The Heat Equation via Fourier Series

The Heat Equation: In class we discussed the flow of heat on a rod of length $L > 0$. Letting $u(x,t)$ be the temperature of the rod at position x and time t, we found the differential equation

$$\frac{\partial u}{\partial t} = \kappa^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \tag{1}$$

where $f(x)$ is the initial temperature distribution and $\kappa > 0$ is a physical constant. You can find another explanation of this equation, based on physics, in the notes for 2FM on Vula (first file of notes, pages 19–20). One can have several different boundary condition at the ends of the rod. The most common are Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

which correspond to setting the ends of the rod in an ice bath to keep the temperature zero there, and Neumann boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0,$$

which correspond to keeping the rod insulated so no heat can enter or leave it. More general boundary conditions are possible.

Let's begin by solving the heat equation with the following initial and boundary conditions:

$$\frac{\partial u}{\partial t} = \kappa^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \quad u(x,0) = \cos\left(\frac{\pi x}{L}\right). \tag{2}$$

We start by guessing a solution of the form $u(x,t) = A(x)B(t)$. Plugging this guess in we see

$$A(x)B'(t) = \kappa^2 A''(x)B(t) \Rightarrow \frac{B'}{B}(t) = \kappa^2 \frac{A''}{A}(x) = -\kappa^2 \tau.$$

The right hand side is independent of t, while the left hand side is independent of x, so τ must be a constant. The minus sign above will be convenient later.

We can now separate the partial differential equation into two ordinary differential equations, which are

$$A'' = -\tau A, \qquad B' = -\kappa^2 \tau B.$$

We first solve the equation for B:

$$B(t) = B_0 e^{-\kappa^2 \tau t},$$

where $B_0 = B(0)$.

Now we turn our attention to the equation for A. We try a solution $A(x) = e^{rx}$ and get the characteristic polynomial $r^2 = -\tau$, so we have different types of solutions for positive and negative τ. We first consider the possibility that $\tau < 0$, which we will see is impossible. In this case

$$A(x) = c_\tau^+ e^{\sqrt{-\tau}x} + c_\tau^- e^{-\sqrt{-\tau}x},$$

and we can try to match the boundary conditions in (2). Then

$$0 = A'(0) = \sqrt{-\tau}(c_\tau^+ - c_\tau^-) \Rightarrow c_\tau^+ = c_\tau^- = c_\tau.$$

On the other hand,

$$0 = A'(L) = \sqrt{-\tau} c_\tau (e^{\sqrt{-\tau}L} - e^{-\sqrt{-\tau}L}).$$

Since $L > 0$, this forces $c_\tau = 0$, and so the temperature distrution $u(x,t) = 0$ for all time t, which is impossible. Thus we see $\tau < 0$ is impossible, and we must have $\tau > 0$.

So far we have

$$A(x) = c_\tau^+ \cos(\sqrt{\tau}x) + c_\tau^- \sin(\sqrt{\tau}x),$$

where $c_\tau^\pm$ are constants. Again, we match the boundary conditions, first at $x = 0$:

$$\frac{\partial u}{\partial x}(0,0) = 0 = B_0\sqrt{\tau}(-c_\tau^+ \sin(0) + c_\tau^- \cos(0)) \Rightarrow c_\tau^- = 0,$$

and so $A(x) = c_\tau^+ \cos(\sqrt{\tau}x)$. We can similarly match the boundary conditions at $x = L$:

$$\frac{\partial u}{\partial x}(L,0) = 0 = -B_0 c_\tau^+ \sqrt{\tau}\sin(\sqrt{\tau}L) \Rightarrow \sqrt{\tau}L = n\pi, n = 0,1,2,3,4,\ldots$$

We now see that

$$\tau = \frac{n^2\pi^2}{L^2}, n = 0,1,2,3,4,\ldots$$

Gathering together our expressions for $A(x)$ and $B(t)$, we get

$$u(x,t) = c_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right),$$

where we've let $c_\tau^+ = c_n$ and folded the constants B_0 and c_n together. Finally, we match our initial conditions at $t = 0$ to get

$$\cos\left(\frac{\pi x}{L}\right) = u(x,0) = c_1 \cos\left(\frac{n\pi x}{L}\right) \Rightarrow n = 1, c_n = 1,$$

and so our solution is

$$u(x,t) = e^{-\frac{\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{\pi x}{L}\right).$$

What if we change the initial conditions slightly to

$$u(x,0) = 1 - \cos\left(\frac{\pi x}{L}\right) + 2\cos\left(\frac{3\pi x}{L}\right)?$$

We can now write the solution as a sum of three solutions, for

$$\tau_0 = 0, \qquad \tau_1 = \frac{\pi^2}{L^2}, \qquad \tau_3 = \frac{9\pi^2}{L^2},$$

so the solution now is

$$u(x,t) = c_0 + c_1 e^{-\frac{\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{\pi x}{L}\right) + c_3 e^{-\frac{9\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{3\pi x}{L}\right).$$

Matching the initial data at $t = 0$ we see $c_0 = 1$, $c_1 = -1$, and $c_3 = 2$.

In general, if we can write the intial data as a sum of cosines, we can solve the differential equation. More precisely, if

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$$

then the solution of (1) with Neumann boudnary data is

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right),$$

with the same coefficients in the sum. There is a similar expression for solutions to the Dirichlet boundary value problem as a sum of sines; it would be a nice exercise to work this out.

Observe that, as $t \to \infty$, all the exponential terms decay, and only the $n = 0$ term remains, So

$$\lim_{t\to\infty} u(x,t) = c_0$$

is a constant. We will see a physical interpretation of this fact later on.

Fourier Series: It would be nice if we could write any reasonable (*i.e.* continuous) function on $[0, L]$ as a sum of cosines, so that then we could solve the heat equation with any continuous initial temperature distribution. In fact, we can, using Fourier series. (This is the reason Joseph Fourier first wrote down his series representation of a general function.)

Basically, the set

$$\left\{ \sqrt{\frac{1}{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, 4, \ldots \right\}$$

forms an orthonormal basis of an appropriate space of functions on the interval $[0, L]$, where the inner product is

$$\langle f, g \rangle = \int_0^L f(x) g(x) dx.$$

This means we can write any continuous function $f(x)$ on $[0, L]$ (and even some functions which are not continuous) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right), \qquad c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Notice that

$$c_0 = \frac{1}{L} \int_0^L f(x) dx$$

is the average value of f over $[0, L]$. It would be worthwhile to verify the following as exercises:

1. For $n = 1, 2, 3, \ldots$
$$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}.$$
2. For $n \neq m$ integers
$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0.$$
3. If
$$f(x) = \begin{cases} -1 & 0 < x < \frac{L}{2} \\ 1 & \frac{L}{2} < x < L \end{cases}$$
then the Fourier series for f is
$$f(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{(2n+1)\pi x}{L}\right), \qquad c_n = \frac{2(-1)^n}{(2n+1)\pi}.$$

Finally, we use Fourier series to solve the heat equation, combining our solution from the beginning. The solution to (1) with Neumann boundary data,

$$\frac{\partial u}{\partial t} = \kappa^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t),$$

is

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\kappa^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right), \qquad c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Observe that, as $t \to \infty$, the $u(x,t)$ converges to a constant

$$\lim_{t\to\infty} u(x,t) = \frac{c_0}{2} = \frac{1}{L}\int_0^L u(x,0)dx.$$

Thus we see that **the limiting temperature distribution is the average value of the initial temperature distibution.**

The case of Dirichlet boundary data: Finally we find the solution to the heat equation of a rod of length $L > 0$ with Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \kappa^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0 = u(L,t), \quad u(x,0) = f(x). \tag{3}$$

Again we separate variables, $u(x,t) = A(x)B(t)$, so that

$$AB' = \kappa^2 A'' B \Rightarrow \frac{B'}{B} = \kappa^2 \frac{A''}{A} = -\kappa^2\tau,$$

where τ is a constant. We have the ODEs

$$B' = -\kappa^2 \tau B, \qquad A'' = -\tau A.$$

The solution to the equation for B is

$$B(t) = e^{-\kappa^2\tau t} B(0).$$

For A, we again try $A(x) = e^{rx}$, and so $r^2 = -\tau$, and we again have to consider the cases where $\tau < 0$ and $\tau > 0$. If $\tau < -0$ then

$$A(x) = c_\tau^+ e^{\sqrt{-\tau}x} + c_\tau^- e^{-\sqrt{-\tau}x}.$$

By the boundary condition at $x = 0$ we have

$$0 = A'(0) = c_\tau^+ + c_\tau^- \Rightarrow c_\tau^+ = -c_\tau^- = c_\tau.$$

Now use the boundary condition at $x = L$:

$$0 = A(L) = c_\tau(e^{\sqrt{-\tau}L} - e^{-\sqrt{-\tau}L}).$$

Since $L > 0$ we must have $c_\tau = 0$, and, just as before, we get a contradiction.

As in the Neumann case, we have

$$A(x) = c_\tau^+ \cos(\sqrt{\tau}x) + c_\tau^- \sin(\sqrt{\tau}x),$$

and we use the boundary conditions. At $x = 0$,

$$0 = A(0) = c_\tau^+.$$

On the other hand,

$$0 = A(L) = c_\tau^- \sin(\sqrt{\tau}L) \Rightarrow \sqrt{\tau}L = n\pi \Rightarrow \tau = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, 3, \ldots$$

We conclude

$$A(x) = \tilde{c}_n \sin\left(\frac{n\pi x}{L}\right),$$

where we've replaced c_τ with $\tilde{c}_n$.

Summing over all n, we have

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{c}_n e^{-\frac{n^2\pi^2\kappa^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right). \tag{4}$$

It remains to find the coefficients $\tilde{c}_n$. This time, we use the fact that

$$\{\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2,3,4,\dots\}$$

is an orthonormal basis of

$$\{f : [0,L] \to \mathbb{R} : \int_0^L f(x)dx = 0\} = \{1\}^{\perp},$$

and so

$$f(x) = \sum_{n=0}^{\infty} \tilde{c}_n \sin\left(\frac{n\pi x}{L}\right), \qquad \tilde{c}_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Plugging this sum for f into (4) we have

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{c}_n e^{-\frac{n^2\pi^2\kappa^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right), \qquad \tilde{c}_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right).$$

Notice that, in this case,

$$\lim_{t\to\infty} u(x,t) = 0$$

for all x. This makes sense, because we've set the temperature at the ends of the rod to zero, so the ends of the rod act as heat sinks and absorb all the heat.