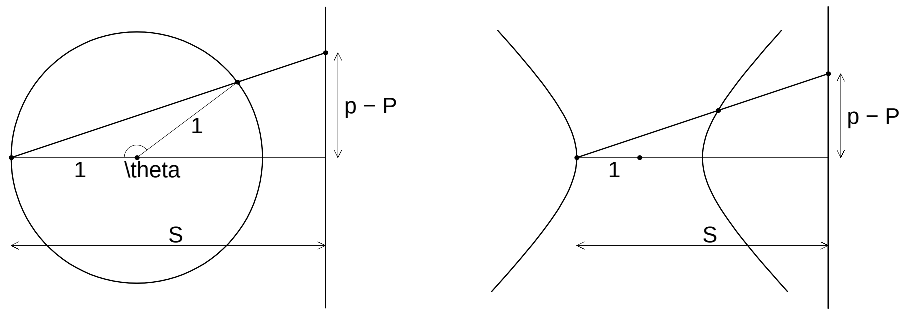
The Szekeres (S) model [3,4] is synchronous, comoving, and irrotational, with a dust equation of state. It is not spherically symmetric, and in fact it has no Killing vectors [5]. The metric is:

$$ds^2 = -dt^2 + \frac{\left(R' - \frac{RE'}{E}\right)^2}{\epsilon + f}\,dr^2 + \frac{R^2}{E^2}(dp^2+dq^2) \tag{5}$$

where $E = E(r,p,q)$ and $\epsilon = -1, 0, +1$. The function $R(t,r)$ has exactly the same solution as for LT. This might suggest it is an areal radius, and in fact it is for the $\epsilon = +1$ case we shall consider here. This metric contains the LT, dust Robertson-walker and Scwarzschild-Kruskal-Szekeres merics as special cases, the $R = R(t)$ case is a regular limit, and its null limit is a generalisation of the Kinnersley rocket metric [8].

1.3. *Riemann Projection*

To understand this better, we look at the the part of the metric that's multiplied by R^2, viz $(dp^2+dq^2)/E^2$. This is just the 2-sphere in funny coordinates, obtained from the Riemann projection:



For the circle, we have

$$\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cot\frac{\theta}{2} = \frac{q-Q}{S} \tag{6}$$

and rotation about the projection axis gives a 2-sphere,

$$\frac{p-P}{S} = \cot\frac{\theta}{2}\cos\phi \tag{7}$$

$$\frac{q-Q}{S} = \cot\frac{\theta}{2}\sin\phi \tag{8}$$

The Riemann hyperbola works similarly. However, in S, the transformation is r-dependent,

$$P = P(r)\ , \qquad Q = Q(r)\ , \qquad S = S(r) \tag{9}$$

so the transformation from (p,q) to (θ,ϕ) introduces cross terms in the metric, such as $dr\,d\theta$. Of course, if $E' = 0$, this transformation recovers the LT model.

1.4. *The Function E*

Thus we have:

$$E(r,p,q) = \frac{S}{2}\left[\left(\frac{p-P}{S}\right)^2 + \left(\frac{q-Q}{S}\right)^2 + \epsilon\right] \tag{10}$$

where ϵ determines the shape of the constant (t,r) surfaces:

$$\epsilon = +1 \quad \to \quad \text{sequence of Riemann spheres} \tag{11}$$

$$\epsilon = -1 \quad \to \quad \text{sequence of Riemann hyperboloids} \tag{12}$$

$$\epsilon = 0 \quad \to \quad \text{sequence of Riemann planes} \tag{13}$$

and ϵ determines the shape of the 2-surfaces that foliate the spatial sections.

Note that $g_{rr} \geq 0$ requires $\epsilon + f \geq 0$, and so

$$f > 0 \quad \to \epsilon \quad = +1, 0, -1 \tag{14}$$

$$f = 0 \quad \to \epsilon \quad = +1, 0 \tag{15}$$

$$-1 < f < 0 \quad \to \epsilon \quad = +1 \tag{16}$$

Clearly, the 3-d geometry determined by f may restrict the possible foliations. For example, you can't foliate closed spatial sections with hyperboloids, but you can foliate a negatively curved space with spheres.

The function E describes a dipole distribution round the 2-sphere at each r value, $(E'/E)_{max} = -(E'/E)_{min}$ located at antipodal points, and $E' = 0$ on a great circle mid way between. In particular, varying only p & q (or θ & ϕ), with t & r constant,

$\frac{E'}{E}$		g_{rr}	ρ
max	$\to$	min	min
min	$\to$	max	max

Naturally, the dipole orientation varies with r.

1.5. *Evolution and Density*

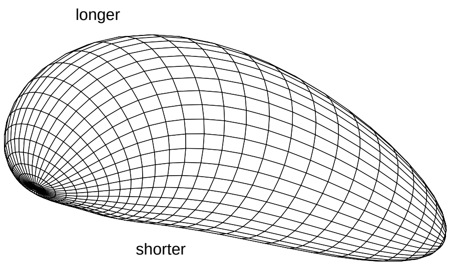
The evolution equation is identical to the LT case, eq(2), and $R(t,r))$ is the only metric function that depends on time. The density is

$$8\pi\rho = \frac{2\left(M' - \frac{3ME'}{E}\right)}{R^2\left(R' - \frac{RE'}{E}\right)} \tag{17}$$

1.6. *Interpretation*

The interpretation (for $\epsilon = +1$) is that the space sections are geometrically a sequence of non-concentric 2-spheres, but the r-dependence of E ensures that the density, curvature, etc are not uniform on each constant (t,r) sphere.

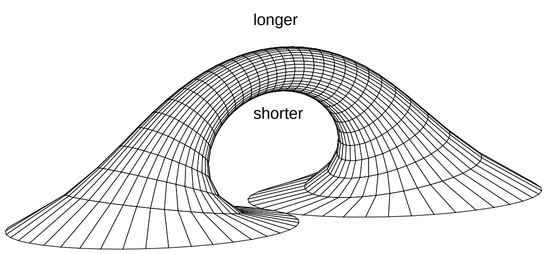
1.7. *Embedding*



We visualise the space sections as embeddings, with one coordinate suppressed, $\theta = \pi/2$, and we note the distance between constatnt r shells varies with (p,q), since $E = E(r,p,q)$

$$ds^2 = \ldots.. \frac{\left(R' - \frac{RE'}{E}\right)^2}{\epsilon + f}\, dr^2 \ldots..$$

1.8. *Wormhole Questions*



A) We know light can't quite get through the Schwarzschild-Kruskal-Szekeres (SKS) wormhole. But, if a SKS type wormhole can be bent round like this, so one side is shorter than the other, does that make it easier for light to get through on the shorter side?

B) Can the two asymptotic regions be spliced together to make a handle topology?

But first we must look at regularity conditions, as these will be central to answering the first question.

2. Regularity

We here select from [9] only certain key results.

2.1. *Regular Origins*

An origin occurs where $R(t, r_o) = 0$ for all t. A closed model would have two such loci. By insisting that the origin should have finite density and curvature, and have the same type of evolution as its immediate neighbourhood, we find that the origin has to be Robertson-Walker like

2.2. *No Shell Crossings*

Shell crossings are where an inner shell of matter runs into an adjacent outer shell. (In S, this does not involve the whole shell at once.) The conditions for avoiding such singularities, over the entire lifetime of the model, can be divided into those which are identical to the LT case [6], and the extra ones required in S. The one that will matter most to us is

$$\max\left(\frac{E'}{E}\right) = \frac{\sqrt{(S')^2 + (P')^2 + (Q')^2}}{S} \leq \frac{M'}{3M} \tag{18}$$

which applies if $M' \geq 0$.

2.3. *Regular Extrema*

A closed model must have at least one maximum in $R(t = \text{constant}, r)$ between its two origins. A wormhole must have a minimum. We call these "bellies" and "necks". In either case, $R'(t, r_m) = 0$ for all t, and $f = -1$ is required to avoid a surface layer.

3. Traversing S Wormholes?

3.1. *LT Wormholes*

The LT model with M constant can describe the full SKS manifold — two asymptotic regions joined by a neck of finite duration — using geodesic coordinates. It is trivial to add matter into such a model, by making M a function of r. It has been established [7] that the effect of introducing matter is to split the event horizons and further reduce the distance through the wormhole that light can travel.

The locus $R = 2M$ becomes an apparent horizon (AH), and the rays go along the AH where $\rho = 0$, but fall inside where $\rho > 0$.

3.2. *S Wormholes*

3.2.1. *Fastest Way Out*

Since S is not spherically symmetric, we first need to find the direction for optimum progress. The null condition

$$k^\alpha k^\beta g_{\alpha\beta} = 0 \tag{19}$$

in S is

$$\frac{\left(R' - \frac{RE'}{E}\right)}{\epsilon + f} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = 1 - \frac{R^2}{E^2} \left[\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2\right] \tag{20}$$

and we see taht, at any given location, the maximum $\frac{\mathrm{d}r}{\mathrm{d}t}$ is obtained if $\frac{\mathrm{d}p}{\mathrm{d}t} = 0 = \frac{\mathrm{d}q}{\mathrm{d}t}$. We call this direction "radial", and we say "rays" for radial null paths. Note that rays would not be geodesic in general. We re-write this, for $\epsilon = +1$, as

$$t'_n = \left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{null} = \frac{\pm 1}{\sqrt{1+f}} \left(R' - \frac{RE'}{E}\right) \tag{21}$$

and the solution for the ray path would be

$$t = t_n(r)\ , \qquad p = p_n(r)\ , \qquad q = q_n(r) \tag{22}$$

but it is $t_n(r)$ we are interested in. We find the fastest rays are where E'/E is maximum.

3.2.2. *Apparent Horizons*

The variation of areal radius along a ray is

$$R_n = R(tn(r), r) \tag{23}$$

$$(R_n)' = \dot{R} t'_n + R' \tag{24}$$

and we define the AH as the locus where $(R_n)' = 0$, which gives

$$\frac{\pm\left(\pm\sqrt{\frac{2M}{R} + f}\,\right)}{\sqrt{1+f}} \left(R' - \frac{RE'}{E}\right) + R' = 0 \tag{25}$$

We find $R \neq 2M$ in general, except
– where $E' = 0$
– at maximum expansion at a neck or belly, $\eta = \pi,\ f = -1$
– near the bang or crunch, $\eta = 0, 2\pi$.

3.2.3. *Bendyness versus Density*

In terms of parameter η the ray equation is

$$\left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_n = \frac{1}{\sqrt{1+f}} \left(\frac{M(1-\cos\eta)}{(-f)\sqrt{1+f}}\right) \times \\ \left\{ -\frac{\sin\eta}{(1-\cos\eta)^2} \frac{(-f)^{3/2} a'}{M} - \left(1 - \frac{3\sin\eta(\eta - \sin\eta)}{(1-\cos\eta)^2}\right) \frac{M'}{M} - \frac{E'}{E} \right\} \tag{26}$$

and the last term, E'/E, is the only non-LT term present. Since the function of η multiplying M'/M is ≥ 1, and the no shell crossing conditions require $|E'/E| \leq |M'/3M|$, this suggests that $E' \neq 0$ gives at most a partial compensation of the effect of non-vacuum, $M' > 0$.

3.2.4. *Key Argument*

We consider the surface

$$R(t,r) = \alpha M(r)\ , \qquad \alpha \text{ constant} \tag{27}$$

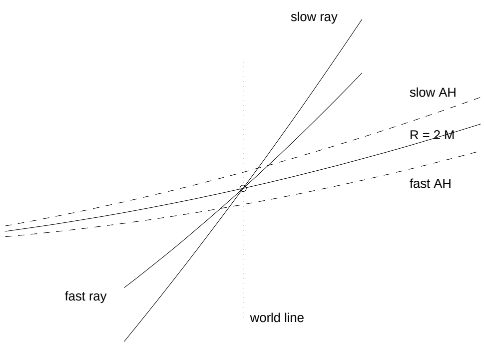
determine its slope

$$\left.\frac{\mathrm{d}t}{\mathrm{d}r}\right|_{R=\alpha M} = \frac{R' - \alpha M'}{\sqrt{\frac{2}{\alpha} + f}} \tag{28}$$

and find where it is null or outgoing timelike. We find

$$M' = 0 \qquad \rightarrow \qquad \alpha \geq 2 \qquad \text{(Schwazschild result)} \tag{29}$$

$$M' \neq 0 \text{and} \frac{E'}{E} \leq \frac{M'}{M} \qquad \rightarrow \qquad \alpha > 2 \tag{30}$$



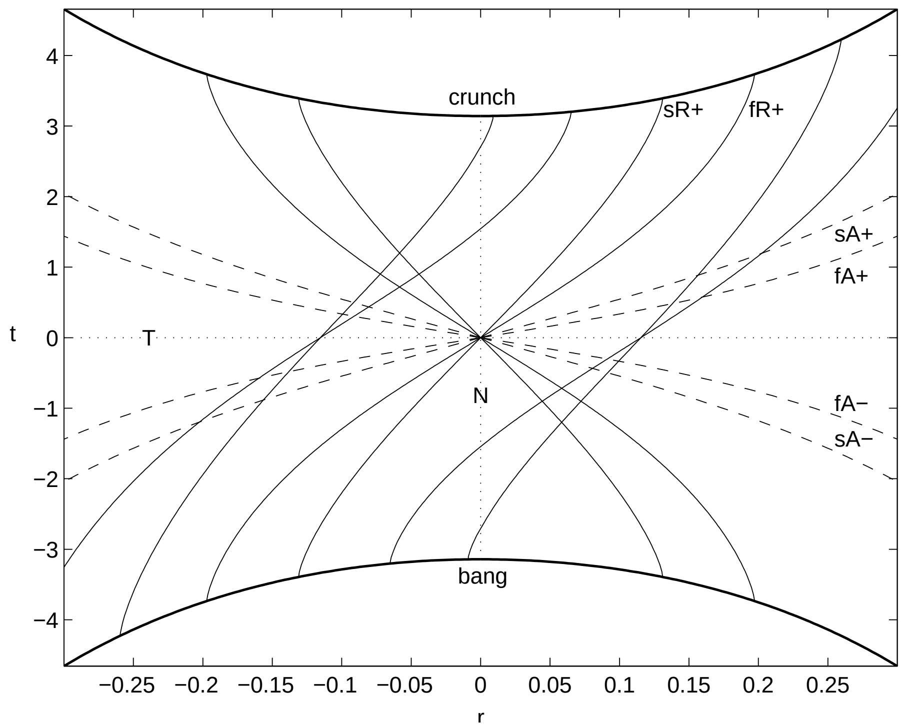
Thus, $R = 2M$ can't be outgoing timelike, and this is true along the whole length of the locus. Since all particle worldlines pass through $R = 2M$, and in the asymptotically flat regions, $R = 2M$ goes to the infinite future or past, thus there is no way for rays to escape falling into the AH.
Actually we have to check the centre of the neck, where $f = -1$, $M' = 0$, $f' = 0$ and $a' = 0$, separately. Taking the limits carefully, we find that the same applies.

3.2.5. *Result*

Where the density is non-zero, $M' > 0$, light travel through wormholes is slower than in vacuum (SKS), and the the bendyness, $E' \neq 0$, can at most only partially compensate for this. Therefore the causality structure of S wormholes is qualitatively the same as in LT wormholes.

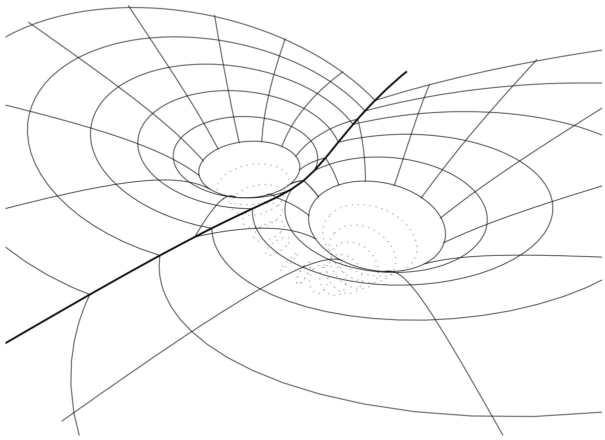
3.2.6. *Numerical Example*

The $(r - t)$ diagram below is for a time-symmetric S model, and shows the fast and slow future apparent horizons (fA+ and sA+), and past apparent horizons, the fast and slow rays that pass through O — the neck at the moment of maximum expansion — towards r increasing (fR+ and sR+), and rays through O going towards r decreasing, as well as rays going through other points. T is the moment of time symmetry which is also the simultaneous time of maximum expansion, and N is the locus of the neck $r = 0$. Note that fA+ & sA+ are two different intersections of the future apparent horizon AH^+ in two different radial directions — the fast & slow poles where E'/E takes extreme values. Note also that there is no origin $R(t, r = r_o) = 0$ in wormhole models.



4. A Handle Topology?

The wormhole topology has two asymptotically flat regions joined by a neck. In S this may be bent round in the embedding. This suggests the possibility that the two asymptotic regions may be joined up. We wish to know if this can be done smoothly? (Of course the embedding is not essential, and the junction does not have to be where the embedding intersects itself.) The sketch below illustrates the concept.



We go about this as follows. We make $r = 0$ at the neck, where $f = -1$, and we make all the arbitrary functions symmetric about $r = 0$, with forms that create a wormhole ($R'(t, r = 0)$ a regular minimum for all t). We cut along a timelike hypersurface Σ,

$$r_\Sigma = Z(p, q) \tag{31}$$

and join it to its mirror image at $r = -Z(p, q)$. We use the Darmois junction conditions, which require continuity of the first and second fundamental forms, and try to solve for the equations with suitable choices for the surface locus and the wormhole shape,

$$Z(p, q)\ ,\quad M(r)\ ,\quad f(r)\ ,\quad a(r)\ ,\quad P(r)\ ,\quad Q(r)\ ,\quad S(r)\ . \tag{32}$$

The two surfaces, being mirror images, are identical, except for the direction of the normal,

$$n^+_\alpha = n^-_\alpha \tag{33}$$

so the first fundamental forms — the intrinsic metrics — are already matched, and the second fundamental forms — the extrinsic curvatures — have opposite signs,

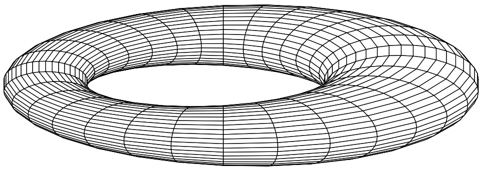
$$K^-_{ij} = K^+_{ij} = -K^-_{ij} = 0 \tag{34}$$

From this we find

$$K_{pt} = \frac{Z_{,p}(R\dot{R}' - \dot{R}R')}{\Delta} = 0 \tag{35}$$

$$K_{qt} = \frac{Z_{,q}(R\dot{R}' - \dot{R}R')}{\Delta} = 0 \tag{36}$$

$$\Delta = \sqrt{R^2(\epsilon + f) + (Z^2_{,p} + Z^2_{,q})(R'E - RE')^2}\ . \tag{37}$$



and these require: *either* $R = R(r)$, i.e. a static model, *or* $R = R(t)$, i.e. a Kantowski-Sachs type model, possibly a torus, *or* $Z = r_\Sigma$ =constant, i.e. a bendy torus topology.

The remaining components, K_{pp}, K_{qq}, K_{pq}, are much longer expressions, that don't involve $\dot{R}$. Superficially it seems that that they should be solvable at any one time, but the evolution equations equations don't preserve the matching.

Thus we conclude a handle topology is not possible.

5. Conculsions

Szekeres (S) models are a generalisation of the spherically symmetric Lemaître-Tolman (LT) models. Both describe inhomogeneous dust distributions, but the former have no Killing vectors. There are 3 arbitrary functions of coordinate radius in LT models (M, f & a), and a further 3 in S models (S, P & Q).

For quasi-spherical S models, the conditions for a regular origin, the conditions for no shell crossings, and the conditions for regular maxima and minima in the spatial sections have been established.

Like LT models, S models can reproduce the Schwarzschild-Kruskal-Szekeres topology of a wormhole connecting two universes, but with non-zero density everywhere. Although the S model's anisotropy makes the proper separation of consecutive shells shorter along certain directions, and null motion faster along those same directions, this is not enough to compensate for the retarding effect of matter, so the causal structure of an S wormhole is the same as that of the corresponding LT model.

On the question of whether the two universes on either side of a wormhole could be joined across a 3-surface to make a handle topology, it has been shown that a smooth junction is not possible at any finite distance, as a surface layer would be created. This conclusion applies to LT models and to the vacuum case — a Schwarzschild wormhole — too.

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