NOTES FOR SECOND YEAR DIFFERENCE AND DIFFERENTIAL EQUATION
PART II: DIFFERENCE EQUATIONS

JESSE RATZKIN

1. Definitions

Recall that a finite sequence has the form $\{y_n\}_{n=0}^N$ and an infinite sequence has the form $\{y_n\}_{n=0}^\infty$. A **difference equation** of order m has the form

$$y_{n+m} = F(n, y_n, y_{n+1}, \ldots, y_{n+m-1}), \tag{1}$$

where F is a given function. In general, our task will be to solve the **initial value problem**

$$y_{n+m} = F(n, y_n, y_{n+1}, \ldots, y_{n+m-1}), \quad y_0 = c+0, y_1 = c_1, \ldots, \quad y_{m-1} = c_{m-1} \tag{2}$$

for a given right hand side F. Notice that we should assign m initial conditions for a difference equation of order m. Another way to think of this is to say that when we're picking solutions of a difference equation of order m, we must make m choices (each of which is an initial condition). In this way, we should imagine the space of solutuons to (1) as being m-dimensional. For instance,

$$y_{n+1} = ny_n, \qquad y_{n+1} = \left(\frac{1}{2}\right)^n y_n$$

are both difference equations of order 1, *i.e.* first order difference equations. It might be instructive to stop reading right now and write out the first several terms of both of these sequences, if we take $y_0 = 1$.

A **solution** to a difference equation is a sequence of number $\{y_n\}_{n=0}^\infty$ such that

$$y_{n+m} = F(n, y_n, y_{n+1}, \ldots, y_{n+m})$$

for all $n = 0, 1, 2, 3, \ldots$

Difference equations come in several different varieties, which we start to list now. If the index n does not appear in the formula for F, then we call the equation **autonomous**; otherwise it is **non-autonomous**. If F is a linear function in the variables $y_n, y_{n+1}, \ldots, y_{n+m-1}$, then we say the difference equation is **linear**, and otherwise we say it is **nonlinear**. Notice that a linear difference equation need not be linear in the n variable. For instance,

$$y_{n+2} = \frac{1}{n^2} + y_n - y_{n+1}$$

Date: 2013.

is a linear, second order, non-autonomous difference equation. A linear difference equation has coefficients in front of each of its terms, which can either be constant (with respect to the index n), or variable (*i.e.* nonconstant). Finally, if the sequence $\{y_n = 0\}_{n=0}^{\infty}$ is a solution to the difference equation, then we say it is **homogeneous**, and otherwise we say it is **non-homogeneous**. Notice that all of these descriptions of the difference equation depend only on the function F, and make no reference at all to the initial value y_0.

One of the main themes we will see is that linear difference (and differential) equations are much easier to solve than nonlinear equations. In fact, we will see that we can often write down solutions to linear difference equations, very explicitly, whereas we often have no hope of writing out an explicit solution to a nonlinear difference equation. This phenomenon persists to differential equations.

2. Some basic examples

It will be worthwhile to consider some examples before we continue. You're already familar with the arithmetic and geometric sequences from MAM1000. A sequence of numbers $\{y_n\}_{n=0}^{\infty}$ is arithmetic if the difference $y_{n+1} - a_n = d$ is the same for all n. For instance, if $d = 2$ and $y_0 = 0$ we obtain the sequence of non-negative even integers:

$$y_0 = 0, y_1 = 2, y_2 = 4, y_3 = 6, \cdots, y_n = 2n, \cdots.$$

This sequence is a solution to the difference equation

$$y_{n+1} = y_n + 2$$

with the initial condition $y_0 = 0$. In this way, we can say the sequence $\{0, 2, 4, 6, \dots\}$ solves the initial value problem

$$y_{n+1} = F(n, y_n) = y_n + 2, \qquad y_0 = 0.$$

This is a linear, first order, non-homogeneous, autonomous difference equation with constant coefficients.

A sequence $\{y_n\}_{n=0}^{\infty}$ is geometric if the ratio $y_{n+1}/y_n = r$ is the same for all n. For instance, the sequence

$$y_0 = 1, y_1 = \frac{1}{2}, y_2 = \frac{1}{4}, y_3 = \frac{1}{8}, \cdots, y_n = \frac{1}{2^n}, \cdots$$

is a geometric sequence with common ratio $r = 1/2$. We can recognize this sequence as a solution to the initial value problem

$$y_{n+1} = F(n, y_n) = \frac{1}{2} y_n, \qquad y_0 = 1.$$

This is a linear, first order, homogeneous, autonomous difference equation with constant coefficients.

Another example you may have seen before is the Fibonacci sequence, which solves the intial value problem

$$y_{n+2} = F(n, y_n, y_{n+1}) = y_n + y_{n+1}, \qquad y_0 = y_1 = 1.$$

The first several terms of the solution are

$y_0 = y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 5, y_5 = 8, y_6 = 13, y_7 = 21, y_8 = 34, y_9 = 55, y_{10} = 89, \ldots$

This is a linear, second order, homogeneous, autonomous difference equation with constant coefficients. We see that the terms of the Fibonacci sequence grow very rapidly.

Here are some more complicated examples of difference equations. The equation

$$y_{n+2} = F(n, y_n, y_{n+1}) = y_n y_{n+1}$$

is second order, nonlinear, homogeneous, and autonomous. The equation

$$y_{n+1} = F(n, y_n) = \frac{1}{n} y_n^2$$

is first order, nonlinear, non-autonomous, and homogeneous. The equation

$$y_{n+2} = F(n, y_n, y_{n+1}) = n - y_n y_{n+1}.$$

is second order, nonlinear, non-autonomous, and non-homogeneous. It can be much more difficult to solve an initial value problem for these latter difference equations!

3. First order, linear, constant coefficient difference equations

We will begin with the simplest difference equations: those which are first order, linear, with constant coefficient. We can write the general form of this difference equation as

$$y_{n+1} = F(n, y_n) = a y_n + g(n), \tag{3}$$

where a is a constant and g is a function. We will first treat the case $g(n) = 0$ for all n; this is the autonomous and homogeneous case. Then we have a geometric sequence, which we can compute explicitly.

Proposition 1. *Let a and c be real numbers with $a \neq 0$. The solution to the initial value problem*

$$y_{n+1} = F(n, y_n) = a y_n, \qquad y_0 = c \tag{4}$$

is $y_n = ca^n$. On the other hand, if $a = 0$ then the solution to the same initial value problem is $y_0 = c$, and $y_n = 0$ for $n \geq 1$.

Observe that, regardless of what a is, the trivial sequence $\{y_n = 0\}_{n=0}^{\infty}$ is the solution to the initial value problem with $c = 0$.

Proof. We use induction. Suppose that we already know

$$y_0 = c, y_1 = ca, y_2 = ca^2, \ldots, y_n = ca^n$$

and we wish to find y_{n+1}. Then by our difference equation we have

$$y_{n+1} = a y_n = a(ca^n) = ca^{n+1},$$

which satisfies our rule. Since we already have $y_0 = c$ (the base case of our induction), this completes the proof. □

We pause here to notice some key differences between solutions, depending on a.

- If $a > 0$ then all the terms in the solution have the same sign (and the same sign as c).
- If $a < 0$ then y_n and y_{n+1} have opposite signs. That is, the terms in the solution flip sign from positive to negative, and back again.
- if $|a| < 1$ then, regardless of what c is, we have

$$\lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |ca^n| = |c| \lim_{n\to\infty} |a|^n = 0.$$

- On the other hand, if $|a| > 1$ then

$$|y_n| = |c||a|^n \to \infty,$$

  and the terms in the solution sequence very quickly become large.
- If $|a| = 1$ then $|y_n| = |c||a|^n = |c|$, and the terms in the solution sequence all have the same size. (Notice that this does not mean the solution is the constant sequence!)

Example: Let $a = 1/3$ and $c = 2$. Then the solution to the inital value problem

$$y_{n+1} = \frac{1}{3} y_n, \qquad y_0 = 2 \tag{5}$$

is given by $y_n = 2 \cdot 3^{-n}$.

This example is very simple, but we can actually use it to find many other solutions. Let's look at an example to make this concrete. Suppose we wish to solve

$$y_{n+1} = \frac{1}{3} y_n + \frac{1}{2}, \qquad y_0 = c, \tag{6}$$

where c is a general, undetermined, number. This difference equation is first order, linear, and non-homogeneous. We begin by trying to find some **particular solution** to the difference equation, without trying to match the initial condition $y_0 = c$, and in this regard the simplest possible solution would be constant: $(y_p)_n = \alpha$ for all n. Let's plug this in and see what α must be:

$$\alpha = (y_p)_{n+1} = \frac{1}{3}(y_p)_n + \frac{1}{2} = \frac{1}{3}\alpha + \frac{1}{2} \Rightarrow \alpha = \frac{3}{4}.$$

Now we have a particular solution $(y_p)_n = 3/4$ for all n. It might be useful to verify this really is a solution, which we can check with the computation

$$\frac{1}{3}(y_p)_n + \frac{1}{2} = \frac{1}{3}\alpha + \frac{1}{2} = \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = \alpha = (y_p)_{n+1}.$$

Attached to (6) we have the associated linear homogeneous difference equation, which we get by setting the non-homogeneous part to 0:

$$(y_h)_{n+1} = \frac{1}{3}(y_h)_n, \quad (y_h)_0 = \bar{c}. \tag{7}$$

First notice that we did not choose the same initial conditions for (7) and for (6); we'll get to why we did this in a bit. Next, notice that we already know the solution to (7), namely $(y_h)_n = \bar{c}3^{-n}$. An interesting thing happens when we add these two solutions together. Indeed, if we let $y_n = (y_h)_n + (y_p)_n$ then we see

$$y_{n+1} = (y_h)_{n+1} + (y_p)_{n+1} = \frac{1}{3}(y_h)_n + \frac{1}{3}(y_p)_n + \frac{1}{2} = \frac{1}{3}[(y_h)_n + (y_p)_n] + \frac{1}{2} = \frac{1}{3}y_n + \frac{1}{2}.$$

Here we have used

$$(y_p)_{n+1} = \frac{1}{3}(y_p)_n + \frac{1}{2}.$$

It remains only to relate c and $\bar{c}$ to match the initial condition. We have

$$c = y_0 = (y_h)_0 + (y_p)_0 = \bar{c} + \alpha \Rightarrow \bar{c} = c - \frac{3}{4}.$$

What we have just done might seem a little arcane, but in fact this trick *always* works!

Theorem 2. *Consider the initial value problem*

$$y_{n+1} = ay_n + b, \qquad y_0 = c. \tag{8}$$

If $a \neq 1$ then the solution is given by

$$y_n = a^n \left(c - \frac{b}{1-a} \right) + \frac{b}{1-a}. \tag{9}$$

Proof. We adapt exactly the same method we used for the previous example. First we look for a particular solution to (8) which does not depend on n. We set $(y_p)_n = \alpha$ for all n, and then

$$\alpha = (y_p)_{n+1} = a(y_p)_n + b = a\alpha + b \Rightarrow \alpha = \frac{b}{1-a}.$$

Notice that we can solve for α precisely because $a \neq 1$. Again, we can check this really is a solution very explicitly:

$$a(y_p)_n + b = a\alpha + b = a \cdot \frac{b}{1-a} + b = \frac{ab + b(1-a)}{1-a} = \frac{b}{1-a} = \alpha = (y_p)_{n+1}.$$

Next we associate to (8) the homogeneous equation

$$(y_h)_{n+1} = a(y_h)_n, \qquad (y_h)_0 = \bar{c},$$

which we already know has the solution $(y_h)_n = a^n\bar{c}$. Again, we get the solution to (8) by summing to get $y_n = (y_h)_n + (y_p)_n$. We now have

$$y_{n+1} = (y_h)_{n+1} + (y_p)_{n+1} = a(y_h)_n + a(y_p)n + b = ay_n + b,$$

and so we do indeed have a solution to our difference equation. It remains to find the constant $\bar{c}$. Evaluating at $n = 0$ we have

$$c = y_0 = (y_h)_0 + (y_p)_0 = \bar{c} + \alpha = \bar{c} + \frac{b}{1-a} \Rightarrow \bar{c} = c - \frac{b}{1-a}.$$

Summing everything together, we obtain exactly (9). $\square$

We must treat the case of $a = 1$ separately, but we can still write out the solution of (8) in this case.

Theorem 3. *The solution of the initial value problem*

$$y_{n+1} = y_n + b, \qquad y_0 = c \tag{10}$$

is given by

$$y_n = c + nb. \tag{11}$$

Proof. We begin by writing out the first several terms of the sequence $\{y_n\}_{n=0}^{\infty}$, which we compute explicitly by using (10) over and over, starting with $y_0 = c$ (which is given). We have

$$y_0 = c, y_1 = c + b, y_2 = c + b + b = c + 2b, y_3 = c + 2b + b = c + 3b, \ldots$$

We see a pattern, which we can verify using induction. Suppose that $y_n = c + nb$. Then

$$y_{n+1} = y_n + b = c + bn + b = c + (n + 1)b,$$

which verifies the induction step. □

We now have a complete description of solutions to (8), for all possible values of a, b, c. It will be useful for us to list some properties of these solutions, and to make sense of these properties we will first need to define some terms. First notice that we have some special solutions, which do not depend on n at all.

Definition 1. *A solution $\{y_n\}_{n=0}^{\infty}$ is called an equilibrium solution if $y_n = \alpha$ for some number α and for all n. That is, an equilibrium solution does not depend on n at all.*

Corollary 4. *If $a \neq 1$ the equilibrium solution of (8) is*

$$\left\{ y_n = \frac{b}{1-a} \right\}_{n=0}^{\infty}.$$

If $a = 1$ then (10) does not have an equilibrium solution, unless $b = 0$, in which case every solution is an equilibrium.

Proof. This follows immediately from (9) and (11). □

We pause here for some applications. First consider a saving account which is compounded monthly. Suppose we deposit R10,000 into an account, which earns interest rate of 3% per month. After n months, how much money is in the account? We can answer this question by solving an initial value problem for a difference equation. Let y_n be the amount in the account after n months, so that

$$y_0 = 10,000, \qquad y_{n+1} = y_n + .03 y_n = 1.03 y_n.$$

The difference equation says that every month the amount of money in the account increases by 3% of its present total. We can solve this difference equation to get

$$y_n = (1.03)^n 10,000.$$

We can make this example a little more general, by saying that after the initial deposit we withdraw R100 per month. Now we change the difference equation to

$$y_{n+1} = 1.03 y_n - 100, \qquad y_0 = 10,000$$

which we still know how to solve. The solution to this initial value problem is

$$y_n = 1.03^n \left(10,000 - \frac{(-100)}{1-1.03}\right) + \frac{(-100)}{1-1.03} \simeq (1.03)^n(6667) + 3333.$$

Next suppose we borrow the large sum of R250, 000 from a bank in order to buy a house. This time, the bank charges 3% interest per month, and each month we pay some fixed amount b, which we will determine just now. If we want to pay off the entire loan in 20 years, how much must we pay each month? We set up the same sort of difference equation as before. Let y_n be the amount of money we still owe after n months, and let b be the (fixed) amount we pay each month. Then we have

$$y_{n+1} = 1.03y_n - b, \qquad y_0 = 250,000, \qquad y_{240} = 0.$$

The last displayed equation above says that we pay of the loan entirely in exactly 20 years. Our general solution has the form

$$y_n = (1.03)^n \left(250,000 - \frac{(-b)}{1-1.03}\right) + \frac{(-b)}{1-1.03},$$

and so we can solve

$$0 = y_{240} = (1.03)^{240} \left[250,000 - \frac{b}{.03}\right] + \frac{b}{.03} \Rightarrow b = \frac{(1.03)^{240} \cdot 7,500}{(1.03)^{240} - 1} \simeq 7,506.$$

A second quick computation reveals that over the life of the loan we end up paying a grand total of R1, 801, 500, which is much more than the original amount we borrowed.

4. General first order difference equations

A general first order difference equation has the form

$$y_{n+1} = F(n, y_n),$$

where F is a function. We will devote this section to a special case, when the system is autonomous. This means we have

$$y_{n+1} = F(y_n). \tag{12}$$

4.1. **Equilibrium solutions.** It is worthwhile to first think of an example:

$$y_{n+1} = y_n - 4y_n^3. \tag{13}$$

We will not find the general solution to this difference equation, but we can easily find the equilibrium solutions. These have the form $y_n = \alpha$ for all n, and so we must have

$$\alpha = y_{n+1} = y_n - 4y_n^3 = \alpha - 4\alpha^3 = \alpha(1-4\alpha^2) \Leftrightarrow \alpha = 0, \pm\frac{1}{2}.$$

In general, we have the following Proposition.

Proposition 5. *The difference equation*

$$y_{n+1} = F(y_n)$$

has $\{y_n = \alpha\}_{n=0}^{\infty}$ *as an equilibrium if and only if* $\alpha = F(\alpha)$.

Proof. If $\alpha = F(\alpha)$ then we have

$$y_{n+1} = \alpha = F(\alpha) = F(y_n),$$

and so $\{y_n = \alpha\}$ is an equilibrium solutions. In the other case, $\alpha \neq F(\alpha)$, and we begin our difference equation with the initial condition $y_0 = \alpha$. Then $y_1 = F(\alpha) \neq \alpha$, and so $\{y_n = \alpha\}$ is not a solution. □

4.2. **Stability of equilibria.** By themselves, equilibria (this is the plural or equilibrium) are not very interesting; they don't ever change at all. However, it **is** very interesting to understand the behavior of solutions to a difference equation of the form (12) near an equilibrium solution. In fact, this is often the most we can do.

Definition 2. *Let $\{y_n = \alpha\}$ be an equilibrium solution of the difference equation* (12)*, so that $\alpha = F(\alpha)$. This solution is* **stable** *if y_0 close to α implies y_n is close to α for all n. More precisely, for every positive ϵ there is a $\delta > 0$ such that $|y_0 - \alpha| < \delta$ implies $|y_n - \alpha| < \epsilon$ for all n. More strongly, an equilibrium is* **asymptotically stable** *if for y_0 close to α we must have $\lim_{n\to\infty} y_n = \alpha$. In more formal language, this says that there is a $\delta > 0$ such that $|y_0 - \alpha| < \delta$ implies $\lim_{n\to\infty} y_n = \alpha$.*

Notice that asymptotically stable implies stable. If an equilibrium is not stable, it is called **unstable**.

Recall that we already found the equilibrium of a linear, first order, difference equation with constant coefficients. This difference equation has the form

$$y_{n+1} = ay_n + b,$$

and (provided $a \neq 1$) the equilibrium solution is $y_n = \frac{b}{1-a}$

Proposition 6. *Provided $a \neq 1$, the equilibrium of* (8) *is stable if and only if $|a| < 1$, in which case it is asymptotically stable.*

Proof. This follows immediately from (9). □

It turns out we can generalize this last proposition to show the following.

Theorem 7. *Let α be an equilibrium solution of the difference equation*

$$y_{n+1} = F(y_n),$$

where F is a differentiable function of y, and $F'(y)$ is continuous. If $|F'(\alpha)| < 1$ then α is an asymptotically stable equilibrimum, while if $|F'(\alpha)| > 1$ then α is unstable. In the case $|F'(\alpha)| = 1$ the test is inconclusive.

The proof goes a bit beyondthe level of this course, but it is a very nice use of the Taylor theorem with remainder.

Again, we discuss some examples here. To begin, we examine the difference equation

$$y_{n+1} = F(y_n) = y_n^2 + y_n - 1$$

This has two equilibrium solutions, which we find by setting

$$\alpha = F(\alpha) = \alpha^2 + \alpha - 1 \Leftrightarrow 0 = \alpha^2 - 1 = (\alpha + 1)(\alpha - 1) \Leftrightarrow \alpha = \pm 1.$$

We see that the equilibrium solutions are precisely $\alpha = 1$ and $\alpha = -1$, and next we compute

$$F'(\alpha) = 2\alpha + 1, \quad F'(1) = 3, \quad F'(-1) = -1.$$

We see from our theorem that $\alpha = 1$ is an unstable equilibrium, but the test is inconclusive for the equilivrium $\alpha = -1$. However, we can draw a cobweb diagram (see the next section) to see that $\alpha = -1$ is stable.

Suppose we want to model the flow of money in a household. We let Y_n represent income, that is the amount of money the members of the household get paid in their jobs; let C_n be the consumption, that is the total amount the members of the household spend; and let I_n be the total amount the members of the household puts into savings and investments. We have the first relation:

$$Y_n = C_n + I_n.$$

Next we suppose that the consumption in month n is a multiple of the income of the previous month:

$$C_{n+1} = aY_n.$$

We combine these two equations to get

$$C_{n+1} = aC_n + I_{n+1}.$$

This is still too general to solve, because we have two unknowns (the consumption and the investment), but now we make the following simplification: we assume that we invest a fixed amount b every month. Now we have

$$C_{n+1} = aC_n + b,$$

which is exactly the difference equation (8). Notice that we must have $a > 0$ because one can only spend a positive amount of money on rent, food, etc. Also notice that $a < 1$ because we never be able to spend more than our income. Thus the present model is always asymptotically stable.

We consider another model which simulates the supply and demand for some product. Suppose that your new company introduces a product, and after n months the new product sells for a price of p_n. The demand after n months is for d_n units, and at the start of the month you deliver s_n units. We suppose that

$$s_n = \alpha + \beta p_n, \qquad d_n = \gamma - \delta p_n,$$

where $\alpha, \beta, \gamma, \delta$ are all positive numbers. We see that γ is the demand for your product when it is free, that (because $\beta > 0$) you can supply more product when the price is higher, and that (also because $\delta > 0$) you sell more product when the price is lower. As a simple model, we set the supply at month $n + 1$ equal to the demand at month n, so that

$$\alpha + \beta p_{n+1} = \gamma - \delta p_n \Leftrightarrow p_{n+1} = \frac{\gamma - \alpha}{\beta} - \frac{\delta}{\beta} p_n = ap_n + b.$$

Here we have set $a = \frac{\gamma-\alpha}{\beta}$ and $b = -\frac{\delta}{\beta}$. We already know that the solution to this difference equation is given by (9), and that we have an equilibrium solution is

$$p_n = \frac{b}{1-a} = \frac{\gamma-\alpha}{\beta-\delta},$$

and that this equilibrium is stable if and only if $1 > |a| = |\delta/\beta|$, *i.e.* if and only if $|\delta| < |\beta|$.

4.3. **The cobweb method.** The cobweb method is a method to graph the solution to a first order, autonomous, difference equation. These difference equations have the form

$$y_{n+1} = F(y_n),$$

where F is a given function. The procedure is as follows:

(1) On your paper, plot the x and y axes, and sketch the graphs $y = x$ and $y = f(x)$.
(2) Plot the point $(y_0, 0)$, and then draw a vertical line connecting it to the graph $y = f(x)$; this vertical line will meet the graph at the point $(y_0, f(y_0)) = (y_0, y_1)$.
(3) Draw a horizontal line connecting the point (y_0, y_1) to the graph $y = x$; this horizontal line will meet the graph at the point (y_1, y_1).
(4) Draw a vertical line to connect the point (y_1, y_1) to the graph $y = f(x)$; this vertical line will meet the graph at the point $(y_1, f(y_1)) = (y_1, y_2)$.
(5) Draw a horizontal line connecting the point (y_1, y_2) to the graph $y = x$; this horizontal line will meet the graph at the point (y_2, y_2).
(6) Repeat as many times as you please.

We make some comments. First, observe that $y = \alpha$ is an equilibrium solution if and only if $f(\alpha) = \alpha$, which happens if and only if the two graphs intersect at $x = y = \alpha$. If you happen to pick $y_0 = \alpha$ then the sequence of points you get using the cobweb method will just be (α, α), repeated over and over again. If α is asymptotically stable, then starting with y_0 near α will produce a sequence of points going towards (α, α).

Exercise: Consider the difference equation $y_{n+1} = y_n^2$, and observe that $\alpha = 0, \pm 1$ are the equilibrium solutions. Start the cobweb method with both $y_0 = 1/2$ and $y_0 = 3/4$, and see what happens. What happens if you start the cobweb method with $y_0 = 1.1$?

5. First order, linear, homogeneous systems

Here we study some properties of first order, linear, homogeneous systems of difference equations with constant coefficients. The general form of this system is

$$y_{n+1} = Ay_n, \quad y_n \in \mathbf{R}^N, \quad A \in M_{N\times N}.$$

5.1. **The two by two case.** In the case $d = 2$ we can write this as
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

This system has some very special solutions given by eigenvectors of the coefficient matrix A. If
$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
is an eigenvector with eigenvalue λ, this means $Av = \lambda v$. Then if we start with the initial condition
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
we have
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Av = \lambda v = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
and
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = A(\lambda v) = \lambda A v = \lambda^2 v = \lambda^2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
and so on. By induction we see that the solution is
$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \lambda^n \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$
In fact, this remains true even in the $N \times N$ case: if $A \in M_{N\times N}$ and $v \in \mathbf{R}^N$ is an eigenvector of A with eigenvalue λ, then $y_n = \lambda^n v$ solves the difference equation $y_{n+1} = Ay_n$.

In the case that the coefficient matrix
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
is diagonalizable, we can find two linearly independent eigenvectors
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$
which in fact form a basis of $\mathbf{R}^2$. (You may want to revise your linear algebra notes to see this.) Then we can write any arbitrary vector w as a linear combination of u and v, say $w = \alpha u + \beta v$. We claim the following theorem:

Theorem 8. *Let*
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
have two linearly independent eigenvectors
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$
with $Au = \mu u$ and $Av = \lambda v$. Also let
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \alpha u + \beta v.$$

Then the solution to the initial value problem

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{14}$$

is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \alpha\mu^n u + \beta\lambda^n v. \tag{15}$$

Proof. We first verify that (15) matches the initial condition. We have

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \alpha u + \beta v = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

as desired. Now we apply A to see that (15) solves (14). Indeed,

$$A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = A(\alpha u + \beta v) = \alpha A u + \beta A v = \alpha\mu u + \beta\lambda v = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Continuing, we see

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix} = A(\alpha\mu^n u + \beta\lambda^n v) = \alpha\mu^n A u + \beta\lambda^n A v = \alpha\mu^{n+1} u + \beta\lambda^{n+1} v,$$

which proves that (15) does in fact solve (14). □

Here is an example.[1] Suppose that during a given year 10% of the people living in the Cape Town city bowl leave for the suburbs, while the remaining 90% stay in the city bowl. At the same time, 2% of the people living in the suburbs move to the city bowl, while 98% of the people living in the suburbs stay. If we let s_n be the number of people living in the suburbs each year, and c_n the number of people living in the city, we have the system of difference equations

$$s_{n+1} = .98 s_n + .1 c_n, \qquad c_{n+1} = .02 s_n + .9 c_n,$$

which we can rewrite in matrix form as

$$\begin{bmatrix} s_{n+1} \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} .98 & .10 \\ .02 & .90 \end{bmatrix} \begin{bmatrix} s_n \\ c_n \end{bmatrix} = A \begin{bmatrix} s_n \\ c_n \end{bmatrix}.$$

With a little bit computuation, we see that the eigenvalues of the matrix A are $\lambda = 1$ and $\mu = .88$, with associated eigenvectors

$$v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad Av = v, \quad u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad Au = .88u.$$

We see by (15) that the general solution to

$$\begin{bmatrix} s_{n+1} \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} .98 & .10 \\ .02 & .90 \end{bmatrix} \begin{bmatrix} s_n \\ c_n \end{bmatrix}$$

is

$$\begin{bmatrix} s_n \\ c_n \end{bmatrix} = \alpha(\lambda)^n v + \beta\mu^n u = \alpha \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \beta(.88)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we must choose the constants α and β to match the initial conditions.

[1] The statistics in this example are all made up for the purposes of explanation.

We can notice something interesting if we take a limit as $n \to \infty$. In this case, we see $\mu^n = (.88)^n \to 0$, and so only the term αv remains in our solution. Thus, in the limit, the ratio of people living in the suburbs to people living in the city is always close to $5 : 1$ after long time, regardless of the initial population distribution.

5.2. **The $N \times N$ case.** Now we continue to the case where the coefficient matrix A is an $N \times N$ matrix. Again, we will only be able to completely solve this when A is diagonalizable, *i.e.* when A has N linearly independent eigenvectors. (To describe the solution in the general case, we would need to write A in Jordan normal form.)

To make the notation a little easier, we write a term in our sequence as a vector y_n, which we think of as a column. Thus when we write y_n we really have

$$y_n = \begin{bmatrix} (y_n)_1 \\ (y_n)_2 \\ \vdots \\ (y_n)_N \end{bmatrix}.$$

As before, we have the system of difference equations

$$y_{n+1} = Ay_n,$$

where each y_n is a vector in $\mathbf{R}^N$ (or in $\mathbf{C}^N$). As before, we have an easy solution if we assume that the initial condition y_0 is an eigenvector of A with eigenvalue λ; in this case, the solution is $y_n = \lambda^n y_0$. We can verfy this solves the difference equation using the fact that y_0 is an eigenvector with eigenvalue λ:

$$Ay_n = A(\lambda^n y_0) = \lambda^n A y_0 = \lambda^{n+1} y_0 = y_{n+1}.$$

The following theorem has exactly the same proof as the previous theorem.

Theorem 9. *Let $A \in M_{N\times N}$ be a matrix with N linearly independent eigenvector $v_1, \ldots, v_N$, such that $Av_i = \lambda_i v_i$ for $i = 1, \ldots, N$. Then the solution to the initial value problem*

$$y_{n+1} = Ay_n, \qquad y_0 = c \in \mathbf{R}^N \tag{16}$$

is given by

$$y_n = a_1 \lambda_1^n v_1 + a_2 \lambda_2^n v_2 + \cdots + a_N \lambda_N^n v_N, \tag{17}$$

where

$$c = a_1 v_1 + a_2 v_2 + \cdots + a_N v_N.$$

We consider another example.[2] Suppose that UCT wishes to gather statistics on its MSc sctudents. A typical MSc is a two-year degree, and so there will be students in their first year, and students in their second year. Students can occasionally take longer than two years to complete their degrees, and we will group them with the remaining second year students. Data shows that in a given academic year 40% of "second year" MSc students will complete their degrees, 30% will remain in the programme without finishing, and 30% will quit their degrees without finishing. Similarly, in a given year, 10% of first year MSc students will finish their degrees,

[2]The statistics in this example are all made up for the purposes of explanation.

50% will move on the the second year of their studies, 20% will need to repeat their first year of studies, and 20% will quit without finishing their degrees. We can represent this data by writing

$$y_{n+1} = Ay_n = \begin{bmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.3 & 1 \end{bmatrix} y_n.$$

Here y_n is a column vector with four entries:

$$y_n = \begin{bmatrix} \text{\# MSc students graduated in year } n \\ \text{\# second year MSc students in year } n \\ \text{\# first year MSc students in year } n \\ \text{\# MSc students who quit in year } n \end{bmatrix}.$$

UCT wants to know (after many years) how many of its MSc students will complete their degrees and how many will quit without finishing.

With a little bit of work we can diagonalize A:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.3 & 1 \end{bmatrix} = QDQ^{-1} \\ &= \begin{bmatrix} 1 & -4 & 19 & 0 \\ 0 & -7 & 40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & -3 & 13 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4/7 & 27/56 & 0 \\ 0 & 1/7 & 5/7 & 0 \\ 0 & 0 & 1/8 & 0 \\ 0 & 3/7 & 29/56 & 1 \end{bmatrix}. \end{aligned}$$

In this notation, the solution to our difference equation is

$$y_n = A^n y_0 = (QDQ^{-1})^n y_0 = QD^nQ^{-1}y_0.$$

We can now study the behavior of this system after many years, by taking a limit as $n \to \infty$. We see

$$\begin{aligned} L &= \lim_{n\to\infty} A^n = \lim_{n\to\infty} (QDQ^{-1})^n = Q(\lim_{n\to\infty} D^n)Q^{-1} \\ &= \begin{bmatrix} 1 & -4 & 19 & 0 \\ 0 & -7 & 40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & -3 & 13 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4/7 & 27/56 & 0 \\ 0 & 1/7 & 5/7 & 0 \\ 0 & 0 & 1/8 & 0 \\ 0 & 3/7 & 29/56 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4/7 & 27/56 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3/7 & 29/56 & 1 \end{bmatrix}. \end{aligned}$$

If we suppose that initially we have the same number of first and second year MSc students, then we can write

$$y_0 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix},$$

and so

$$\begin{aligned} y_\infty &= \lim_{n\to\infty} y_n = \lim_{n\to\infty}(A^n y_0) = (\lim_{n\to\infty} A^n)y_0 \\ &= Ly_0 = \begin{bmatrix} 1 & 4/7 & 27/56 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3/7 & 29/56 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 59/112 \\ 0 \\ 0 \\ 53/112 \end{bmatrix}. \end{aligned}$$

Thus we see that, after many years, we would expect that about 52.7% of MSc students finsh their degrees, and the remaining students quit without finishing.

6. Second order, linear, constant coefficient difference equations

In this section we consider second order, linear, constant coefficient difference equations, which all have the form

$$y_{n+2} = -a_1 y_{n+1} - a_0 y_n + g(n),$$

where a_0 and a_1 are numbers and g is a function of n. We can rearrange this to read

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = g(n). \tag{18}$$

Notice that, because we have a second order difference equation, we shoul assign two initial conditions to solve an initial value problem. In other words, the space of solutions to this difference equation (without prescribing any of the initial data) should be two-dimensional.

6.1. The homogeneous case.

We begin with the homogeneous case, which occurs when $g(n) = 0$ for all n. In this case, (18) becomes

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0. \tag{19}$$

We guess that a solution has the form $y_n = r^n$ for some $r \in \mathbf{R}$ or $r \in \mathbf{C}$. Substituting this into (19) we have

$$0 = r^{n+2} + a_1 r^{n+1} + a_0 r^n = r^n(r^2 + a_1 r + a_0) \Leftrightarrow 0 = r^2 + a_1 r + a_0.$$

Thus we see $y_n = r^n$ solves (19) if and only if

$$r^2 + a_1 r + a_0 = 0. \tag{20}$$

We call (20) the **characteristic equation** associated to (19).

Here is a quick example. Consider

$$y_{n+2} + 5y_{n+1} + 6y_n = 0.$$

In this case, the characteristic equation is

$$0 = r^2 + 5r + 6 = (r+2)(r+3) \Leftrightarrow r = -2, r = -3.$$

Thus we obtain two (linearly independent) solutions

$$(y_1)_n = (-2)^n, \qquad (y_2)_n = (-3)^n.$$

How do we choose between them? The answer to this question is given by our choice of initial conditions.

Before we go further, we need a quick lemma.

Lemma 10. *Let $\{(y_1)_n\}$ and $\{(y_2)_n\}$ solve the difference equation*

$$y_{n+2} + a_1 y_{n+1} + a_0 y_0 = 0.$$

Then so does $\{\alpha(y_1)_n + \beta(y_2)_n\}$ for any pair of real numbers α and β.

Proof. We compute:

$$\begin{aligned} 0 &= \alpha[(y_1)_{n+2} + a_1(y_1)_{n+1} + a_0(y_1)_n] + \beta[(y_2)_{n+2} + a_1(y_2)_{n+1} + a_0(y_2)_n] \\ &= \alpha(y_1)_{n+2} + \beta(y_2)_{n+2} + a_1\alpha(y_1)_{n_1} + a_1\beta(y_2)_{n+1} + a_0\alpha(y_1)_n + a_0\beta(y_2)_n. \end{aligned}$$

□

Remark 1. *This lemma seems very simple, but it is actually very important. It is called the* **principle of superposition**.

Suppose we want to solve the initial value problem

$$y_{n+2} + 5y_{n+1} + 6y_n = 0, \quad y_0 = 1, \quad y_1 = 2.$$

We already know two solutions of the difference equation, $(-2)^n$ and $(-3)^n$, and we know that (by the principle of superposition) $y_n = \alpha(-2)^n + \beta(-3)^n$ is also a solution. In fact, because we only a have second order equation, this is a complete list of all possible solutions to the difference equation. It remains to find α and β to match the initial conditions. We have

$$1 = y_0 = \alpha + \beta, \quad 2 = y_1 = -2\alpha - 3\beta \Rightarrow \alpha = 5, \beta = -4.$$

We have just found our solution: $y_n = 5(-2)^n - 4(-3)^n$.

We have the following general theorem.

Theorem 11. *Consider the difference equation*

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0,$$

and suppose the characteristic equation $r^2 + a_1 r + a_0 = 0$ has two distinct roots r_1 and r_2. Then the general solution has the form $y_n = \alpha r_1^n + \beta r_2^n$, where α and β can be any numbers you like. In this case one can solve the intitial value problem

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0, \quad y_0 = c_0, \quad y_1 = c_1$$

by chooing α and β appropriately.

Proof. We have just seen that both r_1^n and r_2^n solve the difference equation, so by the superposition principle so does $y_n = \alpha r_1^n + \beta r_2^n$. It remains to see that one can solve any initial value problem, so long as $r_1 \neq r_2$. Substituting, we have

$$c_0 = \alpha + \beta, \quad c_1 = \alpha r_1 + \beta r_2,$$

which we can rewrite as

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

This matrix equation always has a unique solution precisely when

$$\det \begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} = r_2 - r_1 \neq 0.$$

□

Here is an example. Consider the initial value problem

$$y_{n+2} + y_{n+1} + \frac{5}{4} y_n = 0, \quad y_0 = -1, \quad y_1 = 1.$$

The characteristic equation is

$$r^2 + r + \frac{5}{4} \Rightarrow r_{\pm} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i.$$

We see that we have two distinct roots, which happen to be complex conjugate numbers. The general solution is then

$$y_n = \alpha \left(-\frac{1}{2} + i\right)^n + \beta \left(-\frac{1}{2} - i\right)^n,$$

and we can determine the coefficients α and β from the initial conditions. Substituting, we see

$$-1 = y_0 = \alpha + \beta, \qquad 1 = y_1 = \alpha \left(-\frac{1}{2} + i\right) + \beta \left(-\frac{1}{2} - i\right),$$

which we can solve to get

$$\alpha = -\frac{1}{2} - \frac{i}{4}, \quad \beta = -\frac{1}{2} + \frac{i}{4} \Rightarrow y_n = \left(-\frac{1}{2} - \frac{i}{4}\right)\left(-\frac{1}{2} + i\right)^n + \left(-\frac{1}{2} + \frac{i}{4}\right)\left(-\frac{1}{2} - i\right)^n.$$

We can, of course, rewrite all these complex numbers in polar form. Then

$$r_{\pm} = \frac{\sqrt{5}}{2}(\cos\theta + i \sin\theta), \quad \tan\theta = -2.$$

Either way, we obtain the same answer in the end.

We next consider the case when the characteristic equation (20) has a repeated root, which we call r_*. This means

$$r^2 + a_1 r + a_0 = (r - r_*)^2 \Leftrightarrow a_1 = -2r_*, a_0 = r_*^2.$$

In this case, we need to find a second linearly independent, other than r_*^n. We guess again, this time with nr_*^n. We check that

$$(n+2)r_*^{n+2} + a_1(n+1)r_* + a_0 n = (n+2)r_*^2 - 2(n+1)r_*^2 + nr_*^2 = 0.$$

We now have two linearly independent solutions, and we can try to write

$$y_n = \alpha r_*^n + \beta n r_*^n = (\alpha + n\beta) r_*^n.$$

Again, we want to show that we can solve any initial value problem using a linear combination of these two solutions. Suppose we want to solve

$$y_{n+2} + a_1 y_{n+1} + a_0 y_0 = 0, \quad y_0 = c_0, \quad y_1 = c_1,$$

where r_* is a double root of the characteristic equation $r^2 + a_1 r + a_0 = 0$. We look for a solution of the form

$$y_n = (\alpha + n\beta)r_*^n.$$

Substituting, we see

$$c_0 = \alpha, \quad c_1 = (\alpha + \beta)r_* \Leftrightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ r_* & r_* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

This last matrix equation always has a unique solution provided

$$\det \begin{bmatrix} 1 & 0 \\ r_* & r_* \end{bmatrix} = r_* \neq 0.$$

We consider another example, and solve the initial value problem

$$y_{n+2} - 4y_{n+1} + 4y_n = 0, \quad y_0 = 2, \quad y_1 = 1.$$

The characteristic equation is

$$0 = r^2 - 4r + 4 = (r-2)^2 \Rightarrow r_* = 2.$$

We see that we have a double root at $r_* = 2$, and so the general solution is

$$y_n = (\alpha + n\beta)2^n.$$

Substituting our initial conditions we see

$$2 = y_0 = \alpha, \quad 1 = y_1 = 2(\alpha + \beta) \Rightarrow \alpha = 2, \beta = -\frac{3}{2}.$$

The solution is therefore

$$y_n = \left(2 - \frac{3}{2}n\right) 2^n.$$

There is one remaining case to consider, namely $r_* = 0$ is a double root of the characteristic equation. In this case we have

$$y_{n+2} = 0, \quad y_0 = c_0, \quad y_1 = c_1,$$

and we can read off that the only solution is $\{c_0, c_1, 0, 0, 0, \dots\}$. We summarize with the following theorem.

Theorem 12. *Consider the difference equation*

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0,$$

with characteristic equation

$$r^2 + a_1 r + a_0 = 0.$$

If the characteristic equation has two distinct roots $r_1 \neq r_2$ then all solutions have the form

$$y_n = \alpha r_1^n + \beta r_2^n$$

for some choice of numbers α and β. If the characteristic equation has a nonzero double root $r_* \neq 0$ then all solutions have the form

$$y_n = (\alpha + n\beta)r_*^n.$$

Finally, if $r_* = 0$ is a double root then the general solution is

$$\{c_0, c_1, 0, 0, 0, \dots\}.$$

6.2. **Nonhomogeneous equations.** In general, it is very difficult to solve

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = g(n)$$

when the right hand side $g(n)$ of the equation is nonzero. However, there is one special case in which we can explicitly write down the solution:

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = b, \tag{21}$$

where b is a constant. In this case, we mimic what we did in the first order case: we find a particular solution of the nonhomogeneous equation, and write the general solution as the sum of this particular solution and the solution to the associated homogeneous equation, which is $y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0$. The simplest possible particular solution is a constant, *i.e.* $(y_p)_n = \alpha$ for all n. Plugging this in, we have

$$\alpha + a_1\alpha + a_0\alpha = \alpha(1 + a_1 + a_0) = b \Rightarrow \alpha = \frac{b}{1 + a_1 + a_0}. \tag{22}$$

This is a valid solution if $1 + a_1 + a_0 \neq 0$. If $1 + a_1 + a_0 = 0$ then we try to find a particular solution of the form $(y_p)_n = \alpha n$. Substituting we have

$$\begin{aligned} b &= (y_p)_{n+2} + a_1(y_p)_{n+1} + a_0(y_p)_n = \alpha(n+2) + \alpha a_1(n+1) + \alpha a_0 n \\ &= \alpha[n + 2 + a_1 n + a_1 + a_0 n] = \alpha[n(1 + a_1 + a_0) + 2 + a_1] \\ &= \alpha[2 + a_1], \end{aligned}$$

where we have used $1 + a_1 + a_0 = 0$. Thus we have

$$\alpha = \frac{b}{2 + a_1},$$

which is a valid solution so long as $a_1 \neq -2$. If we're very unlucky, then $1+a_1+a_0 = 0$ and $a_1 = -2$, so we must have $a_0 = 1$. In this case our difference equation reads

$$y_{n+2} - 2y_{n+1} + y_n = b,$$

and we look for a solution of the form $(y_p)_n = \alpha n^2$. Substituting, we have

$$\begin{aligned} b &= (y_p)_{n+2} - 2(y_p)_{n+1} + (y_p)_n = \alpha(n+2)^2 - 2\alpha(n+1)^2 + \alpha n^2 \\ &= \alpha[n^2 - 4n + 4 - 2(n^2 + 2n + 1) + n^2] = 2\alpha, \end{aligned}$$

and so we have $\alpha = b/2$ as our solution.This now lists a particular solution of (21) in all possible cases. As we did before, we can find the general solution to this difference equation by summing this particular solution we have just found with the general solution to the associated homogeneous equation:

$$y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0.$$