NOTES FOR SECOND YEAR DIFFERENTIAL EQUATION PART III: FIRST ORDER ORDINARY SYSTEMS

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1. Introduction

A first order system of differential equations is a system of n first order ODEs. In general, these can be coupled together. Here are some examples systems of two ODEs for two unknown functions $y_1(x)$ $y_2(x)$.

(1) $y_1' = y_2, \quad y_2' = 2xy_1$
(2) $y_1' = y_1 + y_2, \quad y_2' = y_1 - y_2$
(3) $y_1' = x^2 y_1, \quad y_2' = (1-x)y_2$

The last example is **decoupled**, because the ODE for y_1 does not involve y_2, and the ODE for y_2 does not involve y_1. The other two systems are **coupled**.

At this point we set some notation and state definitions. We will consider a vector-valued function y of the real variable x, and write

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}.$$

We can now write our system of ODEs as

$$y' = F(x,y), F : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n. \tag{1}$$

In components, (1) looks like

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}' = \begin{bmatrix} F_1(x, y_1, \dots, y_n) \\ F_2(x, y_1, \dots, y_n) \\ \vdots \\ F_n(x, y_1, \dots, y_n) \end{bmatrix}. \tag{2}$$

Now the appropriate intial value is a vector $y(0) = y_0 \in \mathbf{R}^n$. As before, we say a vector-valued function $y(x)$ solves (1) if $y'(x) = F(x, y(x))$ for all x. We say that (1) is **decoupled** if each F_j depends only on x and y_j, and otherwise, it is a **coupled** system. We also have the same definitions as before. An ODE system is **linear** if each F_j is a linear function of $y_1, y_2, \dots, y_n$, otherwise it is nonlinear. An ODE system is **autonomous** if none of the F_j's depend on x.

The examples we wrote out at the start of this section were all 2×2 systems of ODEs. Written in vector notation, they look like

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(1)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ 2xy_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2x & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(2)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1 + y_2 \\ y_1 - y - 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(3)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} x^2 y_1 \\ (1-x)y_2 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & 1-x \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

These systems we have written out all have something in common: the right hand side F is always a linear function of y. We have written each system in terms of matrix multiplication to emphasize the fact that each system is linear.

We can see an example of a nonlinear ODE system, namely

$$y_1' = y_2^2, \qquad y_2' = y_1^2.$$

This is a nonlinear, coupled 2×2 system of ODEs, which we can write in vector notation as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2^2 \\ y_1^2 \end{bmatrix}.$$

However, because this system is not linear, we cannot write it as a matrix product, as we did with the linear systems above.

We also see a way to turn second order scalar ODEs into first order systems. Consider the second order, linear ODE

$$y'' + p_1(x)y' + p_0(x)y = q(x),$$

and make the substitution

$$y_1(x) = y(x), \qquad y_2(x) = y'(x).$$

Then the second order scalar ODE becomes the first order system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ q - p_1 y_2 - p_0 y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -p_0 & -p_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ q \end{bmatrix}.$$

In fact, we can use the same technique to convert any kth-order ODE into a $k \times k$ system of first-order ODEs.

2. Linear systems

A homogeneous, linear system of ODEs has the form

(3) $$y' = A(x)y, \qquad y(x) \in \mathbf{R}^n, \quad A(x) \in M_{n\times n}.$$

We refer to the matrix A as the coefficient matrix of the system, and observe that the entries in A are allowed to be functions in general. Written out in components,

(3) looks like

$$
\begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}' = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}. \tag{4}
$$

Mostly we will write out 2×2 systems, which look like

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},
$$

and we can illustrate most of the general properties with these examples.

There is an example which is very easy to solve, namely when the coefficient matrix A is diagonal. In this case, the system decouples completely, and we end up with n independent, linear, first order ODEs. More explicitly, if A is diagonal, then we have

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}' = \begin{bmatrix} a_{11}(x) & 0 & \cdots & 0 \\ 0 & a_{22}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn}(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tag{5}
$$

which we can rewrite (after multiplying the matrices out) as

$$
y_1' = a_{11}(x)y_1, \quad y_2' = a_{22}(x)y_2, \cdots, \quad y_n' = a_{nn}(x)y_n. \tag{6}
$$

The fact that the system decouples is equivalent to saying y_j' depends only on y_j, and not on any of the other y's. Each of these ODEs is separable (and also linear), so they are all easy solve. We obtain

$$
y_1(x) = c_1 e^{\int a_{11}(x)dx}, \quad y_2(x) = c_2 e^{\int a_{22}(x)dx}, \ldots, \quad y_n(x) = c_n e^{\int a_{nn}(x)dx}, \tag{7}
$$

where $c_1, c_2, \ldots, c_n$ are constants of integration.

There is one choice of initial conditions for which we can always solve.

Lemma 1. *The initial value problem*

$$
y'(x) = A(x)y(x), \qquad y(0) = 0
$$

has the solution $y(x) = 0$ *for all* x.

Proof. Observe that

$$
y' = \frac{d}{dx}(0) = 0 = A(x) \cdot 0 = A(x)y(x).
$$

□

Remark 1. *Observe that* $y = 0$ *is always an equilibrium solution of the ODE system* $y' = Ay$.

Definition 1. *The equilibrium solution* $y \equiv 0$ *is* **stable** *if for every* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $|y(0)| < \delta$ *implies* $|y(x)| < \epsilon$ *for all* x*. In other words, if you start with initial conditions close enough to* 0 *then you stay near* 0*. The equilibrium* 0 *is* **asymptotically stable** *if there is* $\delta > 0$ *such that* $|y(0)| < \delta$ *implies* $\lim_{x\to\infty} |y(x)| = 0$*. In other words, if you start with initial conditions close enough to* 0 *then the solution in fact limits towards* 0*. If* 0 *is neither stable nor asymptotically stable for the ODE system* $y' = Ay$ *then* 0 *is* **unstable**.

We will see that we cannot solve coupled systems of ODEs in general, but we can say something about the solutions in many cases. We will also see below a nice way to characterize when 0 is stable or unstable, at least in the case when A is a constant matrix.

2.1. **Linear systems with constant coefficients.** In this section we only consider the case when the coefficient matrix A has constant entries. In other words, we assume each a_{ij} is a constant. The system (3) is now $y' = Ay$, where A is a fixed $n \times n$ matrix. We will start with the initial value problem

$$y' = Ay, \quad y(x) \in \mathbf{R}^n, \quad A \in M_{n\times n}, \quad y(0) = v. \tag{8}$$

Before stating some results, it would be nice to recall some facts from linear algebra. A nonzero vector $v \in \mathbf{R}^n$ is an eigenvector of A with eigenvalue $\lambda \in \mathbf{R}$ if $Av = \lambda v$. Notice that λ is allowed to be zero, but v is not. One can find eigenvalues by solving the equation

$$\det(A - \lambda I) = 0,$$

where I is the $n \times n$ identity matrix, and then find associated eigenvectors by solving the system of equations $Av = \lambda v$. A matrix A is **diagonalizable** if one can find a basis of $\mathbf{R}^n$ consisting of eigenvectors of A. In other words, A is diagonalizable if and only if one can find n linearly indenpendent vectors $\{v_1, \ldots, v_n\}$ such that $Av_j = \lambda_j v_j$.

Proposition 2. *We consider the initial value problem* (8)*, and suppose that* v *is an eigenvector of* A *with eigenvalue* λ*. That is,* $Av = \lambda v$*. Then the solution of* (8) *is*

$$y(x) = e^{\lambda x} v. \tag{9}$$

Proof. We need to check that $y(x) = e^{\lambda x} v$ solves the ODE and has the correct initial condition. First we take a derivative:

$$\frac{d}{dx} y(x) = \frac{d}{dx}\left(e^{\lambda x} v\right) = \lambda e^{\lambda x} v = e^{\lambda x}(Av) = Ay.$$

Here we have used the fact that v is an eigenvector of A with eigenvalue λ. Next we evaluate at $x = 0$:

$$y(0) = e^{\lambda \cdot 0} v = v.$$

□

Proposition 3. *Suppose the initial vector v in (8) is a linear combination of eigenvectors of A:*

$$v = c_1 v_1 + \cdots c_k v_k, \qquad A v_j = \lambda_j v_j.$$

Then the solution to the initial value problem is

$$y(x) = c_1 e^{\lambda_1 x} v_1 + \cdots + c_k e^{\lambda_k x} v_k.$$

Proof. We compute much as in the last case:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(c_1 e^{\lambda_1 x} v_1 + \cdots + c_k e^{\lambda_k x} v_k\right) \\
&= c_1 \lambda_1 e^{\lambda_1 x} v_1 + \cdots + c_k \lambda_k e^{\lambda_k x} \\
&= c_1 e^{\lambda_1 x} A v_1 + \cdots + c_k e^{\lambda_k x} A v_k \\
&= A\left(c_1 e^{\lambda_1 x} v_1 + \cdots + c_k e^{\lambda_k x} v_k\right) \\
&= Ay.
\end{aligned}$$

Here we have used the fact that $Av_j = \lambda_j v_j$, and the fact that matrix multiplication is linear. Thus y satisfies the correct ODE. We evaluate at $x = 0$ as well:

$$y(0) = c_1 e^{\lambda_1 \cdot 0} v_1 + \cdots + c_k e^{\lambda_k \cdot 0} v_k = a_1 v_1 + \cdots + a_k v_k = v.$$

□

Theorem 4. *Let $A \in M_{n \times n}$ and consider the intial value problem*

$$y' = Ay, \quad y(0) = y_0.$$

Suppose that A has a basis of eigenvectors $\{v_1, v_2, \ldots, v_n\}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that $Av_j = \lambda_j v_j$. Because $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$, we can write

$$v = y(0) = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$$

for some coefficients $c_1, \ldots, c_n$. Then the solution to our inital value problem is

$$y(x) = c_1 e^{\lambda_1 x} v_1 + c_2 e^{\lambda_2 x} v_2 + \cdots + c_n e^{\lambda_n x} v_n.$$

Proof. If A has a basis of eigenvectors (*i.e.* if A is diagonalizable) then we can write as a linear combination of eigenvectors. The theorem now follows immediately from the previous proposition. □

Example: We consider

$$y' = Ay, \qquad A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}.$$

One can check that A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$, with

$$Av_1 = A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3v_1$$

and

$$Av_2 = A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -2v_1.$$

Suppose we want to solve the initial value problem

$$y' = Ay, \quad y(0) = \begin{bmatrix} -5 \\ 5 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -4v_1 + 3v_2.$$

The solution is

$$y(x) = -4e^{3x}v_1 + 3e^{-2x}v_2.$$

Observe that as $x \to \infty$, the v_2 term becomes very small, so when x is very large the solution $y(x)$ is very close to a multiple of the eigenvector v_1.

We pause here to recall our definitions of stability. Roughly speaking, 0 is stable if any solution starting near enough to 0 stays near 0, and it is asymptotically stable if any solution starting near enough to 0 tends towards 0. Formally, 0 is stable if for every $\epsilon > 0$ there is $\delta > 0$ such that $|y(0)| < \delta$ implies $|y(x)| < \epsilon$ for all ϵ. Similarly, 0 is asymptotically stable if there is $\delta > 0$ such that $|y(0)| < \delta$ implies $\lim_{x\to\infty} |y(x)| = 0$.

Now apply this criterion to the system

$$y' = Ay = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} y.$$

If we choose the initial condition

$$y(0) = c_1v_1 + c_2v_2$$

then the solution is

$$y(x) = c_1e^{3x}v_1 + c_2e^{-2x}v_2.$$

In particular, of $c_1 \neq 0$ then $|y(x)|$ will grow exponentially because of the $e^{3x}v_1$ term. We see that in this case 0 must be unstable, because A has one positive eigenvalue $\lambda_1 = 3$. This one positive eigenvalue forces the solution to grow exponentially.

Exercise: Sketch some generic solutions of the system

$$y' = Ay = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$$

on the $y_1 - y_2$ plane. This sketch is usually called a **phase portrait** of the system $y' = Ay$.

We're ready to state a theorem.

Theorem 5. *Let $A \in M_{n\times n}$ have real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and consider the system of ODEs $y' = Ay$. Then 0 is a stable equilibrium if and only if $\lambda_j \leq 0$ for all $j = 1, 2, \dots n$, and 0 is asymptotically stable if and only if $\lambda_j < 0$ for all $j = 1, 2, \dots, n$.*

Proof. We prove this theorem in the case that A is diagonalizeable, *i.e.* A has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ with $Av_j = \lambda_j v_j$. However, the statement of the theorem is true in general. Let $y_0 = y(0)$ be any initial condition, and find coefficients $c_1, c_2, \dots, c_n$ such that

$$y_0 = c_1v_1 + c_2v_2 + \cdots c_nv_n \Rightarrow y(x) = c_1e^{\lambda_1 x}v_1 + c_2e^{\lambda_2 x}v_2 + \cdots + c_ne^{\lambda_n x}v_n.$$

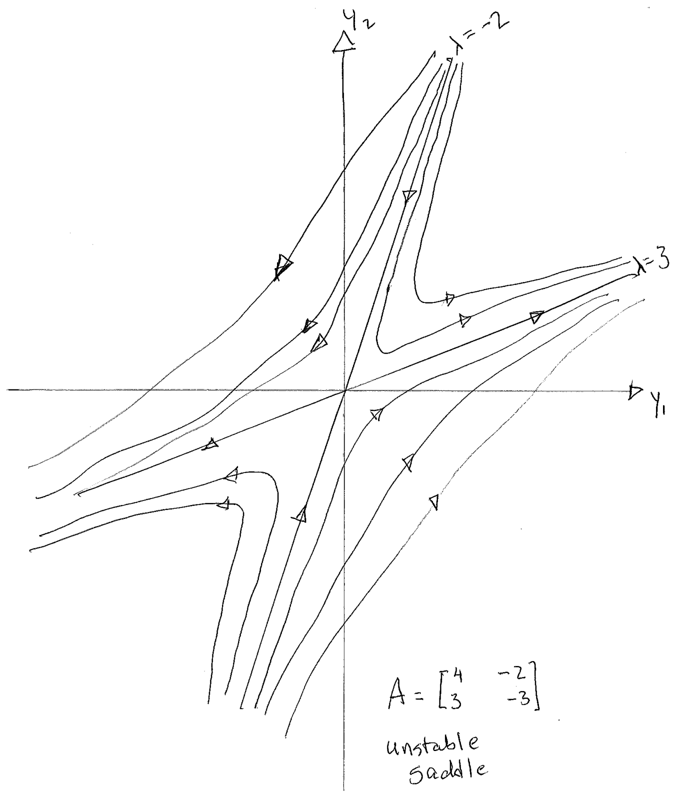

FIGURE 1. The is the phase portrait for the system $y' = Ay$ where A has eigenvalues 3 and -2, so the origin is an unstable saddle.

If $\lambda_j < 0$ for all j, we see that **all** solutions $y(x)$ decay exponentially to 0, regardless of where they start. In particular, in this case 0 is asymptotically stable. If $\lambda_j \leq 0$ then for $x > 0$ we have $e^{\lambda_j x} \leq 1$, and so

$$|y(x)| = \left|c_1 e^{\lambda_1 x} v_1 + \cdots + c_n e^{\lambda_n x} v_n\right| \leq |c_1||v_1| + |c_2||v_2| + \cdots + |c_n||v_n|.$$

In particular, if we choose $|c_j|$ small enough for all j, then y stay near 0, and so 0 is stable. On the other hand, if one eigenvalue is positive, say $\lambda_n > 0$, then we can choose $y_0 = c_n v_n$, and the solution $y(x) = c_n e^{\lambda_n x} v_n$ will have an exponentially growing length, no matter how small we choose c_n. Therefore, in this case 0 is unstable. □

We will later see a more general statement, when the eigenvalues of A are allowed to be complex numbers.

Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} y.$$

With a little work, we compute that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. We have one positive eigenvalue, so 0 is unstable. A little more work tells us that the two eigenvectors are

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad Av_1 = -v_1, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Av_2 = 3v_2.$$

Thus, any solution has the form

$$y(x) = c_1 e^{-x} v_1 + c_2 e^{3x} v_2 \to c_2 e^{3x} v_2$$

as $x \to \infty$. After a long enough period of time, the solution will be very close to a scalar multiple of v_2. The phase portrait of this system looks like a saddle.

Definition 2. *Let $A \in M_{2\times 2}$ have real eigenvalues $\lambda_1 < 0$ and $\lambda_2 > 0$. (That is, A has one positive eigenvalue and one negative eigenvalue.) Then we call 0 a* **saddle point** *of the system $y' = Ay$.*

Observe that a saddle point is always unstable!
Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} y.$$

With a little work, we compute that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -4$. This time we have two negative eigenvalues, so we conclude that 0 is asymptotically stable. With a little more work, we compute that the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Av_1 = -v_1, \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad Av_2 = -4v_2.$$

Thus, any solution has the form

$$y(x) = c_1 e^{-x} v_1 + c_2 e^{-4x} v_2 \to 0$$

as $x \to \infty$. Notice that the v_2 component tends to zero much faster than the v_1 component, which we can see in the phase portrait of the system.

Definition 3. *Let $A \in M_{2\times 2}$ have two negative eigenvalues. Then 0 is called a* **sink** *of the system $y' = Ax$.*

Observe that a sink is always asymptotically stable!

Example: Consider the system of ODEs

$$y' = Ay = \frac{1}{5} \begin{bmatrix} 17 & -3 \\ -2 & 18 \end{bmatrix} y.$$

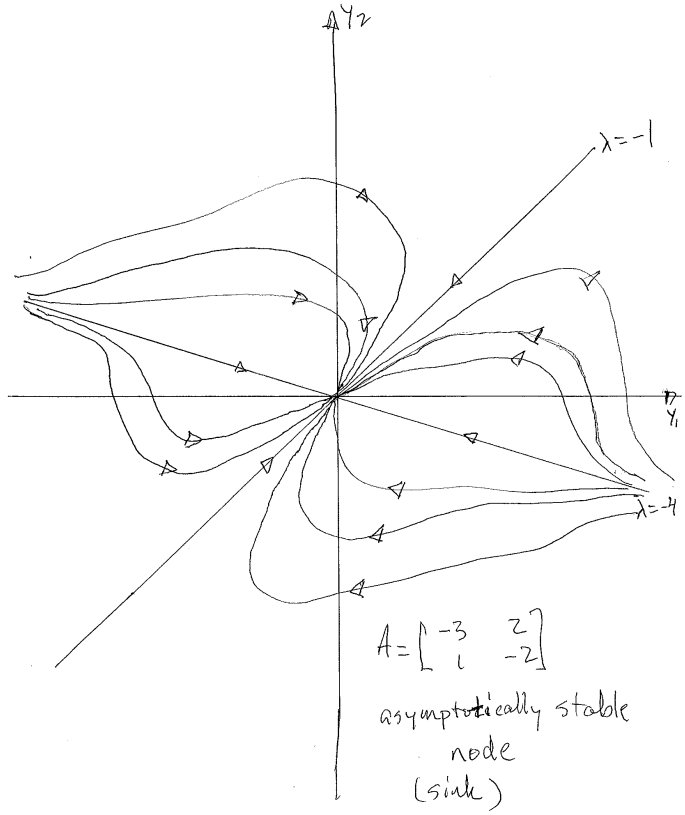

FIGURE 2. The is the phase portrait for the system $y' = Ay$ where A has eigenvalues -1 and -4, so the origin is an asymptotically stable (sink) node.

With a little work, we can compute that the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 4$. We have two positive eigenvalues, so 0 is unstable. With a little more work, we compute the eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad Av_1 = 3v_1, \quad v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad Av_2 = 4v_2.$$

Any solution then has the form

$$y(x) = c_1 e^{3x} v_1 + e^{4x} v_2,$$

which grows exponentially as $x \to \infty$. Notice that the v_2 term grows much faster than the v_1 term, as we can see from the phase portrait.

Definition 4. *Let $A \in M_{2\times 2}$ have two positive eigenvalues. Then 0 is called a* **source** *of the system $y' = Ay$.*

Observe that a source is always unstable!

Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} y.$$

Again, we do a little bit of work and see that the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -1$, so 0 is stable but not asymptotically stable. With a little more work, we find the eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Av_1 = 0, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Av_2 = -2v_2.$$

Any solution of this system now has the form

$$y(x) = c_1 v_1 + c_2 e^{-2x} v_2 \to c_1 v_1$$

as $x \to \infty$. We see that, regardless of where we start, the limiting behavior is that the solution becomes as close as you please to a multiple of v_1.

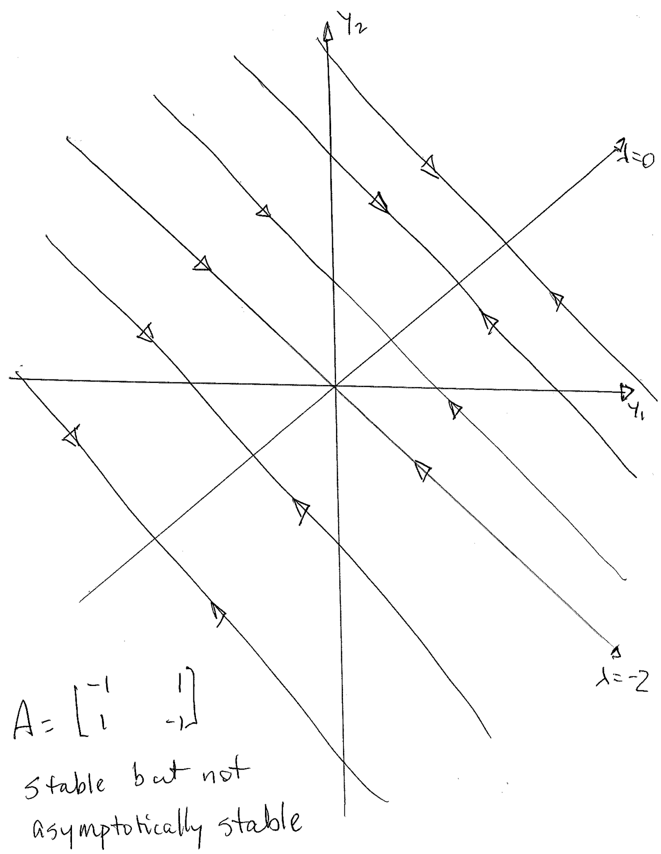

FIGURE 3. The is the phase portrait for the system $y' = Ay$ where A has eigenvalues 0 and -2, so the origin is stable but not asymptotically stable.

We have seen examples of matrices which are not diagonalizable, that is, matrices which do not have a basis of eigenvectors. We illustrate this with an example.

Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} y.$$

We find that $\lambda = 2$ is the only eigenvalue (it is a repeated root of the characteristic polynomial), but it only has one eigenvector associated it:

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Av = 2v.$$

So we have one solution to the system:

$$y(x) = ce^{2x} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where c is an arbitrary real number. We need to find another solution, and we look for a solution of the form

$$y(x) = xe^{2x}v = xe^{2x} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

However, a computation tells us

$$y' = (1+2x)e^{2x}v \neq 2xe^{2x}v = xe^{2x}Av = Ay,$$

so this guess doesn't work. For our next guess, we choose

$$y(x) = xe^{2x}v + e^{2x}w = xe^{2x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2x} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where $w \in \mathbf{R}^2$ is some fixed vector we must find. Now setting $y' = Ay$ forces

$$\begin{aligned} y' &= e^{2x}(1+2x)v + 2e^{2x}w = Ay = A(xe^{2x}v + e^{2xw}) \\ &= 2xe^{2x}v + e^{2x}Aw. \end{aligned}$$

Collecting terms and cancelling, we see

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = v = (A - 2I)w = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Solving, we find $w_1 + w_2 = -1$, so we can choose

$$w = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow y(x) = xe^{2x}v + e^{2x}w = xe^{2x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2x} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Notice that we have a choice in solving for w, but any choice we make in this step will give us a solution. In fact, any two choices will differ by a multiple of v. (Challenge problem: prove this last statement.)

Theorem 6. *Let $A \in M_{2\times 2}$ have a repeated eigenvalue λ and only one eigenvector v (up to scalar multiples). Then the general solution to the system of ODES $y' = Ay$ has the form*

$$y(x) = c_1 e^{\lambda x} v + c_1 \left[x e^{\lambda x} v + e^{\lambda x} w \right], \qquad \text{where } (A - \lambda I)w = v.$$

In this case, we call w a generalized eigenvector of A associated to the eigenvalue λ. The equilibrium solution $y = 0$ is stable if and only if $\lambda \leq 0$.

Remark 2. *Be careful to only do this when A is not diagonalizable!*

Finally, we treat the case when A has complex eigenvalues.

Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} y.$$

Computing, we find that the eigenvalues of A are $\lambda_{\pm} = \pm 2i$, and we have eigenvectors

$$v_{\pm} = \begin{bmatrix} \pm 2i \\ 1 \end{bmatrix}, \quad Av_{+} = 2iv_{+}, \quad Av_{-} = -2iv_{-}.$$

Thus we can write our solution as

$$y(x) = c_{+} e^{2ix} v_{+} + c_{-} e^{-2ix} v_{-} = c_{+} e^{2ix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} + c_{-} e^{-2ix} \begin{bmatrix} -2i \\ 1 \end{bmatrix}.$$

We use Euler's formula: if $\theta \in \mathbf{R}$ then $e^{i\theta} = \cos\theta + i\sin\theta$ to rewrite our expression for y as

$$y(x) = k_1 \cos(2x) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - k_1 \sin(2x) \begin{bmatrix} 2 \\ 0 \end{bmatrix} + k_2 \cos(2x) \begin{bmatrix} 2 \\ 0 \end{bmatrix} + k_2 \sin(2x) \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $k_1 = c_{+} + c_{-}$ and $k_2 = -1(c_{+} - c_{-})$.

We see immediately that the solution $y(x)$ is periodic, with period π. In particular, the equilibrium $y_0 = 0$ is stable, but not asymptotically stable. If you trace out the solution curves on the $y_1 - -y_2$ plane you will find they are ellipses.

Theorem 7. *Let $A \in M_{2\times 2}$ have complex eigenvalues $\lambda_{\pm} = a \pm ib$, with eigenvectors $v_{\pm} = u \pm iw$, where a and b are real numbers and u and w are vectors in $\mathbf{R}^2$. Then the general solution of the ODE system $y' = Ay$ is*

$$y(x) = e^{ax} \left[k_1 \cos(bx) u - k_1 \sin(bx) w + k_2 \cos(bx) w + k_2 \sin(bx) u \right].$$

In particular, the equilibrium $y = 0$ is stable if and only if $a \leq 0$ and asymptotically stable if and only if $a < 0$.

Example: Consider the system of ODEs

$$y' = Ay = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} y.$$

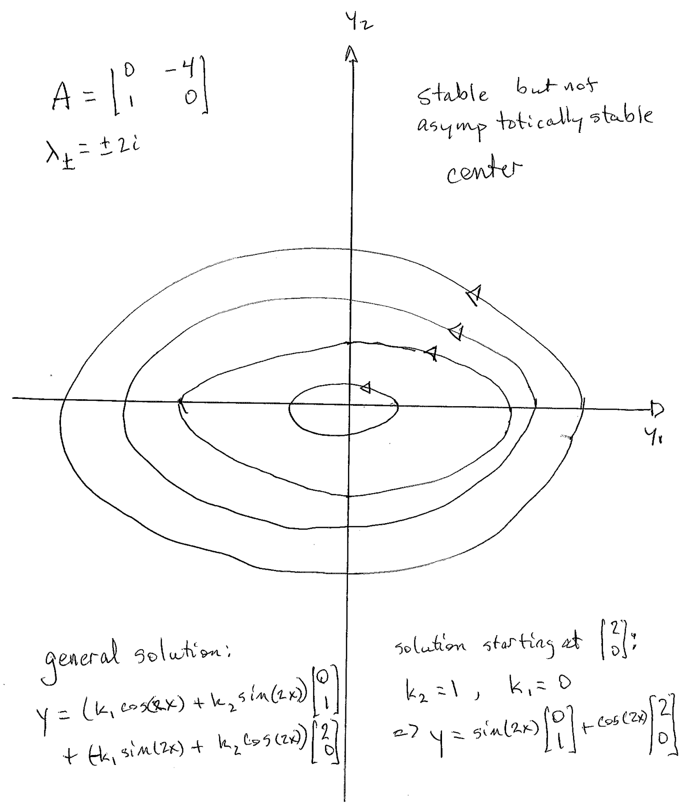

FIGURE 4. The is the phase portrait for the system $y' = Ay$ where A has eigenvalues $\pm 2i$, so the origin is a stable center (but not asymptotically stable).

Computing, we find that the eigenvalues of A are $\lambda_\pm = -1/2 \pm i$, and we have eigenvectors

$$v_\pm = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}, \quad Av_+ = \left(-\frac{1}{2} + i\right) v_+, \quad Av_- = \left(-\frac{1}{2} - i\right) v_-.$$

We can now read off that the general solution of our ODE system is

$$y(x) = e^{-x/2} \left[(k_1 \cos x + k_2 \sin x) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-k_1 \sin x + k_2 \cos x) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right].$$

We see in particular that the equilibrium $y = 0$ is asymptotically stable, and in fact all solutions spiral in to the origin.

We can now summarize the stability properties of $y = 0$ for a linear system $y' = Ay$ with constant coefficients.

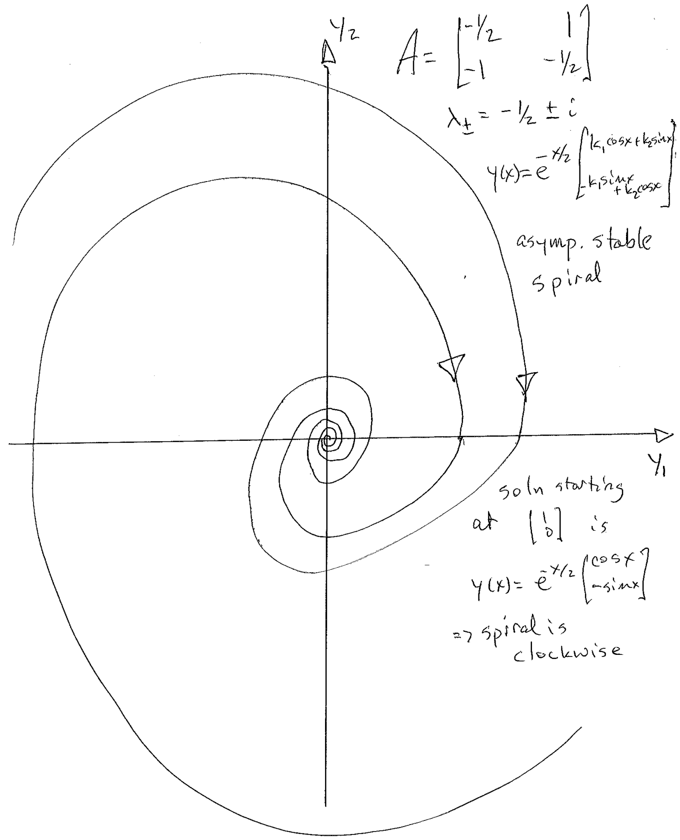

FIGURE 5. The is the phase portrait for the system $y' = Ay$ where A has eigenvalues $-1/2 \pm i$, so the origin is an asymptotically stable spiral.

Theorem 8. *Let $A \in M_{n\times n}$ be a fixed matrix whose entries are real numbers, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. Consider the system of ODEs $y' = Ay$ and observe that $y_0 = 0$ is an equilibrium solution. This equilibrium is stable if and only if the real part of λ_j is non-positive for every $j = 1, 2, \dots, n$. The solution $y_0 = 0$ is asymptotically stable if and only if the real part of λ_j is negative for every $j = 1, 2, \dots, n$.*

In fact, there is a very nice way to write out all solutions to the system of ODES

$$y' = Ay$$

in a closed form. To make sense of this, we need some more tools from linear algebra. We state the following theorem without proof.

Theorem 9. *For any $A \in M_{n\times n}$, the power series*

$$\sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

converges absolutely. We call the resulting sum the matrix exponential of A, and write it as e^A.

The matrix exponential has many of the nice properties of the usual exponential function, but not all of them. For instance, e^A is always an invertible matrix, even if A is not invertible, and $(e^A)^{-1} = e^{-A}$. However, it is **not** always true that $e^{A+B} = e^A e^B$.

Example: If $D \in M_{n\times n}$ is diagonal then e^D is easy to compute. We have shown in this case that

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & d_n^k \end{bmatrix},$$

so we can sum these to get

$$e^D = \begin{bmatrix} e^{d_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{d_n} \end{bmatrix}.$$

In the 2×2 case, we can write this as

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \Rightarrow e^D = \begin{bmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{bmatrix}.$$

Example: If A is diagonalizable, that is $A = Q^{-1}DQ$, where D is diagonal, we can also compute e^A. First we show that

$$A = Q^{-1}DQ \Rightarrow A^k = Q^{-1}D^kQ.$$

This is true by definition for $k = 1$. Suppose for some positive integer k we have $A^k = Q^{-1}D^kQ$. Then we have

$$A^{k+1} = AA^k = Q^{-1}DQQ^{-1}D^kA = Q^{-1}DD^kQ = Q^{-1}D^{k+1}Q,$$

and so the general formula is true by induction. Taking a limit of partial sums, we see

$$e^A = e^{Q^{-1}DQ} = \sum_{k=0}^{\infty} \frac{1}{k!}(Q^{-1}DQ)^k = \sum_{k=0}^{\infty} \frac{1}{k!} Q^{-1}D^kQ = Q^{-1}e^DQ.$$

Here we compute with an explicit example. Let

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} = Q^{-1}DQ,$$

where

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then

$$e^A = Q^{-1}e^D Q = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} e^3 & 0 \\ 0 & e^{-2} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6e^3 - e^{-1} & 3e^3 - 3e^{-2} \\ -2e^3 + 2e^{-2} & -e^3 + 6e^{-2} \end{bmatrix}.$$

Challenge problem: Can you find conditions on A and B such that $e^{A+B} = e^A e^B$?

We can use matrix exponentials to solve linear systems of the form (3) when A is constant.

Theorem 10. *Let $A \in M_{n\times n}$ be a fixed coefficient matrix. Then the solution to the initial value problem*

$$y' = Ay, \quad y(0) = y_0$$

is

$$y(x) = e^{xA}y_0.$$

Proof. We will only prove this theorem in the case that A is diagonalizable, that is $A = Q^{-1}DQ$, even though it is true in general. In this case, we know that the columns of Q are the eigenvectors of A, with eigenvalues equal to the corresponding diagonal entries of D. Multiply the ODE on the left by Q, so that we have

$$Qy' = QAy = Q(Q^{-1}DQ)y = DQy.$$

Now change varibles, and let $z = Qy$, so that we now have the ODE system

$$z' = Dz,$$

which we write in components as

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}' = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

This is now a decoupled system, which we can write out and solve explicitly: for $j = 1, 2, \ldots, n$, we have

$$z_j' = d_{jj}z_j \Rightarrow z_j(x) = z_j(0)e^{d_{jj}x}.$$

Putting everything together, we have the vector

$$z(x) = Qy(x) = e^{xD}(Qy_0)$$

solving the system

$$Qy' = DQy.$$

Multiplying by Q^{-1} on the left, we recover

$$y = Q^{-1}z = Q^{-1}e^{xD}Qy_0 = e^{xA}y_0,$$

which solves

$$y' = Q^{-1}DQy = Ay,$$

as we wished to prove. □

2.2. **General properties.** In this section describe some general properties of linear systems $y' = Ay$, where now A is allowed to depend on x. First of all, one can reuse our proof using Picard iteration for scalar, first order ODE, to prove existence and uniqueness.

Theorem 11. *Let $A : \mathbf{R} \to M_{n\times n}$ be continuous and consider the initial value problem*

$$y'(x) = A(x) \cdot y(x), \quad y(0) = c_0.$$

There is an $\epsilon > 0$ such that this initial value problem has a unique solution in the interval $-\epsilon < x < \epsilon$.

Of course, we expect to find n independent solutions to a given system of ODEs $y' = Ay$, where $A : \mathbf{R} \to M_{n\times n}$. This means that at any value x_0 we expect to find n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$, and we should be able to solve any initial value problem using a linear combination of these solutions: $y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots c_n y_n(x)$. However, we don't automatically know that these solutions **stay** linearly independent for all values of x. It turns out that they do.

Definition 5. *For each x let $A(x) \in M_{n\times n}$, and suppose the function $x \mapsto A(x)$ is at least differentiable. Suppose that $y_1(x), y_2(x), \ldots, y_n(x)$ all solve $y_j'(x) = A(x)y_j(x)$, and define the* **Wronskian determinant** *to be*

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \end{bmatrix}.$$

Theorem 12. *The Wronskian determinant satisfies*

$$W' = \mathrm{tr}(A)W.$$

We can write this out explicitly when $n = 2$. Let

$$A = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}, \qquad \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{bmatrix}.$$

Because the columns of $\begin{bmatrix} y_1 & y_2 \end{bmatrix}$ correspond to solutions of the system $y' = Ay$, we have

$$\begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix},$$

which we can write as

$$\alpha' = a\alpha + b\gamma, \quad \beta' = a\beta + b\delta, \quad \gamma' = c\alpha + d\gamma, \quad \delta' = c\beta + d\delta.$$

Next we write out

$$\begin{aligned} W' &= \frac{d}{dx} \det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = (\alpha\delta - \beta\gamma)' \\ &= \alpha'\delta + \alpha\delta' - \beta'\gamma - \beta\gamma' \\ &= (a\alpha + b\gamma)\delta + \alpha(c\beta + d\delta) - (a\beta + d\delta)\gamma - \beta(c\alpha + d\gamma) \\ &= a\alpha\delta + d\alpha\delta - a\beta\gamma - d\beta\gamma \\ &= (a + d)(\alpha\delta - \beta\gamma) = \mathrm{tr}(A)W. \end{aligned}$$

This theorem is called Abel's theorem, and is proved by writing out the derivative of a determinant in terms of sum of determinants of $(n-1) \times (n-1)$ cofactors. We will see another version of Abel's theorem later, when we discuss second order (scalar) ODEs. We can use Abel's theorem to conclude the following corollary.

Corollary 13. *The Wronskian is either never zero or always zero. As a consequence, either the solutions $y_1, y_2, \ldots, y_n$ are always linearly independent or never linearly independent.*

Proof. The ODE

$$W' = W \operatorname{tr}(A) \Rightarrow \frac{d}{dx} \ln(W(x)) = \operatorname{tr}(A(x)) \Rightarrow W(x) = ce^{\int_{x_0}^{x} \operatorname{tr}(A)(t)dt}$$

for some constant c. If $c \neq 0$, then W is never zero (because an exponential is never zero). On the other hand, if $c = 0$ the W is identically zero. The second statement of the corollary follows immediately from the fact that the determinant of a matrix is invertible if and only if its columns are linearly independent. $\square$

We summarize the properties of the system $y' = A(x)y$, where $A : \mathbf{R} \to M_{n\times n}$ is a continuous, matrix-valued function, with the following:

- The set of solutions is an n-dimensional vector space.
- Let $\{y_1(x), \ldots, y_n(x)\}$ and let $W(x)$ be the determinant of the matrix with y_j as its jth column. Then $W' = \operatorname{tr}(A)W$.
- Let $\{y_1(x), \ldots, y_n(x)\}$ solve $y' = A(x)y$. Then $\{y_1(x), \ldots, y_n(x)\}$ is linearly independent for all x if and only if $\{y_1(0), \ldots, y_n(0)\}$ is linearly independent.

2.3. **Non-homogeneous linear systems.** In this section we consider

$$y' = A(x)y + g(x),$$

where g is a vector valued function. An example we will consider at the end of the section is

$$y' = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} y + \begin{bmatrix} e^{-x} \\ x \end{bmatrix}.$$

Associated to the non-homogeneous system we have the homogeneous system $y' = A(x)y$, where we set $g = 0$.

Theorem 14. *The general solution of $y' = A(x)y + g(x)$ is $y = y_p + y_h$, where y_p is* **any** *particular solution of the non-homogeneous system and y_h is general solution of the associated homogeneous system $y' = Ay$.*

Proof. Let y_p be some solution of $y' = Ay + g$ and suppose y is any other solution of the same system. Then

$$(y - y_p)' = y' - y_p' = (Ay + g) - (Ay_p + g) = A(y - y_p),$$

so $y - y_p = y_h$ must solve the homogeneous system $y' = Ay$. $\square$

By this theorem, we can find the general solution of $y' = Ay + g$ by following the steps below.

(1) Find the general solution y_h of the associated homogeneous system $y' = Ay$.

(2) Find some particular solution y_p of $y' = Ay + g$, regardless of initial conditions.
(3) The general solution of $y' = Ay + g$ is then $y = y_h + y_p$.

Example: Find the general solution of

$$y' = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} y + \begin{bmatrix} e^{-x} \\ x \end{bmatrix}.$$

By our work above, we know A has eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad Av_1 = 3v_1, \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad Av_2 = -2v_2,$$

so the general solution of the associated homogeneous system is

$$y_h = c_1 e^{3x} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2x} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We also found that $A = Q^{-1}DQ$ where

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

Notice that the columns of Q are the eigenvectors of A; this is always the case when A is diagonalizable, *i.e.* when A has a basis of eigenvectors. Now our non homogeneous system reads

$$y' = Q^{-1}DQy + g(x).$$

Multiply on the left by Q and make the change of variables $z = Qy$, so that we obtain

$$z' = Qy' = DQy + Qg = Dz + Qg,$$

which we write as

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 3z_1 \\ -2z_2 \end{bmatrix} + \begin{bmatrix} 2e^{-x} + x \\ e^{-x} + 3x \end{bmatrix}.$$

This system is now decoupled, and has become two separate linear, first order ODEs, each of which we can solve. We write these out separately as

$$z_1' = 3z_1 + 2e^{-x} + x, \qquad z_2' = -2z_2 + e^{-x} + 3x,$$

which have the solutions

$$z_1 = -\frac{1}{2}e^{-x} - \frac{1}{3}x - \frac{1}{9} + c_1 e^{3x}, \qquad z_2 = e^{-x} + \frac{3}{2}x - \frac{3}{4} + c_2 e^{-2x},$$

or (in matrix notation)

$$z = e^{-x} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + x \begin{bmatrix} -1/3 \\ 3/2 \end{bmatrix} + \begin{bmatrix} -1/9 \\ -3/4 \end{bmatrix} + c_1 e^{3x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2x} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Finally, we recover y by writing $y = Q^{-1}z$.

In the case of this particular system, there is an alternative method to find a particular solution. Notice that the inhomogeneous terms are all formed by exponentials and linear functions, so we should guess that y_p would a sum of exponentials and linear functions. This is because the derivative of an exponential is another exponential, and the derivative of a linear function is a constant. Thus we should guess

$$y_p = \begin{bmatrix} a_1 e^{-x} + b_1 x + c_1 \\ a_2 e^{-x} + b_2 x + c_2 \end{bmatrix}.$$

Now plug this into the system of ODEs:

$$y_p' = e^{-x} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and

$$\begin{aligned} Ay_p + g &= \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a_1 e^{-x} + b_1 x + c_1 \\ a_2 e^{-x} + b_2 x + c_2 \end{bmatrix} \\ &= e^{-x} \begin{bmatrix} 4a_1 - 2a_2 + 1 \\ 3a_1 - 3a_2 \end{bmatrix} + x \begin{bmatrix} 4b_1 - 2b_2 \\ 3b_1 - 3b_2 + 1 \end{bmatrix} + \begin{bmatrix} 4c_1 - 2c_2 \\ 3c_1 - 3c_2 \end{bmatrix}. \end{aligned}$$

One can then equate $y_p' = Ay_p + g$ and solve for the unknown coefficients $a_1, a_2, b_1, b_2, c_1, c_2$. This method is called the **method of undetermined coefficients**, and we will see it again later on in the course.

3. Nonlinear systems

In this section we will study some of the properties of nonlinear systems of ODEs. Much as we did for difference equations, we will concentrate on finding equilibrium solutions an analysing their stability properties. Also, for the sake of simplicity, we will concentrate on 2×2 systems (*i.e.* systems of two equations in two unknown functions). However, many of the properties we will discuss carry over to larger $n \times n$ systems, with more equations and unknown functions.

Our general setting will be a vector-valued ODE of the form

$$y'(x) = F(x, y(x)), \quad x \in \mathbf{R}, \quad y(x) \in \mathbf{R}^n, \quad F : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n.$$

We will only discuss the autonomous case, that is

$$y : \mathbf{R} \to \mathbf{R}^n, \quad F : \mathbf{R}^n \to \mathbf{R}^n, \quad y' = F(y).$$

At this point, we first list some examples.

Example: Here we just write down a particular example, which we will later see (in tutorials) can model the populations of a pair of competing species. We consider

$$y_1' = y_1(1 - y_1 - y_2), \quad y_2' = y_2 \left(\frac{3}{4} - y_2 - \frac{1}{2} y_1 \right) \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1(1 - y_1 - y_2) \\ y_2 \left(\frac{3}{4} - y_2 - \frac{1}{2} y_1 \right) \end{bmatrix}.$$

Example: We consider a pendulum. We let a mass m swing free on a rod of length l (of negligible mass), and use θ to denote the angle the rod makes with the veritcal direction. The force of gravity pull the mass downward, so that we have

$$-mgl \sin\theta = ml^2 \frac{d^2\theta}{dt^2} \Leftrightarrow \theta'' = -\frac{g}{l} \sin\theta.$$

We can convert this to a first order system by letting $\phi_1 = \theta$ and $\phi_2 = \theta'$, so that we have the system of ODEs

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' = \begin{bmatrix} \phi_2 \\ -\frac{g}{l}\sin(\phi_1) \end{bmatrix}.$$

This is a nonlinear system of first order ODEs in the unknown functions ϕ_1 and ϕ_2. It is coupled, because the ODE for ϕ_1 involves ϕ_2, and the ODE for ϕ_2 involves ϕ_1.

Example: We can refine the previous example by introducing some dampling. This means we take say there is some friction in the pivot point of the pendulum, so that the original ODE becomes

$$\theta'' = -\frac{g}{l}\sin\theta - \frac{c}{ml}\theta'.$$

(The factor in front of the θ' term is chosen to make certain things turn out nicer.) We can again make the substitution $\phi_1 = \theta$ and $\phi_2 = \theta'$ to obtain the system

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' = \begin{bmatrix} \phi_2 \\ -\frac{g}{l}\sin\phi_1 - \frac{c}{ml}\phi_2 \end{bmatrix}.$$

3.1. **Equilibria.** In this section we discuss equilibrium solutions of autonomous systems of first order ODEs. Again, these systems all have the form

$$y'(x) = F(y(x)), \quad y : \mathbf{R} \to \mathbf{R}^n, \quad F : \mathbf{R}^n \to \mathbf{R}^n.$$

Definition 6. *A solution $y(x)$ of $y' = F(y)$ is an* **equilibrium solution** *if y is independent of x, i.e. y is some constant vector c. One can also call and equilibrium solution a* **fixed point** *of the system.*

Theorem 15. *A constant vector $c \in \mathbf{R}^n$ is an equilibrium if and only if $F(c) = 0$.*

Proof. An equilibrium solution $y(x)$ must be constant, so that $y' = 0$. If this equilibrium is the constant vector c, then we have

$$0 = \frac{d}{dx}(c) = F(c),$$

which proves that and equilibrium must satisfy $F(c) = 0$. Now suppose that $F(c) = 0$, and set $y(x) = c$ for all x. Then we have

$$F(c) = 0 = \frac{d}{dx}(c),$$

so $y \equiv c$ is indeed an equilibrium solution. □

Example: In this example we find the fixed points of the competing species system we wrote down in the previous section. We have

$$y_1' = y_1(1 - y_1 - y_2), \quad y_2' = y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1(1 - y_1 - y_2) \\ y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \end{bmatrix},$$

from which we can read off

$$F(y) = F\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1(1 - y_1 - y_2) \\ y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \end{bmatrix}.$$

To find the fixed points we must find the zeroes of F:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = F(y) = \begin{bmatrix} y_1(1-y_1-y_2) \\ y_2\left(\frac{3}{4}-y_2-\frac{1}{2}y_1\right) \end{bmatrix}. \Leftrightarrow 0 = y_1(1-y_1-y_2), \quad 0 = y_2\left(\frac{3}{4}-y_2-\frac{1}{2}y_1\right).$$

We have several cases to consider. First, if $y_1 = 0$, then the second equation reads

$$0 = y_2\left(\frac{3}{4}-y_2\right) \Rightarrow y_2 = 0 \text{ or } y_2 = \frac{3}{4},$$

so we have the fixed points

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3/4 \end{bmatrix}.$$

Next we set

$$y_2 = 0 \Rightarrow y_1(1-y_1) = 0 \Rightarrow y_1 = 0 \text{ or } 1.$$

Finally, we consider the case that neither $y_1 = 0$ nor $y_2 = 0$, so we must have

$$1 - y_1 - y_2 = 0, \quad \frac{3}{4} - y_2 - \frac{1}{2}y_1 = 0 \Rightarrow y_1 = \frac{1}{2} = y_2.$$

Putting this all together, we have the following four fixed points:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3/4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Example: Next we find the fixed points of a free pendulum. The free pendulum satisfies

$$\theta'' = -\frac{g}{l}\sin\theta \Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' = \begin{bmatrix} \phi_2 \\ -\frac{g}{l}\sin\phi_1 \end{bmatrix}$$

where $\phi_1 = \theta$ and $\phi_2 = \theta'$. To find the fixed points we set

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = F(\phi) = \begin{bmatrix} \phi_2 \\ \frac{g}{l}\sin\phi_1 \end{bmatrix} \Leftrightarrow \phi_2 = 0, \quad \sin\phi_1 = 0.$$

Thus the equilibrium solutions are

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \Leftrightarrow \theta = k\pi,$$

where k is any integer. We can interpret this physically as saying that the equilibria correspond to the pendulum either hanging straight down or pointing straight up.

Exercise: Show that the damped pendulum,

$$\theta'' = -\frac{g}{l}\sin\theta - \frac{c}{ml}\theta' \Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' = \begin{bmatrix} \phi_2 \\ -\frac{g}{l}\sin\phi_1 - \frac{c}{ml}\phi_2 \end{bmatrix}$$

has exactly the same fixed points as the free pendulum.

3.2. **Stability.** To analyze the stability of the system $y' = F(y)$ near a fixed point $y = c$, we would like to find the linear system which best approximates $y' = F(y)$ near $y = c$. We can do this by replacing F with its first order Taylor polynomial, which is exactly the linear function approximating F the best. In this section we will only treat the case of 2×2 systems, but the theory for larger systems is nearly identical.

Here we write

$$y' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = F(y) = \begin{bmatrix} F_1(y_1, y_2) \\ F_2(y_1, y_2) \end{bmatrix}.$$

Let $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ be a fixed point, and write out the linear approximation of F near $y = c$:

$$F(y) \simeq F(c) + DF|_{y=c}(y - c) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(c_1, c_2) & \frac{\partial F_1}{\partial y_2}(c_1, c_2) \\ \frac{\partial F_2}{\partial y_1}(c_1, c_2) & \frac{\partial F_2}{\partial y_2}(c_1, c_2) \end{bmatrix} \begin{bmatrix} y_1 - c_1 \\ y_2 - c_2 \end{bmatrix}.$$

If we let $u_1 = y_1 - c_1$ and $u_2 = y_2 - c_2$ and replace F with its linearization we now have the first order system of ODEs

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where all partial derivatives of F_1 and F_2 are evaluated at (c_1, c_2).

Example: We linearize the system

$$y_1' = y_1(1 - y_1 - y_2), \quad y_2' = y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1(1 - y_1 - y_2) \\ y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \end{bmatrix}$$

about each of its fixed points. First we compute some partial derivatives:

$$\frac{\partial F_1}{\partial y_1} = 1 - 2y_1 - y_2, \quad \frac{\partial F_2}{\partial y_1} = -\frac{1}{2}y_2, \quad \frac{\partial F_1}{\partial y_2} = -y_1, \quad \frac{\partial F_2}{\partial y_2} = \frac{3}{4} - 2y_2 - \frac{1}{2}y_1.$$

Next we evaluate the matrix of partial derivatives of F at each fixed point. We have

$$\left.\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix}, \quad \left.\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}\right|_{(0,3/4)} = \begin{bmatrix} 1/4 & 0 \\ -3/8 & -3/2 \end{bmatrix}$$

and

$$\left.\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}\right|_{(1,0)} = \begin{bmatrix} -21 & -1 \\ 0 & 1/4 \end{bmatrix}, \quad \left.\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}\right|_{(1/2,1/2)} = \begin{bmatrix} -1/2 & -1/2 \\ -1/4 & -3/2 \end{bmatrix}.$$

For instance, linearizing about $(0, 0)$ we obtain the system $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} u_1 \\ (3/4)u_2 \end{bmatrix}$, which is a source node, but linearlizing about $(1, 0)$ we obtain the system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} -21 & -1 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This latter system has a coefficient matrix with one positive and one negative eigenvalue, so we expect the phase portrait to have a saddle point there. This is, in fact, exactly what happens. We can also see that $(0, 3/4)$ is another saddle point (with one positive and one negative eigenvalue). With a little computation, we also see that the eigenvalues of the coefficient matrix at $(1/2, 1/2)$ are

$$\lambda_\pm = \frac{-7 \pm \sqrt{29}}{8} < 0,$$

so this point turns out to be a sink node.

Theorem 16. *Let $c \in \mathbf{R}^n$ be a fixed point of the first order, autonomous system $y' = F(y)$ of n equations in n unknown functions $y_1, \dots, y_n$, and let $A \in M_{n\times n}$ be the matrix*

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}.$$

If A has an eigenvalue λ_j with the real part of λ_j positive, then the system $y' = F(y)$ is unstable near c. On the other hand, if the real part of λ_j is negative for **all** *eigenvalues λ_j of A, then $y' = F(y)$ is asymptotically stable near c. If the real parts of all the eigenvalues of A are zero then the test is inconclusive.*

Example: We can see from this theorem that the system

$$y_1' = y_1(1 - y_1 - y_2), \quad y_2' = y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1(1 - y_1 - y_2) \\ y_2\left(\frac{3}{4} - y_2 - \frac{1}{2}y_1\right) \end{bmatrix}$$

is asymptotically stable near the fixed point $(1/2, 1/2)$, and unstable near the fixed points $(0, 0)$, $(1, 0)$, and $(0, 3/4)$.

Example: We have seen that the fixed points of the system corresponding to the free pendulum are $(k\pi, 0)$ where k is an integer. We linearlize the system to obtain

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(k\pi) & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (-1)^{k+1}\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

In this case, the eigenvalues are given by $\lambda_\pm^2 = (-1)^{k+1}\frac{g}{l}$. If k is odd we have one positive and one negative eigenvalue, so the system is unstable near these fixed points. If k is even then we have pure imaginary eigenvalues, and the test is inconclusive.

Exercise: Do the same stability analysis for the fixed points of the damped pendulum:

$$\theta'' = -\frac{g}{l}\sin\theta - \frac{c}{ml}\theta' \Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}' = \begin{bmatrix} \phi_2 \\ -\frac{g}{l}\sin\phi_1 - \frac{c}{ml}\phi_2 \end{bmatrix}.$$

We close this section of notes with a table listing the stability properties of fixed points for the case of 2×2 systems. In this table, λ_1 and λ_2 are the eigenvalues of either the coefficient matrix (in the case of linear systems) or the linearization F at the fixed point.

	linear stability	systems type	nonlinear stability	systems type
λ_1, λ_2 both positive	unstable	source node	unstable	source node
λ_1, λ_2 both negative	asymp. stable	sink node	asymp. stable	sink node
$\lambda_1 < 0$, $\lambda_2 > 0$	unstable	saddle	unstable	saddle
$\lambda_1 > 0$, $\lambda_2 = 0$	unstable	line of fixed points	unstable	inconclusive
$\lambda_1 < 0$, $\lambda_2 = 0$	stable	line of fixed points	inconclusive	inconclusive
$\lambda_\pm = a + ib$, $a > 0$	unstable	spiral	unstable	spiral
$\lambda_\pm = a + ib$, $a < 0$	asymp. stable	spiral	asymp. stable	spiral
$\lambda_\pm = ib$	stable, not asymp.	center	inconculsive	inconclusive