Wronskians in Second Order ODE

In these notes we discuss the Wronskian of a second order, linear differential equation. We start with the equation

$$y'' + a(x)y' + b(x)y = 0. \tag{1}$$

We can be slightly more general, and also consider equations of the form

$$c(x)y'' + a(x)y' + b(x)y = 0,$$

but we recover (1) when we divide by $c(x)$, and so (1) covers all the interesting cases we want to examine.

Let y_1 and y_2 solve (1), and define

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}.$$

Observe that $W \neq 0$ precisely when the 2×2 matrix above is nonsingular, that is, precisely when y_1 and y_2 are linearly independent solutions. We state this as a theorem.

Theorem 1 *The Wronskian of two solutions of a differential equation is nonzero precisely when those two functions are linearly independent.*

Next, we find an equation for the Wronskian itself. Take a derivative:

$$\begin{aligned} W' &= (y_1y_2' - y_2y_1')' = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' \\ &= y_1y_2'' - y_2y_1'' = y_1(-ay_2' - by_2) - y_2(-ay_1' - by_1) \\ &= -a(y_1y_2' - y_2y_1') = -aW, \end{aligned} \tag{2}$$

or

$$W' + aW = 0.$$

This is a separable equation (or one we can solve using integrating factors, either way you get the same answer), and we see

$$W(x) = ce^{-\int_0^x a(s)ds}.$$

Observe that W is an exponential, so the only way it can vanish is if $c = 0$. We summarize this in a theorem.

Theorem 2 *The Wronskian W of $y'' + a(x)y' + b(x)y = 0$ satisfies the equation*

$$W' + a(x)W = 0,$$

and so

$$W(x) = ce^{-p(x)}, \qquad p(x) = \int_0^x a(s)ds$$

for some constant c determined by initial conditions. In particular, either W is never zero or it is identically zero.

In fact, if we're given one solution $y_1(x)$ we can use the Wronskian to find a second, linearly independent solution $y_2(x)$ (and thus all possible solutions, by taking linear combinations). We illustrate this with an example.

Consider the equation

$$y'' - 3x^2y' - 6xy = 0, \tag{3}$$

and verify that

$$y_1 = e^{x^3}$$

is a solution. We'd like to fine a second, linearly independent solution $y_2(x)$. The Wronskian

$$W(x) = y_1 y_2' - y_2 y_1'$$

satisfies the differential equation

$$W' - 3x^2 W = 0 \Rightarrow \frac{d}{dx}(\ln(W)) = \frac{W'}{W} = 3x^2. \tag{4}$$

We can directly integrate this equation to get $W(x) = ce^{x^3}$; in fact, we can rescale (multiplying y_2 by the appropriate constant) to make $c = 1$, and so

$$W = e^{x^3} = y_1 y_2' - y_2 y_1' = e^{x^3}(y_2' - 3x^2 y_2) \Rightarrow y_2' - 3x^2 y_2 = 1.$$

This last equation has the solution

$$y_2(x) = e^{x^3}\left(\int_0^x e^{-s^3}\, ds + k\right),$$

where k is a constant. It remains to find k, which we do by matching the initial conditions. We have, using the fundamental theorem of calculus,

$$1 = W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = y_2'(0) = e^0(e^{-0} + k) = 1 + k.$$

Thus $k = 0$ and

$$y_2(x) = e^{x^3}\int_0^x e^{-s^3}\, ds.$$

I'll end these notes with advice/remarks. In the last week or so, I've seen several students pull out formulas for solutions to differential equations using several determinants. These formulas involving determinants are essentially Cramer's rule for the solution to a linear system of equation, so they're correct. However, let's think about what you're doing. You're **using a complicated formula to solve a system of two linear equations with two unknows.** Why? Why are you making your life more complicated than you need to? Doing this goes against the most fundamental thing I want to teach you, which is

Think before you compute.