NOTES FOR SECOND YEAR DIFFERENCE AND DIFFERENTIAL EQUATION
PART I: INTRODUCTION AND BACKGROUND

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1. Motivations

In this course we will learn about difference equations and differential equations. These are very good tools to understand dynamical systems, *i.e.*, systems that change in time. One can use a difference equation to model a system that changes with discrete time steps, such as the amount owed on a loan with interest compounded monthly. One can use a differential equation to model a system that changes continuously with time, such as the amount owned on a loan with interest compounded continuously. One can also use difference and differential equations to model other phenomena. For instance, suppose your company produces an electronic gadget. How does the number of gadgets you sell depend on the price and the demand for your product? Can we model your sales using a difference or differential equation? We will see examples later on in the notes.

These is no general rule to find the solution of a general difference or differential equation. Indeed, it can sometimes be a little difficult to determine **if a solution actually does exist!** Thus, we will learn a variety a methods that will work in different cases. One of the main difficulties in this course will be determining which method to use to solve a given problem. For this reason, you will need to practice using the various techniques we learn.

2. Background

We will need to recall some basic things you learned in previous courses.

Our first main topic will be difference equations, which are equations for sequences. Recall that a **sequence** is a list of numbers. A finite sequence is a list with finitely many numbers, such as $\{1, 2, 3, 4, 5\}$ or $\{0, -1, 1, -2, 2, -3, 3\}$. An infinite sequence is a list of infinitely many numbers, and is usually written with elipses (those little dots). For instance, the sequence of positive, even integers is written $\{2, 4, 6, 8, \dots, \}$, and the sequence of negative integer powers of 2 is written

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \right\}.$$

If we denote the nth term of a sequence by a_n, then we write the sequence as $\{y_n\}$. If $\{y_n\}$ has $N+1$ terms (*i.e.* $N+1$ numbers) in it, we can further write this sequence as $\{y_n\}_{n=0}^{N}$. If $\{y_n\}$ has infinitely many terms, we write $\{y_n\}_{n=0}^{\infty}$. We will

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refer to the subscript n as the **index** of the term y_n, and think of it as a counter telling us where we are in the sequence. We will find it sometimes convenient to start the sequence with the index 1 (or even something else) instead of 0, and in this case we can write $\{y_n\}_{n=1}^{N}$, and so on.

Our second main topic will be differential equations. For this topic we will need to recall the derivative of a function, and also partial derivative of functions of several variables. If f is a function of one variable, which we call t, then the derivative measures the rate of change of f with respect to t:

$$f'(t_0) = \left.\frac{df}{dt}\right|_{t=0} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

Similarly, the partial derivative of a function of several variables measures its rate of change with respect to one of its independent variables. It would be a good idea to revise some basics of differentiation from MAM1000.

3. Other sources

There are many good books on differential equations. I particularly like *Elementary Differential Equations* by Boyce and DiPrima. This book is available fro three hour loan in the main library, from the short loans desk. There are also notes by Dr. A. Weltman, which are available on Vula.