NOTES FOR SECOND YEAR DIFFERENTIAL EQUATION PART VI: FOURIER SERIES AND THE HEAT EQUATION

JESSE RATZKIN

1. Introduction

In this set of notes we introduce Fourier series and use them to solve the heat equation.

The heat equation is your first example of a partial differential equation (PDE), that is, an equation for an unknown function of several variables, which involves partial derivatives. If u is a function of the variables x and t, then a first order PDE for u has the form

$$0 = F\left(x, t, u(x,t), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \tag{1}$$

and a second order PDE for u has the form

$$0 = F\left(x, t, u(x,t), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}\right). \tag{2}$$

For the sake of brevity, we will often indicate partial derivatives with subscripts, such as $\frac{\partial u}{\partial x} = u_x$ and $\frac{\partial^2 u}{\partial t^2} = u_{tt}$.

The usual definitions apply to PDEs, as they did to ODE. A function u solves a PDE of the form (**??**) or (**??**) if the function $u(x,t)$ satisfies (**??**) or (**??**) for all x and t. A PDE is linear if the function F is linear in u and all its derivatives. A PDE is autonomous if F does not depend explicitly on x or t.

We will see more about PDEs later on, but first we need to introduce Fourier series.

2. Fourier series

It would be nice if we could write any reasonable (*i.e.* continuous) function on the closed interval $[-L, L]$ as a sum of sines and cosines. Joseph Fourier was the first person to do this, and his motivation was finding solutions to the heat equation. Later on we will see why writing a general function as a sum of sines and cosines makes solving the heat equation easier.

Date: 2013.

2.1. **Full Fourier series over $[-L, L]$.** First we need to recall some machinery from linear algebra. The space of continuous function on $[-L, L]$ carries the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx,$$

and we need to remember that a list of functions $\{f_1, f_2, \dots\}$ is an **orthonormal set** if

$$\langle f_i, f_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 1. *The set*

$$\left\{ \sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi x}{L}\right), \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots \right\}$$

is an orthonormal set.

Proof. We first check orthogonality. Remember the angle sum formulas for sin and cos:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi, \qquad \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

and

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi, \qquad \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

We combine these to read

$$\cos\theta\cos\phi = \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)), \quad \sin\theta\sin\phi = \frac{1}{2}(\cos(\theta + \phi) - \cos(\theta - \phi))$$

and

$$\sin\theta\cos\phi = \frac{1}{2}(\sin(\theta + \phi) - \sin(\theta - \phi)).$$

Next we compute: for $n \neq m$ we have

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right) dx = 0$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) - \cos\left(\frac{(n-m)\pi x}{L}\right) dx = 0.$$

Both of these hold because $\cos\left(\frac{k\pi x}{L}\right)$ is a $\frac{L}{k}$-periodic function with average value 0. Similarly, we have

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(n+m)\pi x}{L}\right) - \sin\left(\frac{(n-m)\pi x}{L}\right) dx = 0$$

for all pairs of integers n and m.

Finally we check that

$$\int_{-L}^{L} dx = 2L \Rightarrow \langle 1, 1 \rangle = 2L,$$

and

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(\frac{2n\pi x}{L}\right) dx = L,$$

and

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} 1 - \cos\left(\frac{2n\pi x}{L}\right) dx = L.$$

This completes the proof. □

It is a little more difficult to prove the following theorem, so we omit the proof here.

Theorem 2. *The set*

$$\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x}{L}\right), \sqrt{\frac{1}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots\right\}$$

is an orthonormal basis for the vector space of continuous functions on $[-L, L]$.

The difficult part in proving the theorem above is to show that

$$\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x}{L}\right), \sqrt{\frac{1}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots\right\}$$

spans the space of continous functions.

Corollary 3. *Let* $f : [-L, L] \to \mathbf{R}$ *be continuous, and for* $n = 0, 1, 2, 3, \ldots$ *define*

$$a_n = \left\langle f, \frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

and

$$b_n \left\langle f, \frac{1}{L}\sin\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

Proof. This follows immediately from the fact that

$$\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x}{L}\right), \sqrt{\frac{1}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots\right\}$$

is an orthonormal basis and that for any orthonormal basis $\{\phi_k\}$ we have

$$f(x) = \sum_k \langle f, \phi_k\rangle \phi_k(x).$$

□

Definition 1. *Let* $f : [-L, L] \to \mathbf{R}$ *be a continuous function. We define the* **Fourier coefficients** *of* f *to be*

$$a_n = \left\langle f, \frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

and

$$b_n \left\langle f, \frac{1}{L}\sin\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,$$

and call

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

the **Fourier series** *representation of* f*.*

Example: Let $f(x) = |x|$ and compute the Fourier series representation of f. First observe that f is an even function, and for every n $\sin\left(\frac{n\pi x}{L}\right)$ is odd. Thus, for every n, b_n is the integral of an odd function over $[-L, L]$, so $b_n = 0$ for all n. Next we compute

$$a_n = \frac{1}{L}\int_{-L}^{L}|x|\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{n^2\pi^2}((-1)^n - 1).$$

This last quantity is 0 for n even and $-\frac{4L}{n^2\pi^2}$ for n odd. We should also do a particular computation for $n = 0$ to get a_0:

$$a_0 = \frac{1}{L}\int_{-L}^{L}|x|dx = \frac{2}{\sqrt{L}}\int_0^L xdx = L.$$

We can thus write the Fourier series of $f(x) = |x|$ as

$$|x| = \frac{L}{2} - \sum_{k=0}^{\infty}\frac{4L}{(2k+1)^2\pi^2}\cos\left(\frac{(2k+1)\pi x}{L}\right).$$

(Here we have used the substitution $n = 2k + 1$ for n odd.)

Example: Let

$$f(x) = \begin{cases} -1 & -L \le x < 0 \\ 1 & 0 \le x \le L. \end{cases}$$

Even though this is not a continuous function, it still has a valid Fourier series representation, which we compute now. First observe that f is odd and $\cos\left(\frac{n\pi x}{L}\right)$ is even, so $a_n = 0$ for all n. It remains to compute b_n, which we find as follows:

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2L}{n\pi}((-1)^n - 1).$$

This last quantity is $\frac{4L}{n\pi}$ when n is odd and 0 when n is even. Thus we can represent

$$f(x) = \sum_{k=1}^{\infty}\frac{4L}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi x}{L}\right),$$

where again we substitute $n = 2k + 1$ for n odd.

It is interesting to oberve that

$$\frac{d}{dx}(|x|) = \begin{cases} -1 & -L < x < 0 \\ 1 & 0 < x < L, \end{cases}$$

and this relation is reflected in the Fourier series, after we differentiate term by term.

2.2. **Fourier half-series.** In this section we develop two Fourier series representations for a function of $[0, L]$.

We start with a function $f(x)$ defined on the interval $0 \le x \le L$. We can extend f to $[-L, L]$ in two ways, either as an even function or as an odd function. Then even extension is defined by saying $f(-x) = f(x)$, and the odd extension is defined by the rule $f(-x) = -f(x)$. If we choose the even extension, we get a Fourier series for f as a sum of cosines, but if we choose the odd extension then we get a Fourier series for f as a sum of sines. In fact, both are valid representations.

Definition 2. *Let $f : [0, L] \to \mathbf{R}$ be continuous and define*

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

and

$$\beta_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Then we can define the even Fourier series representation of f to be

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{L}\right)$$

and the odd Fourier series representation of f to be

$$f(x) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right).$$

Exercise: Verify that if

$$f(x) = \begin{cases} -1 & 0 \le x \le \frac{L}{2} \\ 1 & \frac{L}{2} < x \le L \end{cases}$$

then the even Fourier series of f is

$$f(x) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)\pi} \cos\left(\frac{(2k+1)\pi x}{L}\right).$$

3. The heat equation

The heat equation is

$$u_t = \kappa^2 u_{xx}.$$

Here u is a twice differentiable function of the two variables x and t, $\kappa > 0$ is a physical parameter, and we usually take $x \in [0, L]$ and $t \geq 0$. The function $u(x,t)$ is supposed to measure the heat of a thin rod (or other, similar, object) at position x and time t. The heat equation thus tells us how the temperature of rod evolves in time, so long as there are no external heat sources.

3.1. **Origin and nature of the heat equation.** We can think of the second derivative as the limit of second difference quotients, so that

$$u_{xx}(x_0) \simeq \frac{\frac{u(x_0+\Delta x)-u(x_0)}{\Delta x} - \frac{u(x_0)-u(x_0-\Delta x)}{\Delta x}}{\Delta x} = \frac{u(x_0+\Delta x) - 2u(x_0) + u(x_0 - \Delta x)}{(\Delta x)^2}.$$

If we now set $u_{xx} = 0$ then we can see

$$u(x_0) \simeq \frac{1}{2}(u(x_0 + \Delta x) + u(x_0 - \Delta x)),$$

which says that $u(x_0)$ is roughly equal to the average value of u at neighboring points.

Using similar reasoning, we can understand why the PDE $u_t = \kappa^2 u_{xx}$ should model the flow of heat. First we need to understand that heat likes to spread out. If we place an ice cube in a hot cup of tea and wait, eventually all the liquid in the cup will be the same temperature. This is because the heat in the hot liquid will flow to the cold region of the ice cube. We can model this as saying that the rate of change of heat should be proportional to the difference between $u(x_0, t_0)$ and the temperature of its (physical) neighbors. Thus we should have

$$u_t = \kappa^2 u_{xx}$$

for some constant of proportionality κ.

If we again think of the heat equation as modeling the temperature of a rod, then we can place two sorts of natural boundary conditions at the endpoints $x = L$ and $x = 0$. **Neumann** boundary conditions are

$$\frac{\partial u}{\partial x}(L, t) = 0 = \frac{\partial u}{\partial x}(0, t)$$

for all $t > 0$, which physically means that we have insulated the ends of the rod (so that no heat can flow in or out). The other sort of natural boundary conditions we have are **Dirichlet** boundary conditions, which are

$$u(L, t) = 0 = u(0, t)$$

for all $t > 0$. Physically this means we're holding the temperature of the ends of rod to be zero, *i.e.* the ends are in an ice bath.

To complete our set of given data, we need to specify an initial temperature distribution, which is a given function $f(x) = u(x, 0)$ for $0 \leq x \leq L$. In general we expect to be able to solve either of the following problems: find $u(x, t)$ such that

$$u_t = \kappa^2 u_{xx}, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(L,t) = 0 = \frac{\partial u}{\partial x}(0,t),$$

or find $u(x, t)$ such that

$$u_t = \kappa^2 u_{xx}, \quad u(x,0) = f(x), \quad u(L,t) = 0 = u(0,t).$$

The first of these has Neumann boundary conditions, and the second has Dirichlet boundary conditions.

At this point, it makes sense to list some basic properties of the heat equation. We see that it involves two derivatives in x and one derivative in t, so we say that $u_t = \kappa^2 u_{xx}$ is a second order PDE. It is also a linear PDE, because we can rewrite the heat equation as

$$0 = u_t - \kappa^2 u_{xx} = F(x, t, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}),$$

and in this case F is linear in u_t and u_{xx}. The heat equation is autonomous (because F does not depend on x or t).

Solutions to the heat equation also obey a very nice physical law. To see this, we first define the thermal energy of the rod (at time t) to be

$$E(t) = \int_0^L u^2(x,t)dx,$$

where u solves the heat equation, with either Dirichlet or Neumann boundary conditions.

Theorem 4. *Let u solve*

$$u_t = \kappa^2 u_{xx}, \quad 0 \leq x \leq L, \quad 0 < t$$

with either Dirichlet or Neumann boundary conditions. Then $E'(t) < 0$ for all t. That is, the thermal energy of the rod is always decreasing.

Proof. We take a derivative, use the heat equation, and integrate by parts to see

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\int_0^L u^2(x,t)dx = \int_0^L \frac{d}{dt}(u^2(x,t))dx = 2\int_0^L u u_t dx \\
&= 2\kappa^2 \int_0^L u u_{xx} dx = 2\kappa^2 \left[u u_x|_{x=0}^L - \int_0^L (u_x)^2 dx \right] \\
&= -2\kappa^2 \int_0^L (u_x)^2 dx < 0.
\end{aligned}$$

Here we have used the fact that u has either Dirichlet or Neumann boundary conditions to eliminate the boundary terms in the integration by parts; either will suffice. □

3.2. **Solutions with Neumann boundary conditions.** In this section, we solve the heat equation with Neumann boundary, that is $0 = u_x(L,t) = u_x(0,t)$.

We start by looking for a solution of the form $u(x,t) = A(x)B(t)$. Then we must have

$$0 = u_t - \kappa^2 u_{xx} = AB' - \kappa^2 A'' B \Rightarrow \frac{B'}{B}(t) = \kappa^2 \frac{A''}{A}(x).$$

As these are two functions of different variables, this is only possible if

$$\frac{B'}{B} = \kappa^2 \frac{A''}{A} = -\tau\kappa^2 = \text{constant},$$

which we can rewrite as

$$B' = -\tau\kappa^2 B, \quad A'' = -\tau A.$$

(The minus sign there will be convenient later on.) At this point, it might be worthwhile to point out the boundary conditions on u. We must have

$$u_x(0,t) = 0 \Rightarrow A'(0) = 0, \quad u_x(L,t) = 0 \Rightarrow A'(L) = 0.$$

Regardless of the sign of τ, the solution for B is

$$B(t) = B_0 e^{-\tau\kappa^2 t},$$

where the constant B_0 is merely the value of B at $t = 0$. On the other hand, the solution for A depends a bit on the sign of τ. If $\tau < 0$ then

$$A(x) = c_+ e^{\sqrt{-\tau}x} + c_- e^{\sqrt{-\tau}x}.$$

At this point we can try to match the boundary conditions on A. We must have

$$0 = A'(0) = \sqrt{-\tau}(c_+ - c_-) \Rightarrow c_+ = c_0$$

and

$$0 = A'(L) = \sqrt{-\tau}c_+(e^{\sqrt{-\tau}L} - e^{-\sqrt{-\tau}L}) \Rightarrow c_+ = 0 = c_-.$$

This can only happen if $A \equiv 0$, which in turn implies $u \equiv 0$, which is not the solution of the heat equation we're looking for. Thus we can rule out $\tau < 0$.

We must therefore have $\tau > 0$, and so

$$A(x) = c_1 \cos(\sqrt{\tau}x) + c_2 \sin(\sqrt{\tau}x).$$

Again, we match boundary conditions, and see

$$0 = A'(0) = \sqrt{\tau}c_2 \Rightarrow c_2 = 0.$$

We conclude that with Neumann boundary conditions, the only possible solution to the heat equation on an interval is

$$u(x,t) = c_1 \cos(\sqrt{\tau}x) e^{-\tau\kappa^2 t},$$

where $\tau > 0$ is a parameter we have yet to discover. This is where we use the other boundary data point, namely

$$0 = A'(L) = -c_1\sqrt{\tau}\sin(\sqrt{\tau}L) \Rightarrow \sqrt{\tau}L = n\pi \Rightarrow \tau = \frac{n^2\pi^2}{L^2}$$

for some integer n. We now have found a solution to the heat equation of the form

$$u(x,t) = c_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right),$$

where c_n is a constant we can determine from the initial condition at $t = 0$.

We do not yet have a way to choose *which* integer n we should use. The answer is that we must choose all possible n, and sum the results. We can do this because the heat equation is a linear, homogeneous equation, and so the principle of superposition applies. Thus, we have just derived the following theorem.

Theorem 5. *The general solution to the heat equation with Neumann boundary conditions, which we can write as*

$$u_t = \kappa^2 u_{xx}, \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t),$$

is

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

Using this theorem is exactly where we use the Fourier series we have just discussed.

Example: Solve the Neumann boundary value problem

$$u_t = \kappa^2 u_{xx}, \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = \left\{ \begin{array}{rl} -1 & 0 \leq x \leq \frac{L}{2} \\ 1 & \frac{L}{2} < x \leq L. \end{array} \right.$$

We know that the solution must have the form

$$u(x,t) = \sum_{n=0} c_n n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right),$$

so it only remains to find the coefficients c_n. Evaluating u at $t = 0$, we see

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = \left\{ \begin{array}{rl} -1 & 0 \leq x \leq \frac{L}{2} \\ 1 & \frac{L}{2} < x \leq L. \end{array} \right.$$

However, we have already written this particular function as a sum of cosines, when we found the even Fourier series representation of it. We have

$$u(x,0) = \left\{ \begin{array}{rl} -1 & 0 \leq x \leq \frac{L}{2} \\ 1 & \frac{L}{2} < x \leq L. \end{array} \right\} = \sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)\pi} \cos\left(\frac{(2k+1)\pi x}{L}\right),$$

and so we conclude that we must have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)\pi} e^{-\frac{(2k+1)^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{(2k+1)\pi x}{L}\right).$$

We see now that we have a standard technique to solve the Neumann boundary value problem on the interval $0 \leq x \leq L$, for $t \geq 0$:

$$u_t = \kappa^2 u_{xx}, \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = f(x).$$

First we find the Fourier coefficients

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

so that

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{L}\right).$$

Then the solution to our heat equation with Neumann boundary data must be given in terms of the Fourier series, as

$$u(x,t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

We can pick out the long-time behavior of u in this case. Observe that as $t \to \infty$ all the exponential terms decay to 0, very rapidly. This means that for all x we ahve

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left[\frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right)\right] = \frac{\alpha}{2} = \frac{1}{L}\int_0^L f(x)dx.$$

We have just proved the following corollary.

Corollary 6. *Let $u(x,t)$ solve*

$$u_t = \kappa^2 u_{xx}, \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = f(x).$$

Then for all $x \in [0,L]$ we have

$$\lim_{t\to\infty} u(x,t) = \frac{\alpha_0}{2} = \operatorname{avg}(f),$$

where $\operatorname{avg}(f)$ *is the average value of the initial temperature distrubution f over the interval* $[0,L]$.

3.3. **Solutions with Dirichlet boundary conditions.** In this section, we solve the heat equation with Neumann boundary, that is $0 = u_x(L,t) = u_x(0,t)$. Indeed, we will see that the solution in the Dirichlet case is very similar to the solution we have just constructed in the Neumann case, with the only difference being that we must replace the cosines with sines.

More specifically, we wish to solve

$$u_t = \kappa^2 u_{xx}, \quad u(0,t) = 0 = u(L,t), \quad u(x,0) = f(t),$$

where $f(x)$ is a given function which measures the initial temperature distribution. We again look for a solution $u(x,t) = A(x)B(t)$, and so we must have

$$AB' = \kappa^2 A'' B \Rightarrow \frac{B'}{B}(t) = \kappa^2 \frac{A''}{A}(x) = -\tau\kappa^2 = \text{ constant}.$$

We now have two ODEs for A and B, which are

$$B' = -\tau\kappa^2 B, \quad A'' = -\tau A.$$

We can solve the ODE for B to give

$$B(t) = B_0 e^{-\tau\kappa^2 t}$$

for some constant $B_0 = B(0)$. Using the same reasoning as before we can eliminate the case $\tau \leq 0$, and so (since $\tau > 0$) we must have

$$B(x) = c_+ \cos(\sqrt{\tau}x + c_- \sin(\sqrt{\tau}x).$$

By our boundary conditions we have

$$0 = B(0) = c_+ \Rightarrow B(x) = c_- \sin(\sqrt{\tau}x)$$

and

$$0 = B(L) = c_- \sin(\sqrt{\tau}L) \Rightarrow \sqrt{\tau}L = n\pi \Rightarrow \tau = \frac{n^2\pi^2}{L^2}.$$

We now conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

It remains to find the coefficients β_n, but these must be the coefficients of the Fourier sine-series for the initial condition f. Evaluating at $t = 0$ we see

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\beta_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right).$$

We make two quick observations. First observe that there is no $n = 0$ term, precisely because $\sin(0) = 0$. We also see that for all x we have

$$\lim_{t\to infty} u(x,t) = \lim_{t\to\infty} \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right) = 0.$$

This says that in the case of Dirichlet boundary conditions the limiting temperature idstribution is always zero. This actually should not be too surprising, because we are (physically) holding the ends of the rod in ice, so all the heat should disipate out the ends of the rods.

Example: Solve the boundary value problem

$$u_t = \kappa^2 u_{xx}, \quad u(0,t) = 0 = u(L,t), \quad u(x,0) = f(x) = \begin{cases} x & 0 \leq x \leq \frac{L}{2} \\ L - x & \frac{L}{2} \leq x \leq L. \end{cases}$$

We know the solution must be

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\beta_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

It remains to compute the coefficients β_n.

We have

$$\begin{aligned}
\beta_n &= \frac{2}{L}\left[\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right] \\
&= \frac{2}{L}\left[\frac{-Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx - \frac{L^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}\right. \\
&\quad \left. + \frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right] \\
&= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\sin(n\pi/2) + \frac{L^2}{n^2\pi^2}\sin(n\pi/2)\right] = \frac{4L}{n^2\pi^2}\sin(n\pi/2).
\end{aligned}$$

To evaluate this last quantity, we need to remember some properties of sin. If n is even, we have $\sin(k\pi)$ for some integer k, which is zero. If n is odd, then either $n = 4k+1$ or $n = 4k+3$, and we get either a $+1$ or a -1, depending. In particular we have

$$\beta_{4k+1} = \frac{4L}{(4k+1)^2\pi^2}, \quad \beta_{4k+3} = \frac{4L}{(4k+3)^2\pi^2} \Rightarrow f(x) = \frac{4L}{\pi^2}\sum_{k=0}^{\infty}\left(\frac{1}{(4k+1)^2} + \frac{1}{(4k+3)^2}\right).$$

Putting everything together, we see

$$\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\frac{4L}{(4k+1)^2\pi^2}e^{-\frac{(4k+1)^2\kappa^2\pi^2 t}{L^2}}\sin\left(\frac{(4k+1)\pi x}{L}\right) \\
&\quad + \sum_{k=0}^{\infty}\frac{4L}{(4k+3)^2\pi^2}e^{-\frac{(4k+3)^2\kappa^2\pi^2 t}{L^2}}\sin\left(\frac{(4k+3)\pi x}{L}\right).
\end{aligned}$$