

NOTES FOR SECOND YEAR DIFFERENCE AND DIFFERENTIAL EQUATION PART III: FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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1. DEFINITIONS

An **ordinary differential equation** of order k has the form

$$(1) \quad F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) = 0,$$

where F is a function of $k+2$ variables. We will usually write ODE as an abbreviation for ordinary differential equation. A function $y(x)$ is a **solution** of the ODE (1) in the interval $a < x < b$ if it satisfies the equation (1) for all x in this interval.

We will usually be interested in solving an **initial value problem**. For an ODE of order k , this has the form

$$(2) \quad \begin{aligned} F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) &= 0 \\ y(x_0) = c_0, \quad \frac{dy}{dx}(x_0) = c_1, \dots, \quad \frac{d^{k-1}y}{dx^{k-1}}(x_0) &= c_{k-1}. \end{aligned}$$

Notice that, for an ODE of order k , one must prescribe k initial conditions.

It is usual to see some examples.

- One can check that $y(x) = \frac{1}{2}x^2$ solves the ODE $(y')^2 - 2y = 0$.
- One can also check that the function $y(x) = \sqrt{1-x^2}$ solves the ODE

$$\frac{d}{dx} \left(\frac{y'}{\sqrt{1+(y')^2}} \right) = 1,$$

at least for $-1 < x < 1$.

- For any real number $k \neq 0$, the functions $y_1(x) = \cos(kx)$ and $y_2(x) = \sin(kx)$ solve the ODE $y'' + k^2y = 0$. The general solution to this ODE is then

$$y(x) = a_1y_1(x) + a_2y_2(x) = a_1 \cos(kx) + a_2 \sin(kx),$$

and one can solve an initial value problem for this ODE by choosing a_1 and a_2 so as to match the initial conditions.

- For any real number $k \neq 0$ the function $y_1(x) = e^{kx}$ and $y_2(x) = e^{-kx}$ solve the ODE $y'' - k^2y = 0$. The general solution to this ODE is then

$$y(x) = a_1y_1(x) + a_2y_2(x) = a_1e^{kx} + a_2e^{-kx},$$

and one can solve an initial value problem for this ODE by choosing a_1 and a_2 so as to match the initial conditions.

As we did with difference equations, we can describe an ODE with various terms. An ODE is **autonomous** if the independent variable x does not appear in the formula for the function F . An ODE is **homogeneous** if $y \equiv 0$ is a solution, *i.e.* $F(x, 0, \dots, 0) = 0$. An ODE is linear if it has the form

$$\begin{aligned} 0 &= F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^k y}{dx^k}\right) \\ &= Q(x) + P_0(x)y + P_1(x)\frac{dy}{dx} + \dots + P_k(x)\frac{d^k y}{dx^k}, \end{aligned}$$

where $Q, P_0, \dots, P_k$ are functions of x . In this case, we will usually assume that $P_k \neq 0$, and rearrange this equation to read

$$q(x) = \frac{d^k y}{dx^k} + p_{k-1}\frac{d^{k-1}y}{dx^{k-1}} + \dots + p_1(x)\frac{dy}{dx} + p_0(x)y(x),$$

where Here $q = -Q/P_k$ and $p_j = P_j/P_k$ for $j = 0, 1, \dots, k-1$. In this form we call $p_0, p_1, \dots, p_{k-1}$ the coefficients of the linear ODE, and we say it has constant coefficients if all of these functions are constants.

For this set of notes we will concentrate on first order ODEs, which (according to (1)) have the form

$$F(x, y, y') = 0.$$

In an associated initial value problem we prescribe one initial value, namely $y(x_0) = c_0$. According to the descriptions above, a first order ODE is autonomous if it has the form $F(y, y') = 0$ and it is linear if it has the form $y' = p(x)y + q(x)$.

2. SEPARABLE EQUATIONS

A **separable** ODE has the form

$$(3) \quad \frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

Notice that we require $g(y) \neq 0$ in order that (3) makes sense. We can write the solution of this equation by integrating and using the Fundamental Theorem of Calculus. Rearrange (3) to read

$$g(y)\frac{dy}{dx} = f(x)$$

and integrate both sides of this equation with respect to x to get

$$(4) \quad \int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} g(y)\frac{dy}{dx}dx = \int_{y(x_0)}^{y(x_1)} g(y)dy.$$

Here we have used the chain rule to change variables within the integral.

The most basic example to consider is

$$y' = ky, y(0) = c,$$

where $k \neq 0$ is a fixed number and c is also a fixed number. In fact, this is the most basic differential equation to understand, full stop. We can solve by writing

$$kx = \int k dx = \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{dy}{y} = \ln y + C.$$

(You might think there should be a second constant of integration from the integral on the left hand side of the equation, but we can absorb it into the constant of integration on the right hand side. After all, the sum of two constants is still a constant.) Solving for y , we find

$$y = e^{kx+C} = ce^{kx},$$

where $c = e^C$. One can even check that this is the correct initial condition:

$$y(0) = ce^0 = c.$$

Here are some more examples. Consider the initial value problem

$$y' = \frac{x}{y}, \quad y(1) = 2.$$

We rewrite the ODE as $yy' = x$ and integrate (starting from $x_0 = 1$):

$$\frac{1}{2}(x_1^2 - 1) = \int_1^{x_1} x dx = \int_{y(1)}^{y(x_1)} yy'(x) dx = \frac{1}{2}(y^2(x_1) - y^2(1)).$$

Substituting the initial condition $y(1) = 2$ and solving for $y(x_1)$ we see

$$y(x_1) = \sqrt{2 + x_1^2 - 1} = \sqrt{1 + x_1^2}.$$

Thus we obtain a formula for our solution, namely $y(x) = \sqrt{1 + x^2}$. Notice that this function exists for all values of x , and so, for the initial condition $y(1) = 2$, we have a global solution. It is easy to check that we do not get a global solution for other choices of initial data. For instance, if we change $y(1) = 1/2$ then the solution is

$$y(x) = \sqrt{x^2 - 3/4},$$

which only exists as a sensible function if $|x| > \sqrt{3}/2$.

Next we consider

$$y' = xy, \quad y(1) = 1.$$

Again, we rewrite the equation and integrate, to obtain

$$\frac{1}{2}(x_1^2 - x_0^2) = \int_{x_0}^{x_1} x = \int_{y(x_0)}^{y(x_1)} \frac{y' dx}{y} = \ln y(x_1) - \ln y(x_0) = \ln \left(\frac{y(x_1)}{y(x_0)} \right).$$

Using the initial condition $y(x_0) = y(1) = 1$ we see

$$\ln y(x) = \frac{1}{2}(x^2 - 1) \Rightarrow y = e^{\frac{1}{2}(x^2 - 1)}.$$

We consider

$$y' = y^2, \quad y(0) = 1.$$

We separate variables to see

$$x = \int dx = \int \frac{dy}{y^2} = -\frac{1}{y} + c \Rightarrow y = \frac{1}{c-x}.$$

Matching the initial condition we see

$$1 = y(0) = \frac{1}{c} \Rightarrow c = 1 \Rightarrow y = \frac{1}{1-x}.$$

This gives us a well-defined solution so long as $x < 1$.

As a final example, we consider the model for logistic growth. This is an ODE which is supposed to model a population in an area which cannot carry more than a certain number of people/fish/animals/etc. For instance, we might suppose that we have a pond with a certain number of fish in it, and the pond can only comfortably hold 10,000 fish. That is, if the number of fish is more than 10,000, they become over-crowded and start to die. If we let $y(t)$ be the number of fish at time t , then we must have $y'(t) < 0$ if $y > 10,000$. On the other hand, when the population of fish is small we expect a linear growth rate for the fish population. For instance, we might expect something like $y'(t) \simeq 10y$ when y is small. We can combine these two phenomenon with the model

$$y' = 10y \left(1 - \frac{y}{10,000} \right).$$

This is a separable equation, which we can rearrange and integrate to solve, giving

$$t + C = \int \frac{dy}{y \left(1 - \frac{y}{10,000} \right)} = \ln y - \ln \left(1 - \frac{y}{10,000} \right) = \ln \left(\frac{y}{1 - \frac{y}{10,000}} \right).$$

Here we have used partial fractions to integrate. We can solve for y by taking an exponential. Let $c = e^C$, so that we have

$$ce^t = \frac{y}{1 - \frac{y}{10,000}} \Rightarrow y = \frac{ce^t}{1 + \frac{ce^t}{10,000}} = \frac{10,000ce^t}{10,000 + ce^t}.$$

If we set the initial population to 5,000, then we find

$$5000 = y(0) = \frac{10,000c}{10,000 + c} \Rightarrow c = 10,000 \Rightarrow y(t) = 10,000 \frac{e^t}{1 + e^t}.$$

We can verify that in this case the population y is strictly increasing, and that it has a horizontal asymptote at $y = 10,000$ (which it never actually reaches).

3. FIRST ORDER LINEAR ODE

As we wrote above, a first order linear ODE has the form

$$y' + p(x)y = q(x)$$

for two give functions p and q . Let's actually start with an example, and consider

$$y' + \frac{1}{x}y = 2.$$

Multiplying by x , we can rewrite this equation as

$$xy' + y = (xy)' = 2x \Rightarrow xy = x^2 + c \Rightarrow y = x + \frac{c}{x}.$$

We can try to mimic this example in general. We start with the ODE

$$y' + p(x)y = q(x),$$

and we'd like to multiply by some factor to make the left hand side into an exact derivative. We call this integrating factor $I(x)$, and we write

$$I(x)y' + I(x)p(x)y = Iq(x).$$

Ideally, we'd like the left hand side of the equation above to be $(Iy)'$, so let's see what that means. We then must have

$$(Iy)' = I'y + Iy' = Iy' + Ipy \Rightarrow I'y = Ipy \Rightarrow \frac{I'}{I} = p.$$

This is assuming that $y \neq 0$. We can integrate this last equation to get

$$\ln I = \int \frac{I'}{I} dx = \int p(x) dx \Rightarrow I(x) = e^{\int p(x) dx}.$$

There is an integration of constant we're ignoring, but adding it back in will only multiply I by a constant factor, so we are in fact safe to ignore it. We now have multiply our original ODE by the integrating factor I to get

$$y'e^{\int p(x) dx} + p(x)ye^{\int p(x) dx} = \frac{d}{dx} \left(ye^{\int p(x) dx} \right) = q(x)e^{\int p(x) dx}.$$

We now integrate with respect to x to solve:

$$y(x) = e^{-\int p(x) dx} \left(c + \int q(x)e^{\int p(t) dt} dx \right).$$

Let's see some more examples. First consider

$$y' + 2xy = x^2, \quad y(0) = 1.$$

The integrating factor is

$$I(x) = e^{\int p(x) dx} = e^{\int 2x dx} = e^{x^2},$$

so we have

$$\left(ye^{x^2} \right)' = x^2 e^{x^2} \Rightarrow y = e^{-x^2} \left(c + \int x^2 e^{x^2} dx \right).$$

We might have trouble evaluating this last integral, but in fact we really don't need to, at least in order to find c . We evaluate at $x = 0$ and write the integral as a definite integral.

$$1 = y(0) = e^{-0^2} \left(c + \int_0^0 x^2 e^{x^2} dt \right) \Rightarrow c = 1 \Rightarrow y(x) = e^{-x^2} \left(1 + \int_0^x t^2 e^{t^2} dt \right).$$

Notice that we needed in the last formula for $y(x)$ we needed to call the integration variable something other than x , so we chose t (not that it really matters what we call the integration variable).

Next consider

$$y' - \tan(x)y = \cos x, \quad y(0) = 2.$$

This time the integrating factor is

$$I(x) = e^{\int \tan x dx} = e^{-\ln \cos x} = \frac{1}{\cos x}.$$

Multiplying by I we now have

$$\left(\frac{y}{\cos x}\right)' = 1 \Rightarrow y = \cos x(x + c).$$

Evaluating at $x = 0$ we now have

$$2 = y(0) = \cos(0)(0 + c) \Rightarrow c = 2,$$

and so the solution to our initial value problem is

$$y(x) = \cos x(x + 2).$$

Sometimes an ODE which appears to be nonlinear is actually linear. Here is an example. Consider the initial value problem

$$y^2 y' + \frac{3y^3}{x} = xy^2, \quad y(1) = 3.$$

This appears to be a nonlinear ODE, because we have various powers of y involved, but if we divide through by y^2 , the ODE becomes

$$y' + \frac{3y}{x} = x,$$

which is in fact a nice linear equation. Of course, we should check that we are not dividing by 0 here. We can see that the ODE is homogeneous, so $y \equiv 0$ is in fact a solution, but it does not match our initial conditions. We will see that the other solutions do our ODE never vanish. Returning to our transformed ODE, we wish to solve

$$y' + \frac{3y}{x} = x.$$

The integrating factor is

$$I = e^{\int (3/x) dx} = e^{3 \ln x} = e^{\ln(x^3)} = x^3,$$

so multiplying by I we now have

$$(x^3 y)' = x^4 \Rightarrow x^3 y = \frac{1}{5} x^5 + c \Rightarrow y = \frac{1}{5} x^2 + \frac{c}{x^3}.$$

We finally find c by matching the initial condition. We have

$$3 = y(1) = \frac{1}{5} + c \Rightarrow c = \frac{14}{5} \Rightarrow y = \frac{1}{5} x^2 + \frac{14}{5x^3}.$$

We'll conclude this section with some applications. First suppose you set up a saving account towards your retirement. You begin the account with R10,000, and

every month you deposit R500. Your account earns 8% per year, compounded continuously. If we let $S(t)$ be the amount you've saved after time t , then the differential equation for S is

$$\frac{dS}{dt} = .08S + 6000, \quad S(0) = 10,000.$$

(Notice that the constant on the right hand side is $6000 = 12 \cdot 500$ because the appropriate time period is one year, not one month.) This is an initial value problem for a linear, first order ODE, and we can find its solution using integrating factors. The integrating factor is

$$I = e^{\int (-.08)dt} = e^{-.08t},$$

and so we have

$$S = ce^{.08t} - 75,000.$$

Using the initial condition we find

$$10,000 = S(0) = c - 75,000 \Rightarrow c = 85,000 \Rightarrow S = 85,000e^{.08t} - 75,000.$$

How much money will we have saved up in 20 years? We evaluate:

$$S(20) = 346,007.76.$$

We can compare this to the amount of money we have actually deposited into the account, which is

$$10,000 + 20 \cdot 6,000 = 130,000.$$

Next we look at mortgages. Suppose you'd like to buy a house, and can afford to spend R7500 per month on mortgage payments. The bank will charge $r\%$ annual interest on our loan, compounded continuously, and we plan to pay off the loan in 20 years. This time, we let $S(t)$ be the amount we owe at time t , and we have the differential equation

$$\frac{dS}{dt} = \frac{r}{100}S - 7500 \cdot 12, \quad S(0) = S_0.$$

Here the initial condition S_0 is the amount we can borrow, which is what we wish to compute. We solve the differential equation as we did before, to get

$$S(t) = ce^{\frac{rt}{100}} + \frac{9,000,000}{r}.$$

This time we determine c by using $S(20) = 0$, namely the fact that we pay off the loan in 20 years. We have

$$0 = S(20) = ce^{r/5} - \frac{9,000,000}{r} \Rightarrow c = -\frac{9,000,000e^{-r/5}}{r}.$$

Thus the initial amount we can borrow is

$$S(0) = \frac{9,000,000}{r}(1 - e^{-r/5}).$$

Finally we plug in some values of r :

$$r = 6 \Rightarrow S(0) = 1,048,208.68, \quad r = 9 \Rightarrow 834,701.11, \quad r = 12 \Rightarrow 681,961.54.$$

4. AUTONOMOUS FIRST ORDER ODE

An autonomous, first order ODE has the form $F(y, y') = 0$. We will see in this section that most such equations are in fact separable, but in this section we will concentrate on the properties of solutions, rather than formulas.

We begin by revisiting the logistic equation. Before, we wrote this as

$$y' = 10y \left(1 - \frac{y}{10,000} \right),$$

and it is supposed to model the population of fish in a pond which should only hold at most 10,000 fish. We can generalize this idea to model a population which increases for small numbers, but decreases when the population is larger than some critical number N , which we call the carrying capacity. We can model this as

$$y' = ky \left(1 - \frac{y}{N} \right),$$

where $k > 0$ gives the rate of increase of y when it is small. We can in fact determine everything we want to know about solutions without actually solving the ODE. First, observe that we have two constant solutions, which correspond to the zeroes of y' :

$$0 = y' = y \left(1 - \frac{y}{N} \right) \Leftrightarrow y = 0 \text{ or } y = N.$$

We can analyze the stability of these constant solutions by looking at the sign of y' . Notice that $y' > 0$ for $0 < y < N$ and $y' < 0$ for $y > N$. Now suppose we start with an initial condition $y_0 > 0$ which is small. Then $y' > 0$, so y will increase, moving away from the constant solution 0, no matter how small y_0 is. On the other hand, suppose y_0 is close to N . If $y_0 < N$ then $y' > 0$ and so y will increase towards N (but never actually reach it). Similarly, if $y_0 > N$ then $y' < 0$ and y will decrease towards N (but never actually reach it). Thus we see that 0 is unstable but N is asymptotically stable, to borrow the terminology we used for difference equations.

At this point, we make some formal definitions. Consider the autonomous ODE $y' = f(y)$. The **equilibrium solutions** of this ODE are the constant solutions.

Lemma 1. *The constant function α is an equilibrium of the ODE $y' = f(y)$ if and only if $f(\alpha) = 0$.*

Proof. The function $y(x) = \alpha$, where α is a constant, is a solution of the ODE if and only if

$$0 = \frac{d}{dx}(\alpha) = y' = f(y) = f(\alpha).$$

□

We have a definition of stability which will look very familiar.

Definition 1. *An equilibrium $y = \alpha$ is **stable** if solutions with initial values near α stay near α for all x . More precisely, for every $\epsilon > 0$ there is $\delta > 0$ such that if $|y_0 - \alpha| < \delta$ and we have $|y(x) - \alpha| < \epsilon$ for all x . An equilibrium is **asymptotically stable** if solutions with initial value close to α actually limit towards α . More precisely, there is $\delta > 0$ such that if $|y_0 - \alpha| < \delta$ then $\lim_{x \rightarrow \infty} y(x) = \alpha$.*

Notice that an equilibrium solution which is asymptotically stable is also stable, but the reverse may not be true. (In other words, one can find stable equilibria which are not asymptotically stable.) An equilibrium which is not stable is called **unstable**.

We have a familiar looking theorem.

Theorem 2. *Let α be an equilibrium solution of the autonomous ODE $y' = f(y)$ (i.e. $f(\alpha) = 0$). This equilibrium is asymptotically stable if $f'(\alpha) < 0$, and unstable if $f'(\alpha) > 0$. This test is inconclusive if $f'(\alpha) = 0$.*

The formal proof of this theorem is a bit beyond the present course, but here is some intuition for why it is true. If $f'(\alpha) < 0$ then f is decreasing near α . We already know that $f(\alpha) = 0$, so this implies $f(y_0) > 0$ for $y_0 < \alpha$ and $f(y_0) < 0$ for $y_0 > \alpha$. Using the differential equation $y' = f(y)$, we see this means that y will be increasing if $y_0 < \alpha$, while y will be decreasing if $y_0 > \alpha$. Thus, we see that if we start with y_0 near α then $y(x)$ will move towards α , both for $y_0 > \alpha$ and for $y_0 < \alpha$, and so we conclude that α is asymptotically stable. The reasoning for the case $f'(\alpha) > 0$ is similar.

As an example, we find and classify all the equilibria of the ODE

$$y' = y \cos y.$$

We have

$$0 = y' = y \cos y \Leftrightarrow y = 0 \text{ or } y = \frac{(2k+1)\pi}{2}.$$

We must evaluate a derivative:

$$\frac{df}{dy} = \frac{d}{dy}(y \cos y) = \cos y - y \sin y.$$

Evaluating, we see $f'(0) = 1 > 0$, and so 0 is unstable. Also,

$$f' \left(\frac{(2k+1)\pi}{2} \right) = -\frac{(2k+1)\pi}{2} \sin \left(\frac{(2k+1)\pi}{2} \right) = (-1)^{k+1} \frac{(2k+1)\pi}{2}.$$

Thus the remaining equilibria are asymptotically stable if k is even and unstable if k is odd.

We just saw at the beginning of this section that the logistic equation

$$y' = ky \left(1 - \frac{y}{N} \right)$$

has two equilibrium solutions: $y = 0$ and $y = N$. The solution $y = 0$ is unstable and the solution $y = N$ is asymptotically stable. We can adapt this model by adding in a threshold T with $0 < T < N$. Here the interpretation is that if the population falls below T then it starts to die out (because there are too few animals to reproduce). Now the differential equation is

$$y' = f(y) = -ky \left(1 - \frac{y}{T} \right) \left(1 - \frac{y}{N} \right).$$

The equilibrium solutions are the zeroes of $f(y)$, which are easy to find; they are $y_0 = 0$, $y_1 = T$ and $y_2 = N$. What about stability? We take a derivative:

$$\frac{df}{dy} = -k \left(1 - \frac{y}{T}\right) \left(1 - \frac{y}{N}\right) + \frac{ky}{T} \left(1 - \frac{y}{N}\right) + \frac{ky}{N} \left(1 - \frac{y}{T}\right).$$

Now evaluate these at each of our equilibria:

$$f'(0) = -k < 0, \quad f'(T) = k \left(1 - \frac{T}{N}\right) > 0, \quad f'(N) = k \left(1 - \frac{N}{T}\right) < 0.$$

Here we have used the fact that $0 < T < N$. We conclude that $y_0 = 0$ and $y_2 = N$ are asymptotically stable, while $y_1 = T$ is unstable.

5. EXISTENCE AND UNIQUENESS (OPTIONAL)

In this section we sketch a proof that, under some conditions, a first order initial value problem always has a unique solution, at least in a small interval around the initial value. The precise theorem is the following:

Theorem 3. *Let $F(x, y)$ be a function of two variables which is continuous and such that $\frac{\partial F}{\partial y}$ exists and is continuous. Then there exists $\epsilon > 0$ such that the initial value problem*

$$(5) \quad \frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0$$

has a unique solution in the interval $x_0 - \epsilon < x < x_0 + \epsilon$.

It is worthwhile to consider two examples showing that the hypotheses of this theorem are as weak as possible. First, consider the initial value problem

$$y' = F(y) = 2\sqrt{y}, \quad y(0) = 0.$$

Observe that $\frac{\partial F}{\partial y}$ is not continuous at $y_0 = 0$ (in fact it does not even exist there). Also, this initial value problem has two solutions, namely $y(0) = x^2$ and $y(x) = 0$. This is exactly why we must assume the function $F(x, y)$ is reasonably nice. Second, consider the initial value problem

$$y' = F(y) = y^2, \quad y(0) = 1.$$

This is a separable ODE, and the general solution is $y(x) = (c - x)^{-1}$, where c is the constant of integration. Matching the initial condition $y(0) = 1$, we see $c = 1$, and so $y(x) = (1 - x)^{-1}$. This solution exists only in the interval $\{x < 1\}$. Therefore, even if the right hand side $F(x, y)$ is a very nice function, we cannot expect that the solution of our initial value problem will exist for all x .

We will prove Theorem 3 in several steps. First we will transform (5) into an integral equation, then we will define a sequence of functions which we would like to converge to the solution, and finally we will prove convergence. Along the way we will need to discuss how one can measure the distance between two functions, and the contraction mapping principle.

Lemma 4. *A function $y(x)$ solves (5) if and only if*

$$(6) \quad y(x) = y_0 + \int_{x_0}^x F(t, y(t)) dt.$$

Proof. If y solves (6) then by the Fundamental Theorem of Calculus we have

$$\frac{dy}{dx} = \frac{d}{dx} \int_{x_0}^x F(t, y(t)) dt = F(x, y(x)),$$

and

$$y(x_0) = y_0 + \int_{x_0}^{x_0} F(t, y(t)) dt = y_0,$$

which means exactly that y solves (5). On the other hand, if y solves (5) then

$$y_0 + \int_{x_0}^x F(t, y(t)) dt = y_0 + \int_{x_0}^x y'(t) dt = y(x_0) + y(x) - y(x_0) = y(x).$$

□

For our second step, we define a sequence of functions. Start with $y_0(x) = y_0$ for all x . Then for $n \geq 0$ define

$$(7) \quad y_{n+1} = y_0 + \int_{x_0}^x F(t, y_n(t)) dt = \Phi(y_n).$$

Here we are thinking of Φ as a function on the space of function. That is, Φ is an operation which takes a certain function, y_n , and turns it into a new function y_{n+1} . Things like Φ are usually called **functionals**. Observe that y solves (6) if and only if $y = \Phi(y)$, so we immediately have the following corollary.

Corollary 5. *The function y solves (5) if and only if $y = \Phi(y)$.*

To complete the proof of Theorem 3 we would now like to show that the sequence of functions $\{y_n\}$ converges to a fixed point of the function Φ ; that is, $y_n \rightarrow y_\infty = \Phi(y_\infty)$. We need some tools.

Definition 2. *Let $[a, b] = \{a \leq x \leq b\}$ be a bounded, closed interval, and let $y_1(x)$ and $y_2(x)$ be two continuous functions on $[a, b]$. Then*

$$\text{dist}(y_1, y_2) = \|y_1 - y_2\| = \max_{x \in [a, b]} |y_1(x) - y_2(x)|.$$

Definition 3. *Let $[a, b] = \{a \leq x \leq b\}$ be a bounded, closed interval, and let $\{y_n(x)\}$ be a sequence of continuous functions on $[a, b]$. We say $y_n \rightarrow y$ pointwise if for each x we have $y_n(x) \rightarrow y(x)$. We say $y_n \rightarrow y$ uniformly if $\|y_n - y\| \rightarrow 0$.*

Notice that uniform convergence implies pointwise convergence, but the reverse implication may not be true. A good example to keep in mind for this phenomenon is $\{y_n(x) = x^n\}$ on the interval $[a, b] = [0, 1]$. We have

$$y_\infty(x) = \lim_{n \rightarrow \infty} y_n(x) = \lim_{n \rightarrow \infty} x^n = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

It is also true, but a little harder to prove, that y_∞ defined above is not the uniform limit of any sequence of continuous functions. Thus, the sequence $\{x^n\}$ converges pointwise but not uniformly.

We will need the following result, which we will not prove.

Theorem 6. *A sequence of functions $\{y_n\}$ on the closed interval $[a, b]$ converges uniformly if and only if it satisfies the Cauchy criterion: for all $\epsilon > 0$ there is N such that $m, n > N$ implies $\|y_m - y_n\| < \epsilon$.*

Definition 4. *A functional Ψ on the space of continuous functions is a contraction if there is k such that $0 < k < 1$ and*

$$\|\Psi(y_1) - \Psi(y_2)\| \leq k\|y_1 - y_2\|$$

for all functions y_1 and y_2 .

We sketch a proof of the contraction mapping principle, which is a very useful theorem.

Theorem 7. *Any contraction has a unique fixed point. That is, if Ψ is a contraction on the space of functions $y : [a, b] \rightarrow \mathbf{R}$ then there is a unique function y_* which satisfies $\Psi(y_*) = y_*$.*

Proof. Choose any continuous function $y_0 : [a, b] \rightarrow \mathbf{R}$ and define the sequence of functions

$$y_1 = \Psi(y_0), \quad y_2 = \Psi(y_1), \quad \dots, y_{n+1} = \Psi(y_n).$$

First notice that

$$\|y_{n+1} - y_n\| = \|\Psi(y_n) - \Psi(y_{n-1})\| \leq k\|y_n - y_{n-1}\|,$$

so by induction

$$\|y_{n+1} - y_n\| \leq k^n \|y_1 - y_0\|.$$

Now set $M = \|y_1 - y_0\|$ and use the triangle inequality. For $n > m$ we have

$$\begin{aligned} \|y_n - y_m\| &\leq \|y_n - y_{n-1}\| + \|y_{n-1} - y_{n-2}\| + \dots + \|y_{m+1} - y_m\| \\ &\leq k^{n-1}\|y_1 - y_0\| + k^{n-2}\|y_1 - y_0\| + \dots + k^m\|y_1 - y_0\| \\ &= M \sum_{j=m}^{n-1} k^j \leq Mk^m \sum_{j=0}^{\infty} k^j = \frac{Mk^m}{1-k}. \end{aligned}$$

Recall that $k < 1$, so, once we choose $\epsilon > 0$, we can choose m large enough so that $\frac{Mk^m}{1-k} < \epsilon$. This shows $\{y_n\}$ satisfies the Cauchy criterion, so by our theorem above it converges uniformly to some function y_* . Now apply Ψ to y_* to see

$$\Psi(y_*) = \lim_{n \rightarrow \infty} \Psi(y_n) = \lim_{n \rightarrow \infty} y_{n+1} = y_*,$$

so y_* is indeed a fixed point. Finally, we prove uniqueness. Suppose there is some other fixed point $y_\dagger$ such that $\Psi(y_\dagger) = y_\dagger$. Then

$$\|y_* - y_\dagger\| = \|\Psi(y_*) - \Psi(y_\dagger)\| \leq k\|y_* - y_\dagger\| \Rightarrow \|y_* - y_\dagger\| = 0 \Rightarrow y_* = y_\dagger.$$

□

Proof of Theorem 3. We first see that we're done if we show the functional Φ defined by (7) is a contraction on continuous function in the closed, bounded interval $[x_0 - \epsilon, x_0 + \epsilon]$, for some small positive ϵ . If Φ is a contraction, then the contraction mapping principle tells us that it has a unique fixed point $y_* = \Phi(y_*)$, which solves (5).

To show that Φ is a contraction, we must estimate $\|\Phi(u) - \Phi(v)\|$ for two functions u and v . First let

$$M = \max \left| \frac{\partial F}{\partial y} \right|.$$

Then we have

$$\begin{aligned} |\Phi(u)(x) - \Phi(v)(x)| &= \left| \int_{x_0}^x F(t, u(t)) dt - \int_{x_0}^x F(t, v(t)) dt \right| \\ &= \left| \int_{x_0}^x F(t, u(t)) - F(t, v(t)) dt \right| \\ &\leq \int_{x_0}^x |F(t, u(t)) - F(t, v(t))| dt \\ &\leq \int_{x_0}^x M \|u - v\| dt = M|x - x_0| \|u - v\|. \end{aligned}$$

Notice that $x \in [x_0 - \epsilon, x_0 + \epsilon]$, so $|x - x_0| < \epsilon$. Now we can choose ϵ small enough so that $\epsilon M \leq \frac{1}{2} < 1$, which implies Φ is a contraction, completing the proof. $\square$

The sequence of functions $y_n = \Phi^n(y_0)$ is usually called the Picard iteration sequence, and this method is called the Picard iteration scheme. It is named after the French mathematician Émile Picard, who lived from 1856 to 1941.