NOTES FOR SECOND YEAR DIFFERENTIAL EQUATION PART IV:SECOND ORDER, LINEAR ODES

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1. Introduction

In this set of notes we examine linear, second order ODEs, concentrating on those with constant coefficients. Our ODEs will have the general form

$$a_2 y'' + a_1 y' + a_0 y = g(x);$$

usually a_2, a_1, a_0 are constants, with $a_2 \neq 0$, but sometimes we will allow the coefficients a_2, a_1, a_0 to depend on x. The right hand side $g(x)$ is a given function. Notice that the ODE involves two derivatives of y, so we should expect to assign two initial values for y. In fact, this is the case, and we will generally solve an initial value problem of the form

$$a_2 y'' + a_1 y' + a_0 y = g(x), \quad y(0) = c_0, \quad y'(0) = c_1,$$

where c_0 and c_1 are given numbers. At this point, we need to remark on one important fact:

Theorem 1. *Any initial value problem*

$$a_2 y'' + a_1 y' + a_0 y = g(x), \quad y(0) = c_0, \quad y'(0) = c_1$$

has a unique solution, at least for x in some small interval containing 0*.*

This fact follows from the corresponding fact for first order systems, because one can transform a second order scalar ODE into a 2×2 first order system of ODEs. To transform this single, second order ODE into a first order system, we let

$$y_1 = y, \quad y_2 = y'$$

so that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}.$$

We will first concentrate on the homogeneous case, that is when $g(x) \equiv 0$, and then describe two methods to solve the non-homogeneous case, when $g \neq 0$.

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2. The homogeneous case

2.1. **Constant coefficients.** In this case, we wish to solve

$$a_2 y'' + a_1 y' + a_0 y = 0. \tag{1}$$

We first try to guess a solution. We do know a function which repeats itself under differentiation: the exponential function. Thus it's reasonable to guess that $y(x) = e^{rx}$ for some number r. Let's substitute this in and see what we get:

$$0 = a_2(e^{rx})'' + a_1(e^{rx})' + a_0 e^{rx} = e^{rx}(a_2 r^2 + a_2 r + a_0) \Rightarrow 0 = a_2 r^2 + a_1 r + a_0.$$

This is a quadratic equation for r, so we should find two roots,

$$r_\pm = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}.$$

Example: We consider

$$y'' - 5y' + 6y = 0,$$

and try to find solutions of the form $y(x) = e^{rx}$. Substituting in, we see

$$0 = (e^{rx})'' - 5(e^{rx})' + 6e^{rx} = e^{rx}(r^2 - 5r + 6) \Rightarrow r^2 - 5r + 6 = 0 \Rightarrow r = 2 \text{ or } r = 3.$$

So we have two possible solutions, $y_1 = e^{2r}$ and $y_2 = e^{3x}$. Which one do we choose? The answer is we must choose both.

Theorem 2. *Suppose that both $y_1(x)$ and $y_2(x)$ solve*

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

Then so does $\alpha_1 y_1 + \alpha_2 y_2$ for any constants α_1 and α_2.

This theorem is called the **principle of superposition**, and it is very important. We will see that we really only need the ODE to be linear and homogeneous in order that the principle of superposition holds.

Proof. Let $y = \alpha_1 y_1 + \alpha_2 y_2$, and observe that

$$\begin{aligned} a_2 y'' + a_1 y' + a_0 y &= a_2(\alpha_1 y_1 + \alpha_2 y_2)'' + a_1(\alpha_1 y_1 + \alpha_2 y_2)' + a_0(\alpha_1 y_1 + \alpha_2 y_2) \\ &= \alpha_1(a_2 y_1'' + a_1 y_1' + a_0 y_1) + \alpha_2(a_2 y_2'' + a_1 y_2' + a_0 y_1) = 0. \end{aligned}$$

□

Corollary 3. *Let V be the set of functions $y(x)$ such that $a_2 y'' + a_1 y' + a_0 y = 0$. Then V is a two-dimensional vector space.*

Proof. The principle of superposition tells us that V is closed under scalar multiplication and addition, so V must be a vector space (in particular, a subspace of the vector space of all continuous functions). It remains to find $\dim(V)$, which is really a count of how many free parameters we get to choose in specifying a solution to the ODE. We have already seen that the solution to the initial value problem

$$a_2 y'' + a_1 y' + a_0 y = 0, \quad y(0) = c_0, \quad y'(0) = c_1$$

exists and is unique, so we (always!) get to choose exactly two parameters in specifying y, namely c_0 and c_1. Thus $\dim(V) = 2$. □

We remark that a handy basis of V might be the solution with $y_1(0) = 1$ and $y_1'(0) = 0$, together with the solution with $y_2(0) = 0$ and $y_2'(0) = 1$.

Now we can find the general solution to the ODE

$$y'' - 5y' + 6y = 0.$$

We already have two solutions $y_1(x) = e^{2x}$ and $y_2x) = e^{3x}$, and we know the set of solutions is a two-dimensional vector space. This means the general solution of our ODE must have the form

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) = \alpha_1 e^{2x} + \alpha_2 e^{3x}.$$

Suppose further that we want to solve an initial value problem, say

$$y'' - 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

We know the solution must have the form $y(x) = \alpha_1 e^{2x} + \alpha_2 e^{3x}$ for some constants α_1 and α_2, so we match the initial conditions. We have

$$2 = y(0) = \alpha_1 + \alpha_2, \quad -1 = y'(0) = 2\alpha_1 + 3\alpha_2 \Rightarrow \alpha_1 = 7, \quad \alpha_2 = -5.$$

We conclude $y(x) = 7e^{2x} - 5e^{3x}$.

Example: We show now that we obtain the characteristic equation after transforming the single equation into a system. Recall that the coefficient matrix is now

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_1} & -\frac{a_1}{a_2} \end{bmatrix}, \qquad \begin{bmatrix} y \\ y' \end{bmatrix}' = A \begin{bmatrix} y \\ y' \end{bmatrix}.$$

The characteristic polynomial is then

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -\frac{a_0}{a_2} & -\lambda - \frac{a_1}{a_2} \end{bmatrix} = \lambda^2 + \frac{a_1}{a_2}\lambda + \frac{a_0}{a_2} = 0,$$

which is exactly the same characteristic polynomial we obtain by plugging in $y = e^{rt}$ to $a_2 y'' + a_1 y' + a_0 y = 0$.

We now outline a general theory.

Theorem 4. *Let $a_2 r^2 + a_1 r + a_0$ be a quadratic with two distinct roots $r_1 \neq r_2$. Then the general solution to the ODE*

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$y(x) = \alpha_1 e^{r_1 x} + \alpha_2 e^{r_2 x}.$$

Proof. We try to find solutions to the ODE of the form $y(x) = e^{rx}$ for some real number r. Substituting, we have

$$0 = a_2(e^{rx})'' + a_1(e^{rx})' + a_0(e^{rx}) = e^{rx}(a_2 r^2 + a_1 r + a_0) \Rightarrow a_2 r^2 + a_1 r + a_0 = 0.$$

By assumption, this quadratic equation has precisely two roots, namely $r = r_1$ and $r = r_2$, so we have now found two linearly independent solutions

$$y_1(x) = e^{r_1 x}, \quad y_2(x) = e^{r_2 x}.$$

Then by the principle of superposition we can write the general solution to our ODE as a linear combination of y_1 and y_2. In other words, our general solution must be of the form

$$y(x) = \alpha_1 e^{r_1 x} + \alpha_2 e^{r_2 x}.$$

□

Example: We solve the initial value problem

$$y'' - 4y' + 3y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

Trying solutions of the form $y(x) = e^{rx}$, so that

$$0 = (e^{rx})'' - 4(e^{rx})' + 3e^{rx} = e^{rx}(r^2 - 4r + 3) = e^{rx}(r-1)(r-3) \Rightarrow r = 1 \text{ or } r = 3.$$

Thus the general solution to our ODE is

$$y(x) = \alpha_1 e^{x} + \alpha_2 e^{3x}.$$

Matching the initial condition, we have

$$-2 = y(0) = \alpha_1 + \alpha_2, \quad 1 = y'(0) = \alpha_1 + 3\alpha_2 \Rightarrow \alpha_1 = -\frac{7}{2}, \quad \alpha_2 = \frac{3}{2}.$$

Thus the solution to our initial value problem is

$$y(x) = -\frac{7}{2}e^{x} + \frac{3}{2}e^{3x}.$$

Example: We solve the initial value problem

$$y'' - y' + \frac{5}{4}y = 0, \quad y(0) = -1, \quad y'(0) = 2.$$

As usual, we try a solution of the form $y = e^{rx}$ to see

$$0 = (e^{rx})'' - (e^{rx})' + \frac{5}{4}e^{rx} \Rightarrow r^2 - r + \frac{5}{4} = 0 \Rightarrow r = \frac{1}{2} \pm i.$$

Our general solution is now

$$y(x) = c_+ e^{(1/2+i)x} + c_- e^{(1/2-i)x} = e^{x/2}[c_+(\cos x + i\sin x) + c_-(\cos x - i\sin x)] = e^{x/2}(k_1\cos x + k_2\sin x),$$

where $k_1 = c_+ + c_-$ and $k_2 = i(c_+ - c_-)$. Again, we find the constants k_1 and k_2 by matching the initial condition:

$$-1 = y(0) = k_1, \quad 2 = y'(0) = \frac{1}{2}k_1 + k_2 \Rightarrow k_2 = \frac{5}{2}.$$

Now our solution in

$$y(x) = e^{x/2}\left[-\cos x + \frac{5}{2}\sin x\right].$$

We have one final case to consider, that of a repeated root in the characteristic polynomial $a_2 r^2 + a_1 r + a_0$.

Theorem 5. *Let $a_2r^2 + a_1r + a_0$ be a quadratic polynomial with a repeated root r_*. Then the general solution to the ODE*

$$a_2y'' + a_1y' + a_0y = 0$$

is

$$y(x) = \alpha_1 e^{r_* x} + \alpha_2 x e^{r_* x}.$$

Proof. From our previous analysis, we already know that $y_1(x) = e^{r_* x}$ is one solution to our ODE, so the only thing that remains is to find a second, linearly independent solution. We try $y_2 = xe^{r_* x}$ and see if it works:

$$\begin{aligned} a_2(xe^{r_*x})'' + a_1(xe^{r_*x})' + a_0(xe^{r_*x}) &= a_2(2r_*e^{r_*x} + r_*^2xe^{r_*x}) + a_1(e^{r_*x} + r_*xe^{r_*x}) + a_0xe^{r_*x} \\ &= xe^{r_*x}(a_2r_*^2 + a_1r_* + a_0) + e^{r_*x}(2a_2r_* + a_1) \\ &= e^{r_*x}(2a_2r_* + a_1). \end{aligned}$$

Here we have used the fact that we know r_* is a root of the quadratic $a_2r^2+a_1r+a_0$. However, we still have not used the fact that r_* is a double root. This can only happen if

$$a_2r^2 + a_1r + a_0 = a_2(r - r_*)^2 = a_2(r^2 - 2r_*r + r_*^2) \Rightarrow a_1 = -2r_*a_2 \Rightarrow 2a_2r_* + a_1 = 0.$$

We conclude that in this special case $y_2 = xe^{r_*x}$ must be a solution, so by the superposition principle the general solution to our ODE is

$$y(x) = \alpha_1 e^{r_* x} + \alpha_2 x e^{r_* x}.$$

□

Notice that this proof **only works when we have a double root.**
Example: We solve the initial value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -3.$$

Again, we try a solution of the form $y = e^{rx}$ and find

$$0 = (e^{rx})'' + 4(e^{rx})' + 4(e^{rx}) = e^{rx}(r^2 + 4r + 4) \Rightarrow r = -2.$$

In this case we have a double root of $r_* = -2$, so the general solution has the form $y(x) = \alpha_1 e^{-2x} + \alpha_2 x e^{-2x}$. We find the coefficients by matching the initial conditions. We have

$$2 = y(0) = \alpha_1, \quad -3 = y'(0) = -2\alpha_1 + \alpha_2 \Rightarrow \alpha_2 = 7.$$

Thus our solution is

$$y(x) = 2e^{-2x} - 7xe^{-2x}.$$

2.2. **General theory.** Recall that we can solve the initial value problem

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad y(0) = c_0, \quad y'(0) = c_1$$

for any choice of initial conditions c_0, c_1, which gives us a two-dimensional family of solutions. By the superposition principle, this two-dimensional family is is a vector space, and by the uniqueness of the solution to our initial value problem this two-dimensional vector space is the entire solution space of our ODE. We summarize what we have found with the following theorem.

Theorem 6. *Let $a_2(x), a_1(x), a_0(x)$ be continuous functions, with $a_2 \neq 0$. Then there is $\epsilon > 0$ such that the set of solutions of*

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \qquad -\epsilon < x < \epsilon$$

is a two-dimensional vector space.

At this point, we simplify the notation a little and divide through by $a_2(x)$, so that our ODE becomes

$$0 = y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = y'' + p_1(x)y' + p_0(x)y, \qquad p_1 = \frac{a_1}{a_2}, \quad p_0 = \frac{a_0}{a_2}.$$

We would like to have a simple test to see if two solutions we find span the solution space. To this end, we let $y_1(x), y_2(x)$ be solutions and define the **Wronskian determinant** of the functions y_1, y_2 by

$$W(y_1, y_2)(x) = W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

This is a somewhat familiar object from our work with first order systems of ODEs.

Lemma 7. *We have $W' = -p_1(x)W$.*

Proof. We differentiate our formula for W and use the fact that both y_1 and y_2 solve the original ODE to see

$$\begin{aligned} W' &= (y_1y_2' - y_2y_1')' \\ &= y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1'' \\ &= y_1(-p_1y_2' - p_0y_2) - y_2(-p_1y_1' - p_0y_1) \\ &= -p_1y_1y_2' + p_1y_2y_1' = -p_1W. \end{aligned}$$

□

Theorem 8. *The Wronskian of two solutions y_1 and y_2 is either always zero or never zero. In particular, either the functions y_1 and y_2 are always independent or never independent.*

Proof. We solve the ODE for W:

$$W' = -p_1W \Rightarrow \frac{W'}{W} = \frac{d}{dx}\ln(W) = -p_1 \Rightarrow W(x) = ce^{-\int p(x)dx}$$

for some constant x. Since the exponential is never zero, we see that W is either never zero (if $c \neq 0$) or always zero (if $c = 0$). □

The importance of this theorem is that we can determine whether y_1 and y_2 form a basis of the solution space by evaluating at a single point, say $x = 0$. In fact, we can also use this theorem to find a second linearly independent solution if we're given a first one (see the tutorial problems).

We close this section with a parallel between the case of second order, linear ODE and first order systems. If

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

then we can tranfrom into a 2×2 flinear, irst order system by letting $z_1 = y$ and $z_2 = y'$, so that

$$z' = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{a_0(x)}{a_2(x)} & -\frac{a_1(x)}{a_2(x)} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = A(x)z.$$

In the case that a_2, a_1, a_0 are all constant, we can solve the first order system using eigenvalues and eigenvectors. In this case, the eigenvalues λ of the coefficient matrix A satisfy

$$0 = \det(A - \lambda I) = \lambda^2 + \frac{a_1}{a_2}\lambda + \frac{a_0}{a_2},$$

which is exactly the same as the characteristic polynomial we found in Theorem 4. In this way we recover exactly the same solutions we found before.

3. The non-homogeneous case

Here we consider initial value problems for non-homogeneous equations. We will discuss two methods, the method of undetermined coefficients, and that of variation of parameters.

One of our guiding principles is that the general solution of a non-homogeneous ODE is the sum of the solution to the corresponding homogeneous equation and a particular solution. We write this as a theorem.

Theorem 9. *The general solution of*

$$a_2y'' + a_1y' + a_0y = g(x)$$

has the form

$$y(x) = y_p(x) + \alpha_1 y_1(x) + \alpha_2 y_2(x),$$

where y_p is **any** *solution of the non-homogeneous equation, and $\alpha_1 y_1 + \alpha_2 y_2$ is the general solution of the corresponding homogeneous equation, $a_2y'' + a_1y' + a_0y = 0$.*

Proof. Again, we use the fact that the initial value problem

$$a_2y'' + a_1y' + a_0y = g(x), \quad y(0) = c_0, \quad y'(0) = c_1 \tag{2}$$

has a unique solution. Let $y_p(x)$ be **any** particular solution to the non-homogeneous ODE, and let $b_0 = y_p(0)$ and $b_1 = y_p'(0)$. Then $y(x)$ solves the initial value problem (2) if and only if $\tilde{y}(x) = y(x) - y_p(x)$ solves the initial value problem

$$a_2\tilde{y}'' + a_1\tilde{y}' + a_0\tilde{y} = 0, \quad \tilde{y}(0) = c_0 - b_0, \quad \tilde{y}(0) = c_1 - b_1. \tag{3}$$

We know the general solution of (3) has the form

$$\tilde{y} = \alpha_1 y_1 + \alpha_2 y_2$$

as detailed in the previous section, so we must have

$$y(x) = y_p(x) + \tilde{y}(x) = y_p(x) + \alpha_1 y_1(x) + \alpha_2 y_2(x).$$

□

3.1. **The method of undetermined coefficients.** In this section we outline the method of undetermined coefficients. Our guiding observation is that some functions repeat themselves under differentiation. For instance, the derivative of an exponential function is another exponential, and the derivative of a polynomial of degree k is another polynomial, of lower degree. If the right hand side $g(x)$ is one of these functions, we can guess a good form for the particular solution y_p.

Example: Consider the initial value problem

$$y'' + 3y' + 2y = e^{3x} + \cos x, \qquad y(0) = 2, \qquad y'(0) = -1. \tag{4}$$

We start by solving the homogeneous equation

$$y'' + 3y' + 2y = 0.$$

Guess a solution of the form $y(x) = e^{rx}$ and plug it in to get

$$0 = (e^{rx})'' + 3(e^{rx})' + 2e^{rx} = e^{rx}(r^2 + 3r + 2) = e^{rx}(r+2)(r+1) \Rightarrow r = -1, -2.$$

Thus the homogeneous solution is

$$c_1 e^{-x} + c_2 e^{-2x}.$$

The general solution to (4) is a sum of the homogeneous solution listed above and a particulart solution y_p. To find y_p we use the method of undetermined coefficients and guess

$$y_p = Ae^{3x} + B\cos x + C\sin x.$$

Plugging this guess into the equation we have

$$\begin{aligned} e^{3x} + \cos x &= y_p'' + 3y_p' + 2y_p \\ &= 9Ae^{3x} - B\cos x - C\sin x + 9Ae^{3x} - 3B\sin x + 3C\cos x \\ &\quad +2Ae^{3x} + 2B\cos x + 2C\sin x \\ &= 20Ae^{3x} + (B + 3C)\cos x + (C - 3B)\sin x. \end{aligned}$$

Matching coefficients of the three different terms we get the three equations

$$1 = 20A, \qquad 1 = B + 3C, \qquad 0 = -3B + C,$$

which have the simultaneous solutions

$$A = \frac{1}{20}, \qquad B = \frac{1}{10}, \qquad C = \frac{3}{10}.$$

Putting this together, we see

$$y = \frac{1}{20}e^{3x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x + c_1 e^{-x} + c_2 e^{-2x}.$$

We now use our initial conditions to find c_1 and c_2. We have

$$\begin{aligned} 2 &= y(0) = \frac{1}{20} + \frac{1}{10} + c_1 + c_2 = \frac{3}{20} + c_1 + c_2 \\ -1 &= y'(0) = \frac{3}{20} + \frac{3}{10} - c_1 - 2c_2. \end{aligned}$$

These equations have the solution

$$c_1 = \frac{9}{4}, \qquad c_2 = -\frac{2}{5}.$$

This general technique works most of the time when the right hand side has a certain form, but there is an important caveat: the right hand side $g(x)$ cannot be a solution to the corresponding homogeneous equation. If g solves the homogeneous equation, we must do something slightly different.

Example: Solve the initial value problem

$$y'' - 5y' + 6y = e^{2x}, \quad y(0) = 1, \quad y'(0) = 3.$$

In this case, we know that the associated homogeneous equation, $y'' - 5y' + 6y = 0$ has the general solution $c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{3x}$. In particular, we see that the right hand side e^{2x} does indeed solve the homogeneous equation, so we cannot hope to solve the non-homogeneous solution with $y_p = Ae^{2x}$. This time we try $y_p = Axe^{2x}$, and plug in our guess. We have

$$(Axe^{2x})'' - 5(Axe^{2x})' + 6(Axe^{2x}) = xe^{2x}(4A - 10A + 6A) + e^{2x}(4A - 5A) = -Ae^{2x}.$$

Matching this to the given right hand side, $g(x) = e^{2x}$, we see $A = -1$, and so $y_p = -xe^{2x}$ and our general solution to the non-homogenoue equation is

$$y = -xe^{2x} + c_1 e^{2x} + c_2 e^{3x}.$$

We evaluate y and y' at $x = 0$ to match the initial conditions, given

$$1 = y(0) = c_1 + c_2, \quad 3 = y'(0) = -1 + 2c_1 + 3c_2 \Rightarrow c_1 = -1, \quad c_2 = 2,$$

and therefore $y = -xe^{2x} - e^{2x} + 2e^{3x}$.

We summarize the possible forms of the particular solution y_p in the table below. Here the model ODE is

$$a_2 y'' + a_1 y' + a_0 y = g(x).$$

If g has several of the terms listed below then we must include each of the corresponding terms for y_p. This table is not quite complete, but it should give you a good idea of the type to equation you can solving using the method of undetermined coefficients.

term in $g(x)$	corresponding term in y_p
αe^{bx}	Ae^{bx}
αe^{bx} when $a_2 b^2 + a_1 b + a_0 = 0$	Axe^{bx}
$\alpha_1 \cos(kx) + \alpha_2 \sin(kx)$	$A_1 \cos(kx) + A_2 \sin(kx)$
$\alpha \cos(kx)$ when $-a_2 k^2 + i a_1 k + a_0 = 0$	$A_1 x \cos(kx) + A_2 x \sin(kx)$
polynomial of degree k	polynomial of degree k
$p(x)e^{bx}$, where p is a polynomical of degree k	$q(x)e^{bx}$ where q is a polynomial of degree k
e^{x^2}	$(A_2 x^2 + A_1 x + A_0)e^{x^2}$

3.2. **The method of variation of parameters.** In the next example we have

$$y'' + 3y' + 2y = \frac{1}{1-x}, \qquad y(0) = 1, \qquad y'(0) = 2. \tag{5}$$

The two homogeneous solutions are the same as before, so we can go straight to finding the particular solution. In this case we can't use the method of undetermined coefficients (why?) so we use variation of parameters. We start with

$$y_p(x) = c_1(x)e^{-x} + c_2(x)e^{-2x}.$$

We compute

$$y_p' = c_1' e^{-x} + c_2' e^{-2x} - c_1 e^{-x} - 2c_2 e^{-2x}.$$

We have some freedom in choosing the functions c_1 and c_2, so we set $c_1' e^{-x} + c_2' e^{-2x} = 0$. Now we take a further derivative to get

$$y_p'' = -c_1' e^{-x} - c_2' e^{-2x} + c_1 e^{-x} + 4c_2 e^{-2x}.$$

Plug all this into (5) to get

$$\begin{aligned}
\frac{1}{1-x} &= y_p'' + 3y_p' + 2y_p \\
&= -c_1' e^{-x} - 2c_2' e^{-2x} + c_1 e^{-x} + 4c_2 e^{-2x} - 3c_1 e^{-x} - 6c_2 e^{-2x} + 2c_1 e^{-x} + 2c_2 e^{-2x} \\
&= -c_1' e^{-x} - 2c_2 e^{-2x}.
\end{aligned}$$

We combine this with the equation $c_1' e^{-x} + c_2' e^{-2x} = 0$ to get a system of two equations in two unknows. Add these two equations together to get

$$\frac{1}{1-x} = -c_2' e^{-2x} \Rightarrow c_2' = \frac{e^{2x}}{x-1} \Rightarrow c_2(x) = \int_0^x \frac{e^{2s}}{s-1} ds.$$

We can evaluate

$$c_2(0) = 0, \qquad c_2'(0) = -1.$$

Next we use our system of equations to find c_1. Indeed,

$$c_1' = -e^{-x} c_2' = \frac{e^x}{1-x} \Rightarrow c_1 = \int_0^x \frac{e^s}{1-s} ds.$$

Again we can evaluate to get

$$c_1(0) = 0, \qquad c_1'(0) = 1.$$

Finally we use the initial conditions of (5) to find our solution. We have

$$y(x) = e^{-x} \int_0^x \frac{e^s}{1-s} ds + e^{-2x} \int_0^x \frac{e^{2s}}{s-1} ds + ae^{-x} + be^{-2x},$$

and we have to find the constants a and b. We evaluate at $x = 0$ to get

$$\begin{aligned} 1 &= y(0) = a + b \\ 2 &= y'(0) = -a - 2b \end{aligned}$$

which has the solution $a = 4$ and $b = -3$.

There is a general recipe for finding solutions to a non-homogeneous ODE using variation of parameters, and here it is. As always, the ODE we wish to solve is

$$a_2 y'' + a_1 y' + a_0 y = g(x).$$

(1) Find the general solution of the associated homogeneous ODE $a_2y'' + a_1y' + a_0y = 0$. This general solution has the form $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants.
(2) Try to find a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, where we replace the constants c_1 and c_2 with unknown functions $c_1(x)$, $c_2(x)$.
(3) Require additionally that $c_1'y_1 + c_2'y_2 = 0$, so that now $y_p' = c_1y_1' + c_2y_2'$.
(4) Now evaluate:

$$\begin{aligned} a_2y_p'' + a_1y_p' + a_0y_p &= a_2(c_1'y_1' + c_2'y_2' + c_1y_1'' + c_2y_2'') + a_1(c_1y_1' + c_2y_2') + a_0(c_1y_1 + c_2y_2) \\ &= a_2(c_1'y_1' + c_2'y_2') + c_1(x)(a_2y_1'' + a_1y_1' + a_0y_1) + c_2(x)(a_2y_2'' + a_1y_2' + a_0y_2) \\ &= a_2(c_1'y_1' + c_2'y_2'). \end{aligned}$$

(5) We conclude with the system of first order ODEs

$$c_1'y_1 + c_2'y_2 = 0, \quad c_1'y_1' + c_2'y_2' = \frac{g(x)}{a_2}.$$

Recall that y_1 and y_2 are now known functions, so the unknown functions we wish to find are $c_1(x)$ and $c_2(x)$.
(6) We can solve this system of linear ODEs for c_1 and c_2 by hand, and then write down y_p. This in turn gives us the general solution of the non-homogeneous ODE we are looking for.

We now have two very different methods to solve non-homogeneous, linear, second order ODEs with constant coefficients. Each method has its advantages and disadvantages, and it is not always easy to decide which to use. Here is a good guideline:

- If the right hand side $g(x)$ appears in the table in the previous section, or is a linear combination of several of these terms, the method of undetermined coefficients will be easier, and you should use it.
- Otherwise you should use variation of parameters, and (if necessary) leave the $c_1(x)$ and $c_2(x)$ as integrals.