Some Examples in Second Order ODE

In class we've seen how to solve certain second order, linear differential equations. In particular, we know how to solve an equation of the form

$$y'' + ay' + by = 0, \qquad y(0) = y_0, \qquad y'(0) = y_1$$

if a and b are constants. Here we consider two initial value problems for inhomogeneous equations.

The first example is

$$y'' + 3y' + 2y = e^{3x} + \cos x, \qquad y(0) = 2, \qquad y'(0) = -1. \tag{1}$$

We start by solving the homogeneous equation

$$y'' + 3y' + 2y = 0.$$

Guess a solution of the form $y(x) = e^{rx}$ and plug it in to get

$$0 = (e^{rx})'' + 3(e^{rx})' + 2e^{rx} = e^{rx}(r^2 + 3r + 2) = e^{rx}(r + 2)(r + 1) \Rightarrow r = -1, -2.$$

Thus the homogeneous solution is

$$c_1 e^{-x} + c_2 e^{-2x}.$$

The general solution to (1) is a sum of the homogeneous solution listed above and a particulart solution y_p. To find y_p we use the method of undetermined coefficients and guess

$$y_p = Ae^{3x} + B\cos x + C\sin x.$$

Plugging this guess into the equation we have

$$\begin{aligned} e^{3x} + \cos x &= y_p'' + 3y_p' + 2y_p \\ &= 9Ae^{3x} - B\cos x - C\sin x + 9Ae^{3x} - 3B\sin x + 3C\cos x + 2Ae^{3x} + 2B\cos x + 2C\sin x \\ &= 20Ae^{3x} + (B + 3C)\cos x + (C - 3B)\sin x. \end{aligned}$$

Matching coefficients of the three different terms we get the three equations

$$1 = 20A, \qquad 1 = B + 3C, \qquad 0 = -3B + C,$$

which have the simultaneous solutions

$$A = \frac{1}{20}, \qquad B = \frac{1}{10}, \qquad C = \frac{3}{10}.$$

Putting this together, we see

$$y = \frac{1}{20}e^{3x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x + c_1 e^{-x} + c_2 e^{-2x}.$$

We now use our initial conditions to find c_1 and c_2. We have

$$\begin{aligned} 2 &= y(0) = \frac{1}{20} + \frac{1}{10} + c_1 + c_2 = \frac{3}{20} + c_1 + c_2 \\ -1 &= y'(0) = \frac{3}{20} + \frac{3}{10} - c_1 - 2c_2. \end{aligned}$$

These equations have the solution

$$c_1 = \frac{9}{4}, \qquad c_2 = -\frac{2}{5}.$$

In the next example we have

$$y'' + 3y' + 2y = \frac{1}{1-x}, \qquad y(0) = 1, \qquad y'(0) = 2. \tag{2}$$

The two homogeneous solutions are the same, so we can go straight to finding the particular solution. In this case we can't use the method of undetermined coefficients (why?) so we use variation of parameters. We start with

$$y_p(x) = c_1(x)e^{-x} + c_2(x)e^{-2x}.$$

We compute

$$y_p' = c_1'e^{-x} + c_2'e^{-2x} - c_1e^{-x} - 2c_2e^{-2x}.$$

We have some freedom in choosing the functions c_1 and c_2, so we set $c_1'e^{-x} + c_2'e^{-2x} = 0$. Now we take a further derivtative to get

$$y_p'' = -c_1'e^{-x} - c_2'e^{-2x} + c_1e^{-x} + 4c_2e^{-2x}.$$

Plug all this into (2) to get

$$\begin{aligned}
\frac{1}{1-x} &= y_p'' + 3y_p' + 2y_p \\
&= -c_1'e^{-x} - 2c_2'e^{-2x} + c_1e^{-x} + 4c_2e^{-2x} - 3c_1e^{-x} - 6c_2e^{-2x} + 2c_1e^{-x} + 2c_2e^{-2x} \\
&= -c_1'e^{-x} - 2c_2e^{-2x}.
\end{aligned}$$

We combine this with the equation $c_1'e^{-x} + c_2'e^{-2x} = 0$ to get a system of two equations in two unknows. Add these two equations together to get

$$\frac{1}{1-x} = -c_2'e^{-2x} \Rightarrow c_2' = \frac{e^{2x}}{x-1} \Rightarrow c_2(x) = \int_0^x \frac{e^{2s}}{s-1} ds.$$

We can evaluate

$$c_2(0) = 0, \qquad c_2'(0) = -1.$$

Next we use our system of equations to find c_1. Indeed,

$$c_1' = -e^{-x}c_2' = \frac{e^x}{1-x} \Rightarrow c_1 = \int_0^x \frac{e^s}{1-s} ds.$$

Again we can evaluate to get

$$c_1(0) = 0, \qquad c_1'(0) = 1.$$

Finally we use the initial conditions of (2) to find our solution. We have

$$y(x) = e^{-x} \int_0^x \frac{e^s}{1-s} ds + e^{-2x} \int_0^x \frac{e^{2s}}{s-1} ds + ae^{-x} + be^{-2x},$$

and we have to find the constants a and b. We evaluate at $x = 0$ to get

$$\begin{aligned}
1 &= y(0) = a + b \\
2 &= y'(0) = -a - 2b
\end{aligned}$$

which has the solution $a = 4$ and $b = -3$.