NOTES FOR SECOND YEAR DIFFERENTIAL EQUATION PART II: FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

JESSE RATZKIN

1. Definitions

An **ordinary differential equation** of **order** k has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) = 0, \tag{1}$$

where F is a function of $k+2$ variables. We will usually write ODE as an abreviation for ordinary differential equation. For instance, a first order ODE has the form $F(x, y, y') = 0$, and a second order ODE has the form $F(x, y, y'y'') = 0$. A function $y(x)$ is a **solution** of the ODE (**??**) in the interval $a < x < b$ if for all x in this interval we have

$$F(x, y(x), y'(x), \dots) = 0,$$

i.e. the identity (**??**) holds when we plug in the particular function $y(x)$.

We will usually be interested in solving an **initial value problem**. For an ODE of order k, this has the form

$$\begin{aligned} F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) &= 0 \\ y(x_0) = c_0, \quad \frac{dy}{dx}(x_0) = c_1, \dots, \quad \frac{d^{k-1}y}{dx^{k-1}}(x_0) &= c_{k-1}. \end{aligned} \tag{2}$$

Notice that, for an ODE of order k, one must prescribe k initial conditions.

We first see some examples.

- One can check that $y(x) = \frac{1}{2}x^2$ solves the ODE $(y')^2 - 2y = 0$ for all $x \in \mathbf{R}$.
- One can also check that the function $y(x) = \sqrt{1 - x^2}$ solves the ODE
$$\frac{d}{dx}\left(\frac{-y'}{\sqrt{1 + (y')^2}}\right) = 1,$$
at least for $-1 < x < 1$.
- For any real number $k \neq 0$, the functions $y_1(x) = \cos(kx)$ and $y_2(x) = \sin(kx)$ solve the ODE $y'' + k^2y = 0$. The general solution to this ODE is then
$$y(x) = a_1y_1(x) + a_2y_2(x) = a_1\cos(kx) + a_2\sin(kx),$$

Date: 2014.

and one can solve the initial value problem for any choice of initial conditions $c_0 = y(0)$ and $c_1 = y'(0)$ for this ODE by choosing a_1 and a_2 so as to match the initial conditions.

- For any real number $k \neq 0$ the function $y_1(x) = e^{kx}$ and $y_2(x) = e^{-kx}$ solve the ODE $y'' - k^2 y = 0$. The general solution to this ODE is then

$$y(x) = a_1 y_1(x) + a_2 y_2(x) = a_1 e^{kx} + a_2 e^{-kx},$$

and one can solve an initial value problem for any choice of initial conditions $c_0 = c(0)$ and $c_1 = y'(0)$ for this ODE by choosing a_1 and a_2 so as to match the initial conditions.

ODEs come in many different varieties, and so we'll need some descriptive terms to classify them. An ODE is **autonomous** if the independent variable x does not appear in the formula for the function F. An ODE is **linear** if it has the form

$$\begin{aligned} 0 &= F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^k y}{dx^k}\right) \\ &= Q(x) + P_0(x) y + P_1(x) \frac{dy}{dx} + \dots + P_k(x) \frac{d^k y}{dy^k}, \end{aligned}$$

where $Q, P_0, \dots, P_k$ are functions of x. In this case, we will usually assume that $P_k \neq 0$, and rearrange this equation to read

$$q(x) = \frac{d^k y}{dx^k} + p_{k-1} \frac{d^{k-1} y}{dx^{k-1}} + \dots + p_1(x) \frac{dy}{dx} + p_0(x) y(x),$$

where Here $q = -Q/P_k$ and $p_j = P_j/P_k$ for $j = 0, 1, \dots, k-1$. In this form we call $p_0, p_1, \dots, p_{k-1}$ the coefficients of the linear ODE, and we say it has constant coefficients if all of these functions are constants.

For this set of notes we will concentrate on first order ODEs, which (according to (**??**)) have the form

$$F(x, y, y') = 0.$$

In an associated initial value problem we prescribe one initial value, namely $y(x_0) = c_0$. According to the descriptions above, a first order ODE is autonomous if it has the form $F(y, y') = 0$ and it is linear if it has the form $y' = p(x) y + q(x)$.

2. Separable equations

A **separable** ODE has the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}. \tag{3}$$

Notice that we require $g(y) \neq 0$ in order that (**??**) makes sense. We can write the solution of this equation by integrating and using the Fundamental Theorem of Calculus. Rearange (**??**) to read

$$g(y) \frac{dy}{dx} = f(x)$$

and integrate both sides of this equation with respect to x to get

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} g(y)\frac{dy}{dx}dx = \int_{y(x_0)}^{y(x_1)} g(y)dy. \tag{4}$$

Here we have used the chain rule to change variables within the integral. Once we evaluate these integrals we can (hopefully) solve for the function $y(x)$. Even if we can't explicitly solve for y, we can often still describe the behavior of the solution.

2.1. **Exponential growth and decay.** The most basic example to consider is that of exponential growth/decay. This ODE is

$$y' = ky, \qquad y(0) = c,$$

where $k \neq 0$ is a fixed number and c is also a fixed number. In fact, this is the most basic differential equation to understand, full stop. In practical terms, this ODE models a situation where the rate of change of y is a fixed constant k times y, *i.e.* the rate of change of y is proportional to y.

We can solve by writing

$$kx = \int kdx = \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{dy}{y} = \ln y + C.$$

(You might think there should be a second constant of integration from the integral on the left hand side of the equation, but we can absorb it into the constant C on the right hand side. After all, the sum of two constants is still a constant.) Solving for y, we find

$$y = e^{kx+C} = ce^{kx},$$

where $c = e^C$. One can even check that this is the correct initial condition:

$$y(0) = ce^0 = c.$$

When $k > 0$ the solution y grows exponentially fast as $t \to +\infty$, so we call this the case of exponential growth. Conversely, when $k < 0$, the solution y decays to 0 exponentially as $t \to +\infty$, so we call this exponential decay.

Example: Radioactive elements decay with a certain rate into their stable counterparts. For instance, radioactive thorium-234 disintegrates, and the rate of disintegration is proportional to the current mass of thorium. In other words, if $y(t)$ is the mass of thorium at time t then $y' = ky$, where $k < 0$. We suppose that we start with 100 mg of thorium, and after a week we discover that 82.04 mg remains. When will we have 50 mg of thorium remaining? We have

$$y(0) = 100, \quad y(7) = 82.04, \quad y' = ky,$$

where k is a (negative) constant we must find. We know that the general solution of our ODE is $y = ce^{kt}$, so we plug this in to find

$$100 = y(0) = c, \qquad 82.04 = y(7) = 100e^{7k} \Rightarrow k = \frac{1}{7}\ln(.8204) \simeq -.02828.$$

Thus we have $y(t) = 100e^{-.02828t}$. To find the time when $y(t) = 50$ we again plug in to see

$$50 = y(t) = 100e^{-.02828t} \Rightarrow \frac{1}{2} = e^{-.02828t} \Rightarrow t = \frac{\ln(1/2)}{-.02828} \simeq 24.5 \text{ days}.$$

Example: Suppose we have a colony of bacteria in a petri dish, whose population doubles every three days. In particular, the population grows exponentially, at some rate we must determine. If we start with a colony of 500 bacteria, how long do we have to wait for the population to reach $1,000,000$? We let $y(t)$ be the population at time t where time is measured in hours. We have the information

$$y(0) = c_0 = 500, \quad y(72) = 2 \cdot y(0) = 1000, \quad y' = ky,$$

where k is a constant we must determine. We have

$$y(t) = 500e^{kt} \Rightarrow 1000 = 500e^{72k} \Rightarrow k = \frac{1}{72}\ln(2) \simeq .009627.$$

Now we can plug this in to see

$$1,000,000 = 500e^{kt} \Rightarrow 2,000 = e^{kt} \Rightarrow t = \frac{1}{k}\ln(2,000) \simeq 789.5 \text{ hours}.$$

This is a little less than 33 days.

The two examples above illustrate the importance of the amount t must increase in order for $y(t)$ to either double or half. In the case of

$$y' = ky, \quad k > 0$$

we call the value $t_0 > 0$ such that $y(t_0) = 2y(0)$ the **doubling time** of y. The doubling time is independent of $y(0)$, and is given by

$$y(t_0) = 2y(0) = y(0)e^{kt_0} \Rightarrow t_0 = \frac{1}{k}\ln(2).$$

Similarly, in the case of

$$y' = ky, \quad k < 0$$

we call the value $t_0 > 0$ such that $y(t_0) = \frac{1}{2}y(0)$ the **half-life** of y. The half-life is independent of $y(0)$, and is given by

$$y(t_0) = \frac{1}{2}y(0) = y(0)e^{kt_0} \Rightarrow t_0 = \frac{1}{k}\ln(1/2) = -\frac{1}{k}\ln(2).$$

We see, in particular, that the half-life/doubling time uniquely determines k, and vice versa.

2.2. **Some other examples.** Here are some more examples. Consider the initial value problem

$$y' = \frac{x}{y}, \quad y(1) = 2.$$

We rewrite the ODE as $yy' = x$ and integrate (starting from $x_0 = 1$):

$$\frac{1}{2}(x_1^2 - 1) = \int_1^{x_1} x dx = \int_{y(1)}^{y(x_1)} yy'(x)dx = \frac{1}{2}(y^2(x_1) - y^2(1)).$$

Substituing the initial condition $y(1) = 2$ and solving for $y(x_1)$ we see

$$y(x_1) = \sqrt{2 + x_1^2 - 1} = \sqrt{1 + x_1^2}.$$

Thus we obtain a formula for our solution, namely $y(x) = \sqrt{1 + x^2}$. Notice that this function exists for all values of x, and so, for the initial condition $y(1) = 2$, we have a global solution. It is easy to check that we do not get a global solution for other choices of initial data. For instance, if we change $y(1) = 1/2$ then the solution is

$$y(x) = \sqrt{x^2 - 3/4},$$

which only exists as a sensible function if $|x| > \sqrt{3}/2$.

Next we consider

$$y' = xy, \qquad y(1) = 1.$$

Again, we rewrite the equation and integrate, to obtain

$$\frac{1}{2}(x_1^2 - x_0^2) = \int_{x_0}^{x_1} x = \int_{y(x_0)}^{y(x_1)} \frac{y' dx}{y} = \ln y(x_1) - \ln y(x_0) = \ln\left(\frac{y(x_1)}{y(x_0)}\right).$$

Using the initial condition $y(x_0) = y(1) = 1$ we see

$$\ln y(x) = \frac{1}{2}(x^2 - 1) \Rightarrow y = e^{\frac{1}{2}(x^2-1)}.$$

We consider

$$y' = y^2, \quad y(0) = 1.$$

We separate variables to see

$$x = \int dx = \int \frac{dy}{y^2} = -\frac{1}{y} + c \Rightarrow y = \frac{1}{c - x}.$$

Matching the initial condition we see

$$1 = y(0) = \frac{1}{c} \Rightarrow c = 1 \Rightarrow y = \frac{1}{1 - x}.$$

This gives us a well-defined solution so long as $x < 1$.

2.3. **The logistic equation.** As a final example, we consider the model for logistic growth. (We will reconsider the logistic equation further in our section on autonomous equations, and see that we do not need to write out the solution in order to understand its behavior.) This is an ODE which is supposed to model a population in an area which cannot carry more than a certain number of people/fish/animals/etc. For instance, we might suppose that we have a pond with a certain number of fish in it, and the pond can only comfortably hold 10,000 fish. That is, if the number of fish is more than 10,000, they become over-crowded and start to die. If we let $y(t)$ be the number of fish at time t, then we must have $y'(t) < 0$ if $y > 10,000$. On the other hand, when the population of fish is small we expect a linear growth rate for the fish population. For instance, we might expect something like $y'(t) \simeq 10y$ when y is small. We can combine these two phenomenon with the model

$$y' = 10y\left(1 - \frac{y}{10,000}\right).$$

This is a separable equation, which we can rearrange and integrate to solve, giving

$$10t + C = \int \frac{dy}{y\left(1 - \frac{y}{10,000}\right)} = \ln y - \ln(10,000 - y) = \ln\left(\frac{y}{10,000 - y}\right).$$

Here we have used partial fractions to integrate. We can solve for y by taking an exponential. Let $c = e^C$, so that we have

$$ce^{10t} = \frac{y}{10,000 - y} \Rightarrow y = \frac{10,000ce^{10t}}{1 + ce^{10t}}.$$

If we set the initial population to $5,000$, then we find

$$5000 = y(0) = \frac{10,000c}{1+c} \Rightarrow c = \frac{1}{2} \Rightarrow y(t) = 10,000\frac{e^{10t}}{2 + e^{10t}}.$$

We can verify that in this case the population y is strictly increasing, and that it has a horizontal asymptote at $y = 10,000$ (which it never actually reaches).

We will return to this model for logistic growth in a later section of these notes.

3. First order linear ODE

As we wrote above, a first order linear ODE has the form

$$y' + p(x)y = q(x)$$

for two give functions p and q. Let's actually start with an example, and consider

$$y' + \frac{1}{x}y = 2.$$

Multiplying by x, we can rewrite this equation as

$$xy' + y = (xy)' = 2x \Rightarrow xy = x^2 + c \Rightarrow y = x + \frac{c}{x}.$$

We can try to mimic this example in general. We start with the ODE

$$y' + p(x)y = q(x),$$

and we'd like to multiply by some factor to make the left hand side into an exact derivative. We call this integrating factor $I(x)$, and we write

$$I(x)y' + I(x)p(x)y = Iq(x).$$

Ideally, we'd like the left hand side of the equation above to be $(Iy)'$, so let's see what that means. We then must have

$$(Iy)' = I'y + Iy' = Iy' + Ipy \Rightarrow I'y = Ipy \Rightarrow \frac{I'}{I} = p.$$

This is assuming that $y \neq 0$. We can integrate this last equation to get

$$\ln I = \int \frac{I'}{I}dx = \int p(x)dx \Rightarrow I(x) = e^{\int p(x)dx}.$$

There is an integration of constant we're ignoring, but adding it back in will only multiply I by a constant factor, so we are in fact safe to ignore it. We now have multiply our original ODE by the integrating factor I to get

$$y'e^{\int p(x)dx} + p(x)ye^{\int p(x)dx} = \frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q(x)e^{\int p(x)dx}.$$

We now integrate with respect to x to solve:

$$y(x) = e^{-\int p(x)dx}\left(c + \int q(x)e^{\int p(t)dt}dx\right).$$

Example: Consider

$$y' + 2xy = x^2, \quad y(0) = 1.$$

The integrating factor is

$$I(x) = e^{\int p(x)dx} = e^{\int 2xdx} = e^{x^2},$$

so we have

$$\left(ye^{x^2}\right)' = x^2e^{x^2} \Rightarrow y = e^{-x^2}\left(c + \int x^2e^{x^2}dx\right).$$

We might have trouble evaluating this last integral, but in fact we really don't need to, at least in order to find c. We evaluate at $x = 0$ and write the integral as a definite integral.

$$1 = y(0) = e^{-0^2}\left(c + \int_0^0 x^2e^{x^2}dt\right) \Rightarrow c = 1 \Rightarrow y(x) = e^{-x^2}\left(1 + \int_0^x t^2e^{t^2}dt\right).$$

Notice that we needed in the last formula for $y(x)$ we needed to call the integration variable something other than x, so we chose t (not that it really matters what we call the integration variable).

Example: Consider

$$y' - \tan(x)y = \cos x, \qquad y(0) = 2.$$

This time the integrating factor is

$$I(x) = e^{-\int \tan x dx} = e^{\ln \cos x} = \cos x.$$

Multiplying by I we now have

$$(y\cos x)' = \cos^2 x \Rightarrow y\cos x = \int \cos^2 xdx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + c \Rightarrow y = \frac{x}{2\cos x} + \frac{\sin(2x)}{4\cos x} + \frac{c}{\cos x}.$$

Evaulating at $x = 0$ we now have

$$2 = y(0) = 0 + 0 + c \Rightarrow c = 2,$$

and so the solution to our initial value problem is

$$y(x) = \frac{x}{2\cos x} + \frac{\sin(2x)}{4\cos x} + \frac{2}{\cos x}.$$

Example: Sometimes an ODE which appears to be nonlinear is actually linear. Consider the initial value problem

$$y^2y' + \frac{3y^3}{x} = xy^2, \qquad y(1) = 3.$$

This appears to be a nonlinear ODE, because we have various powers of y involved, but if we divide through by y^2, the ODE becomes

$$y' + \frac{3y}{x} = x,$$

which is in fact a nice linear equation. Of course, we should check that we are not dividing by 0 here. We can see $y \equiv 0$ is in fact a solution, but it does not match our initial conditions. We will see that the other solutions do our ODE never vanish. Returning to our transformed ODE, we wish to solve

$$y' + \frac{3y}{x} = x.$$

The integrating factor is

$$I = e^{\int (3/x)dx} = e^{3\ln x} = e^{\ln(x^3)} = x^3,$$

so multiplying by I we now have

$$(x^3y)' = x^4 \Rightarrow x^3y = \frac{1}{5}x^5 + c \Rightarrow y = \frac{1}{5}x^2 + \frac{c}{x^3}.$$

We finally find c by matching the initial condition. We have

$$3 = y(1) = \frac{1}{5} + c \Rightarrow c = \frac{14}{5} \Rightarrow y = \frac{1}{5}x^2 + \frac{14}{5x^3}.$$

We'll conclude this section with some applications.

Example: First suppose you set up a saving account towards your retirement. You begin the account with R10,000, and every month you deposity R500. Your account earns 8% per year, compounded continuously. If we let $S(t)$ be the amount you've saved after time t, then the differential equation for S is

$$\frac{dS}{dt} = .08S + 6000, \quad S(0) = 10,000.$$

(Notice that the constant on the right hand side is $6000 = 12 \cdot 500$ because the appropriate time period is one year, not one month.) This is an initial value problem for a linear, first order ODE, and we can find its solution using integrating factors. The integrating factor is

$$I = e^{\int (-.08)dt} = e^{-.08t},$$

and so we have

$$S = ce^{.08t} - 75,000.$$

Using the initial condition we find

$$10,000 = S(0) = c - 75,000 \Rightarrow c = 85,000 \Rightarrow S = 85,000e^{.08t} - 75,000.$$

How much money will we have saved up in 20 years? We evaluate:

$$S(20) = 346,007.76.$$

We can compare this to the amount of money we have actually deposited into the account, which is

$$10,000 + 20 \cdot 6,000 = 130,000.$$

Example: Next we look at mortgages. Suppose you'd like to buy a house, and can afford to spend R7500 per month on mortgage payments. The bank will charge $r\%$ annual interest on our loan, compounded continuously, and we plan to pay off the loan in 20 years. This time, we let $S(t)$ be the amount we owe at time t, and we have the differential equation

$$\frac{dS}{dt} = \frac{r}{100}S - 7500 \cdot 12, \quad S(0) = S_0.$$

Here the initial condition S_0 is the amount we can borrow, which is what we wish to compute. We solve the differential equation as we did before, to get

$$S(t) = ce^{\frac{rt}{100}} - \frac{9,000,000}{r}.$$

This time we determine c by using $S(20) = 0$, namely the fact that we pay off the loan in 20 years. We have

$$0 = S(20) = ce^{r/5} - \frac{9,000,000}{r} \Rightarrow c = \frac{9,000,000e^{-r/5}}{r},$$

and so

$$S(t) = \frac{9,000,000}{5}(e^{\frac{r(t-20)}{5}} - 1).$$

Finally we plug in some values of r:

$$r = 6 \Rightarrow S(0) = 1,048,208.68, \quad r = 9 \Rightarrow 834,701.11, \quad r = 12 \Rightarrow 681,961.54.$$

Example: Finally we consider a water tank with salt. Suppose that at time 0 a tank contains y_0 g of salt dissolved in 100 liters of water. A salt water mixturee is poured into the tank at 3 liters per minute, and each 3 liters contains $\frac{3}{4}$ g of salt. At the same time, a well-mixed solution is poured out of the tank through a spigot at the bottom of the tank, also at a rate of 3 litters per minute. We want to find a formula for the amount of salt $y(t)$ in the tank at time t. We first need to write out a differential equation for y. In each minute, 3/4 g salt enters the tank, and $\frac{3}{100}y$ g salt leaves the tank, so we have

$$y' = \frac{3}{4} - \frac{3}{100}y, \quad y(0) = y_0.$$

This is a first order, linear ODE, and its solution is

$$y(t) = 25 + ce^{-.03t},$$

where c is a constant we determine by matching the initial condition:

$$y_0 = y(0) = 25 + c \Rightarrow c = y_0 - 25 \Rightarrow y(t) = 25(1 - e^{-.03t}) + y_0e^{-.03t}.$$

Observe that when $t \to +\infty$ we have $y \to 25$, regardless of the value of y_0.

4. Autonomous first order ODE

An autonomous, first order ODE has the form $F(y, y') = 0$. We will see in this section that most such equations are in fact separable, but in this section we will concentrate on the properties of solutions, rather than formulas.

We begin by revisiting the logistic equation. Before, we wrote this as

$$y' = 10y\left(1 - \frac{y}{10,000}\right),$$

and it is supposed to model the population of fish in a pond which should only hold at most $10,000$ fish. We can generalize this idea to model a population which increases for small numbers, but decreases when the population is larger than some critical number N, which we call the **carrying capacity**. We can model this as

$$y' = ky\left(1 - \frac{y}{N}\right),$$

where $k > 0$ gives the rate of increase of y when it is small. We can in fact determine everything we want to know about solutions without actually solving the ODE. First, observe that we have two constant solutions, which correspond to the zeroes of y':

$$0 = y' = y\left(1 - \frac{y}{N}\right) \Leftrightarrow y = 0 \text{ or } y = N.$$

We can analyze the stability of these constant solutions by looking at the sign of y'. Notice that $y' > 0$ for $0 < y < N$ and $y' < 0$ for $y > N$. Now suppose we start with an initial condition $y_0 > 0$ which is small. Then $y' > 0$, so y will increase, moving away from the constant solution 0, no matter how small y_0 is. On the other hand, suppose y_0 is close to N. If $y_0 < N$ then $y' > 0$ and so y will increase towards N (but never actually reach it). Similarly, if $y_0 > N$ then $y' < 0$ and y will decrease towards N (but never actually reach it). Thus we see that solutions which start near 0 go away from it, but solutions which start near N actually go towards it.

At this point, we make some formal definitions. Consider the autonomous ODE $y' = f(y)$. The **equilibrium solutions** of the ODE $y' = f(y)$ are the constant solutions.

Lemma 1. *The constant function α is an equilibrium of the ODE $y' = f(y)$ if and only if $f(\alpha) = 0$.*

Proof. The function $y(x) = \alpha$, where α is a constant, is a solution of the ODE if and only if

$$0 = \frac{d}{dx}(\alpha) = y' = f(y) = f(\alpha).$$

□

Definition 1. *An equilibrium $y = \alpha$ is* **stable** *if solutions with initial values near α stay near α for all x. More precisely, for every $\epsilon > 0$ there is $\delta > 0$ such that if $|y_0 - \alpha| < \delta$ and we have $|y(x) - \alpha| < \epsilon$ for all x. An equilibrium is* **asymptotically stable** *if solutions with initial value close to α actually limit towards α. More precisely, there is $\delta > 0$ such that if $|y_0 - \alpha| < \delta$ then $\lim_{x\to\infty} y(x) = \alpha$.*

Notice that an equilibrium solution which is asymptotically stable is also stable, but the reverse may not be true. (In other words, one can find stable equilibria which are not asymptotically stable.) An equilibrium which is not stable is called **unstable**.

We have the following theorem.

Theorem 2. *Let α be an equilibrium solution of the autonomous ODE $y' = f(y)$ (i.e. $f(\alpha) = 0$). This equilibrium is asymptotically stable if $\frac{df}{dy}(\alpha) < 0$, and unstable if $\frac{df}{dy}(\alpha) > 0$. This test is inconclusive if $\frac{df}{dy}(\alpha) = 0$.*

The formal proof of this theorem is a bit beyond the present course, but here is some intuition for why it is true. If $\frac{df}{dy}(\alpha) < 0$ then f is decreasing near α. We already know that $f(\alpha) = 0$, so this implies $f(y_0) > 0$ for $y_0 < \alpha$ and $f(y_0) < 0$ for $y_0 > \alpha$. Using the differential equation $y' = f(y)$, we see this means that y will be increasing if $y_0 < \alpha$, while y will be decreasing if $y_0 > \alpha$. Thus, we see that if we start with y_0 near α then $y(x)$ will move towards α, both for $y_0 > \alpha$ and for $y_0 < \alpha$, and so we conclude that α is asymptotically stable. The reasoning for the case $\frac{df}{dy}(\alpha) > 0$ is similar.

We just saw at the beginning of this section that the logistic equation

$$y' = ky\left(1 - \frac{y}{N}\right)$$

has two equilibrium solutions: $y = 0$ and $y = N$. The solution $y = 0$ is unstable and the solution $y = N$ is asymptotically stable. We can adapt this model by adding in a threshold T with $0 < T < N$. Here the interpretation is that if the population falls below T then it starts to die out (because there are too few animals to reproduce). Now the differential equation is

$$y' = f(y) = -ky\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{N}\right).$$

The equilibrium solutions are the zeroes of $f(y)$, which are easy to find; they are $y_0 = 0$, $y_1 = T$ and $y_2 = N$. What about stability? We take a derivative:

$$\frac{df}{dy} = -k\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{N}\right) + \frac{ky}{T}\left(1 - \frac{y}{N}\right) + \frac{ky}{N}\left(1 - \frac{y}{T}\right).$$

Now evaluate these at each of our equlibria:

$$f'(0) = -k < 0, \quad f'(T) = k\left(1 - \frac{T}{N}\right) > 0, \quad f'(N) = k\left(1 - \frac{N}{T}\right) < 0.$$

Here we have used the fact that $0 < T < N$. We conclude that $y_0 = 0$ and $y_2 = N$ are asymptotically stable, while $y_1 = T$ is unstable.

Example: In this example we consider a critical threshold model for a population. This means that the population will decay to 0 if it is below some critical threshould, but will grow very fast if it above this threshold. We can model this behavior with the ODE

$$y' = -k\left(1 - \frac{y}{T}\right)y,$$

where $k > 0$ is a growth rate and $T > 0$ is the threshold. We see that the equilibria are

$$0 = y' = -ky\left(1 - \frac{y}{T}\right) \Leftrightarrow y = 0 \text{ or } y = T.$$

Moreover, we see $y' < 0$ when $0 < y < T$ and $y' > 0$ when $y > T$. Alternatively, we can compute a derivative, and see

$$\frac{d}{dy}\left[-ky\left(1 - \frac{y}{T}\right)\right] = -k + \frac{2ky}{T}, \quad \frac{df}{dy}(0) = -k < 0, \quad \frac{df}{dy}(T) = k > 0.$$

Thus $y = 0$ is an asymptotically stable equilibrium and $y = T$ is unstable.

Example: Find and classify all the equilibria of the ODE

$$y' = y \cos y.$$

We have

$$0 = y' = y\cos y \Leftrightarrow y = 0 \text{ or } y = \frac{(2k+1)\pi}{2}.$$

We must evaluate a derivative:

$$\frac{df}{dy} = \frac{d}{dy}(y\cos y) = \cos y - y \sin y.$$

Evaluating, we see $f'(0) = 1 > 0$, and so 0 is unstable. Also,

$$f'\left(\frac{(2k+1)\pi}{2}\right) = -\frac{(2k+1)\pi}{2}\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k\frac{(2k+1)\pi}{2}.$$

Thus the remaining equilibria are asymptotically stable if k is even and unstable if k is odd.

5. Other methods

5.1. **Exact equations.** Sometimes we can recognize the left hand side of an ODE as the derivative of a function.

Example: Consider

$$2x + y^2 + 2xyy' = 0.$$

This first order ODE is not separable, linear, or autonomous, so we haven't developed a method yet to handle it. However, is we let $g(x, y) = x^2 + xy^2$ we observe that

$$\frac{\partial g}{\partial x} = 2x + y^2, \qquad \frac{\partial g}{\partial y} = 2xy,$$

so that we can write our ODE as

$$\frac{\partial}{\partial x}(x^2 + xy^2) + \frac{\partial}{\partial y}(x^2 + xy^2)y' = 0.$$

If we further let $y = y(x)$ then the equation above becomes

$$\frac{d}{dx}(g(x, y(x))) = \frac{d}{dx}(x^2 + x(y(x))^2) = 0,$$

so that

$$x^2 + xy^2 = c,$$

where c is a constant.

The equation we just examined has the form

$$M(x, y) + N(x, y)y' = 0.$$

Theorem 3. *Let*

$$M(x, y) + N(x, y)y' = 0 \tag{5}$$

where M and N have continuous partial derivatives, and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{6}$$

*Then there exists a function $g(x, y)$ such that $M = \frac{\partial g}{\partial x}$ and $N = \frac{\partial g}{\partial y}$, and the solutions of (**??**) are given by $g(x, y(x)) = c$ for some constant c. Moreover, (**??**) is a necessary condition for the existence of solutions of this form.*

We sketch some ideas behind the proof of this theorem. First observe that, if $M = \frac{\partial g}{\partial x}$ and $N = \frac{\partial g}{\partial y}$ then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial N}{\partial x},$$

so (**??**) is a necessary condition in order that g exists. However, if (**??**) holds, we can recover g by choosing a base-point (x_0, y_0) and integrating, first along in the x-direction and then in the y-direction. Essentially, the heart of this theorem is the fact (which you saw in the 2AC module) that the curl of a vector field is zero if and only if it is a gradient.

Example: Find solutions of

$$y \cos x + 2xe^y + (\sin x + x^2 e^y - 1)y' = 0.$$

We let

$$M = y \cos x + 2xe^y, \qquad N = \sin x + x^2 e^y - 1$$

and verify that

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y = \frac{\partial N}{\partial x}.$$

Thus we must be able to find $g(x, y)$ such that

$$M = y \cos x + 2xe^y = \frac{\partial g}{\partial x}, \quad N = \sin x + x^2 e^y - 1 = \frac{\partial g}{\partial y}.$$

Integrate the first of these equations with respect to x to obtain

$$g = y \sin x + x^2 e^y + h(y),$$

where h is a function of the single variable y we must determine. Now use the second equation to see

$$\frac{\partial g}{\partial y} = \sin x + x^2 e^y + h' = \sin x + x^2 e^y - 1 \rightarrow h' = -1.$$

We conclude that $h = -y + c$ for some constant c, so we have

$$y \sin x + x^2 e^y - y = c,$$

which determines the solutions of our original ODE.

5.2. **Changes of variables.** Sometimes a change of variables will make an ODE which appears difficult into something easier to solve. This is the case if the equation has the form

$$y' = f(x,y) = F\left(\frac{y}{x}\right). \tag{7}$$

In this case, we introduce the new dependent variable $v = \frac{y}{x}$, so that

$$y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Now we substitute into (**??**) to get

$$v + xv' = F(v), \tag{8}$$

which is now a linear, first order ODE which we can solve.

Example: We solve

$$y' = \frac{y^2 + 2xy}{x^2} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}.$$

Making the substitution $v = \frac{y}{x}$ we obtain

$$xv' + v = v^2 + 2v \Rightarrow v' = \frac{v^2 + v}{x}.$$

This is a separable equation, so integrating we have

$$\int \frac{dx}{dx} = \int \frac{dv}{v(v+1)} = \int \left(\frac{1}{v} - \frac{1}{v+1}\right) dv,$$

which in turn gives

$$\ln|x| + \ln|c| = \ln|v| - \ln|v+1| = \ln\left|\frac{v}{v+1}\right| \Rightarrow cx = \frac{v}{v+1}.$$

Finally we substitute back to get

$$cx = \frac{y/x}{(y/x)+1} = \frac{y}{x+y} \Rightarrow y = \frac{cx^2}{1-cx}.$$

6. Existence and uniqueness (optional)

In this section we sketch a proof that, under some conditions, a first order initial value problem always has a unique solution, at least in a small interval around the initial value. The precise theorem is the following:

Theorem 4. *Let $F(x,y)$ be a function of two variables which is continuous and such that $\frac{\partial F}{\partial y}$ exists and is continuous. Then there exists $\epsilon > 0$ such that the initial value problem*

$$\frac{dy}{dx} = F(x,y), \qquad y(x_0) = y_0 \tag{9}$$

has a unique solution in the interval $x_0 - \epsilon < x < x_0 + \epsilon$.

It is worthwhile to consider two examples showing that the hypotheses of this theorem are as weak as possible. First, consider the initial value problem

$$y' = F(y) = 2\sqrt{y}, \qquad y(0) = 0.$$

Observe that $\frac{\partial F}{\partial y}$ is not continuous at $y_0 = 0$ (in fact it does not even exist there). Also, this initial value problem has two solutions, namely $y(0) = x^2$ and $y(x) = 0$. This is exactly why we must assume the function $F(x, y)$ is reasonably nice. Second, consider the initial value problem

$$y' = F(y) = y^2, \quad y(0) = 1.$$

This is a separable ODE, and the general solution is $y(x) = (c-x)^{-1}$, where c is the constant of integration. Matching the initial condition $y(0) = 1$, we see $c = 1$, and so $y(x) = (1 - x)^{-1}$. This solution exists only in the interval $\{x < 1\}$. Therefore, even if the right hand side $F(x, y)$ is a very nice function, we cannot expect that the solution of our initial value problem will exist for all x.

We will prove Theorem **??** in several steps. First we will tranform (**??**) into an integral equation, then we will define a sequence of funtions which we would like to converge to the the solution, and finally we will prove convergence. Along the way we will need to discuss how one can measure the distance between two functions, and the contraction mapping principle.

Lemma 5. *A function $y(x)$ solves* (**??**) *if and only if*

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))dt. \tag{10}$$

Proof. If y solves (**??**) then by the Fundamental Theorem of Calculus we have

$$\frac{dy}{dx} = \frac{d}{dx}\int_{x_0}^{x} F(t, y(t))dt = F(x, y(x)),$$

and

$$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t, y(t))dt = y_0,$$

which means exactly that y solves (**??**). On the other hand, if y solves (**??**) then

$$y_0 + \int_{x_0}^{x} F(t, y(t))dt = y_0 + \int_{x_0}^{x} y'(t)dt = y(x_0) + y(x) - y(x_0) = y(x).$$

□

For our second step, we define a sequence of functions. Start with $y_0(x) = y_0$ for all x. Then for $n \geq 0$ define

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt = \Phi(y_n). \tag{11}$$

Here we are thinking of Φ as a function on the space of function. That is, Φ is an operation which takes a certain function, y_n, and turns it into a new function y_{n+1}. Things like Φ are usually called **functionals**. Observe that y solves (**??**) if and only if $y = \Phi(y)$, so we immediately have the following corollary.

Corollary 6. *The function y solves (**??**) if and only if $y = \Phi(y)$.*

To complete the proof of Theorem **??** we would now like to show that the sequence of functions $\{y_n\}$ converges to a fixed point of the function Φ; that is, $y_n \to y_\infty = \Phi(y_\infty)$. We need some tools.

Definition 2. *Let $[a, b] = \{a \leq x \leq b\}$ be a bounded, closed interval, and let $y_1(x)$ and $y_2(x)$ be two continuous functions on $[a, b]$. Then*

$$\mathrm{dist}(y_1, y_2) = \|y_1 - y_2\| = \max_{x \in [a,b]} |y_1(x) - y_2(x)|.$$

Definition 3. *Let $[a, b] = \{a \leq x \leq b\}$ be a bounded, closed interval, and let $\{y_n(x)\}$ be a sequence of continuous functions on $[a, b]$. We say $y_n \to y$ pointwise if for each x we have $y_n(x) \to y(x)$. We say $y_n \to y$ uniformly if $\|y_n - y\| \to 0$.*

Notice that uniform convergence implies pointwise convergence, but the reverse implication may not be true. A good example to keep in mind for this phenomenon is $\{y_n(x) = x^n\}$ on the interval $[a, b] = [0, 1]$. We have

$$y_\infty(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

It is also true, but a little harder to prove, that y_∞ defined above is not the uniform limit of any sequence of continuous functions. Thus, the sequence $\{x^n\}$ converges pointwise but not uniformly.

We will need the following result, which we will not prove.

Theorem 7. *A sequence of functions $\{y_n\}$ on the closed inteval $[a, b]$ converges uniformly if and only it satisfies the Cauchy criterion: for all $\epsilon > 0$ there is N such that $m, n > N$ implies $\|y_m - y_n\| < \epsilon$.*

Definition 4. *A functional Ψ on the space of continuous functions is a contraction if there is k such that $0 < k < 1$ and*

$$\|\Psi(y_1) - \Psi(y_2)\| \leq k\|y_1 - y_2\|$$

for all functions y_1 and y_2.

We sketch a proof of the contraction mapping principle, which is a very useful theorem.

Theorem 8. *Any contraction has a unique fixed point. That is, if Ψ is a contraction on the space of functions $y : [a, b] \to \mathbf{R}$ then there is a unique function y_* which satisfies $\Psi(y_*) = y_*$.*

Proof. Choose any continuous function $y_0 : [a, b] \to \mathbf{R}$ and define the sequence of functions

$$y_1 = \Psi(y_0), \quad y_2 = \Psi(y_1), \quad \ldots, y_{n+1} = \Psi(y_n).$$

First notice that

$$\|y_{n+1} - y_n\| = \|\Psi(y_n) - \Psi(y_{n-1})\| \leq k\|y_n - y_{n-1}\|,$$

so by induction

$$\|y_{n+1} - y_n\| \leq k^n\|y_1 - y_0\|.$$

Now set $M = \|y_1 - y_0\|$ and use the triangle inequality. For $n > m$ we have

$$\begin{aligned} \|y_n - y_m\| &\leq \|y_n - y_{n-1}\| + \|y_{n-1} - y_{n-2}\| + \cdots + \|y_{m+1} - y_m\| \\ &\leq k^{n-1}\|y_1 - y_0\| + k^{n-2}\|y_1 - y_0\| + \cdots + k^m\|y_1 - y_0\| \\ &= M \sum_{j=m}^{n-1} k^j \leq M k^m \sum_{j=0}^{\infty} k^j = \frac{Mk^m}{1-k}. \end{aligned}$$

Recall that $k < 1$, so, once we choose $\epsilon > 0$, we can choose m large enough so that $\frac{Mk^m}{1-k} < \epsilon$. This shows $\{y_n\}$ satisfies the Cauchy criterion, so by our theorem above it converges uniformly to some function y_*. Now apply Ψ to y_* to see

$$\Psi(y_*) = \lim_{n\to\infty} \Psi(y_n) = \lim_{n\to} y_{n+1} = y_*,$$

so y_* is indeed a fixed point. Finally, we prove uniqueness. Suppose there is some other fixed point $y_\dagger$ such that $\Psi(y_\dagger) = y_\dagger$. Then

$$\|y_* - y_\dagger\| = \|\Psi(y_*) - \Psi(y_\dagger)\| \leq k\|y_* - y_\dagger\| \Rightarrow \|y_* - y_\dagger\| = 0 \Rightarrow y_* = y_\dagger.$$

□

Proof of Theorem **??**. We first see that we're done if we show the functional Φ defined by (**??**) is a contraction on continuous function in the closed, bounded interval $[x_0 - \epsilon, x_0 + \epsilon]$, for some small positive ϵ. If Φ is a contraction, then the contraction mapping principle tells us that it has a unique fixed point $y_* = \Phi(y_*)$, which solves (**??**).

To show that Φ is a contraction, we must estimate $\|\Phi(u) - \Phi(v)\|$ for two functions u and v. First let

$$M = \max \left| \frac{\partial F}{\partial y} \right|.$$

Then we have

$$\begin{aligned} |\Phi(u)(x) - \Phi(v)(x)| &= \left| \int_{x_0}^{x} F(t, u(t))dt - \int_{x_0}^{x} F(t, v(t))dt \right| \\ &= \left| \int_{x_0}^{x} F(t, u(t)) - F(t, v(t))dt \right| \\ &\leq \int_{x_0}^{x} |F(t, u(t)) - F(t, v(t))|dt \\ &\leq \int_{x_0}^{x} M\|u - v\|dt = M|x - x_0|\|u - v\|. \end{aligned}$$

Notice that $x \in [x_0 - \epsilon, x_0 + \epsilon]$, so $|x - x_0| < \epsilon$. Now we can choose ϵ small enough so that $\epsilon M \leq \frac{1}{2} < 1$, which implies Φ is a contraction, completing the proof. □

The sequence of functions $y_n = \Phi^n(y_0)$ is usually called the Picard iteration sequence, and this method is called the Picard iteration scheme. It is named after the French mathematician Émile Picard, who lived from 1856 to 1941.