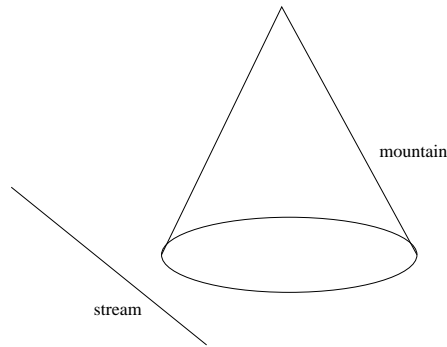


Fractals, Part I

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1 Introduction: What is a fractal?

For many years, people tried to model physical objects and phenomena using regular geometric objects. One can try to model a mountain with a cone, or a riverbank with a line segment or a circular arc. In a way, this is a nice choice of modeling; cones, lines and circles are relatively simple objects. Here is a possible picture of one of these modeled landscapes.



Have you ever seen a landscape like this?

Unfortunately, riverbanks rarely form straight lines and I have yet to see a mountain which looks like a cone. Here are some pictures of natural objects, like mountains and riverbanks, which have very complicated structure.

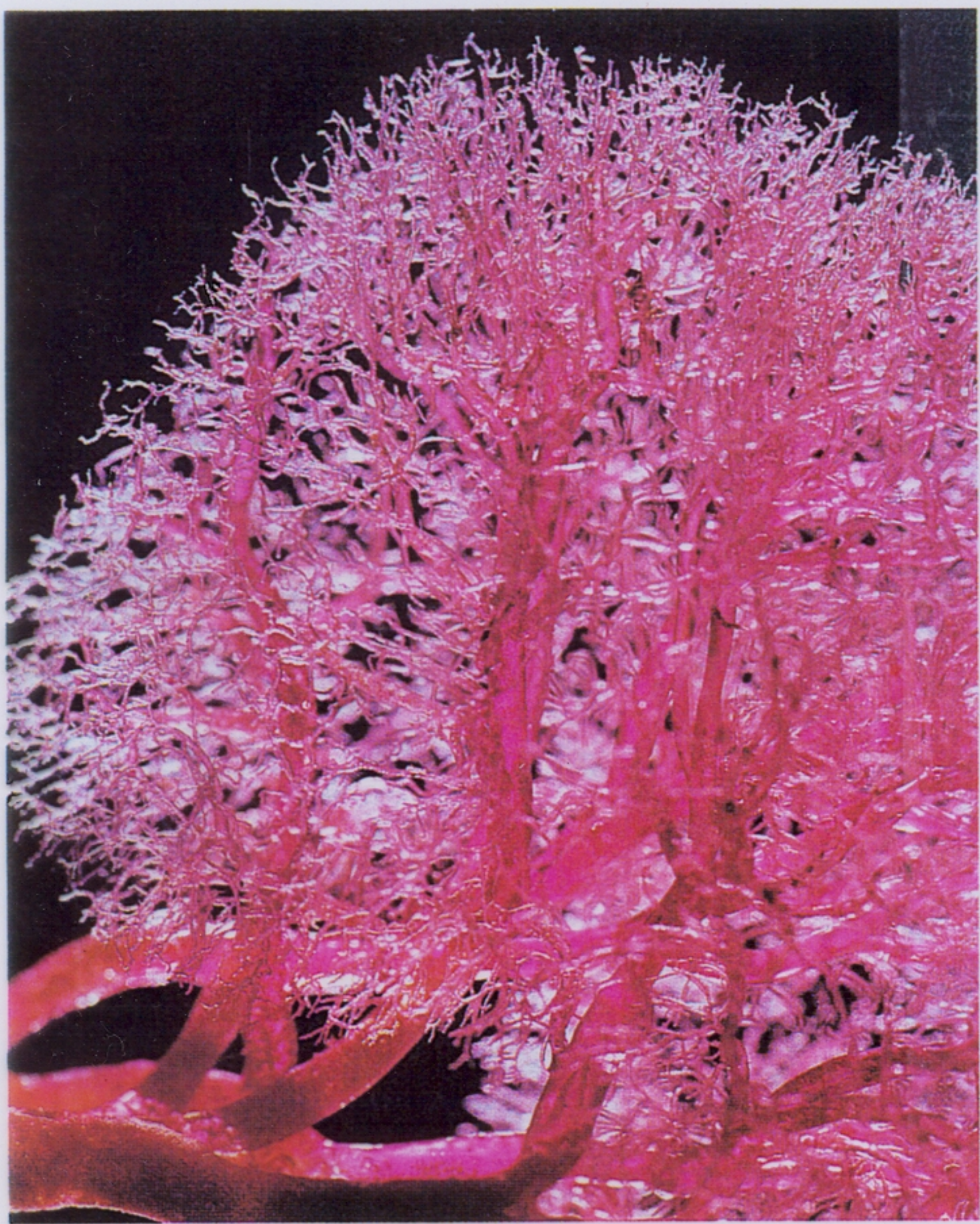


Plate 2: Cast of a child's kidney, venous and arterial system, © Manfred Kage, Institut für wissenschaftliche Fotografie.



Plate 4: Wadi Hadramaut, Gemini IV image, © Dr. Vehrenberg KG.

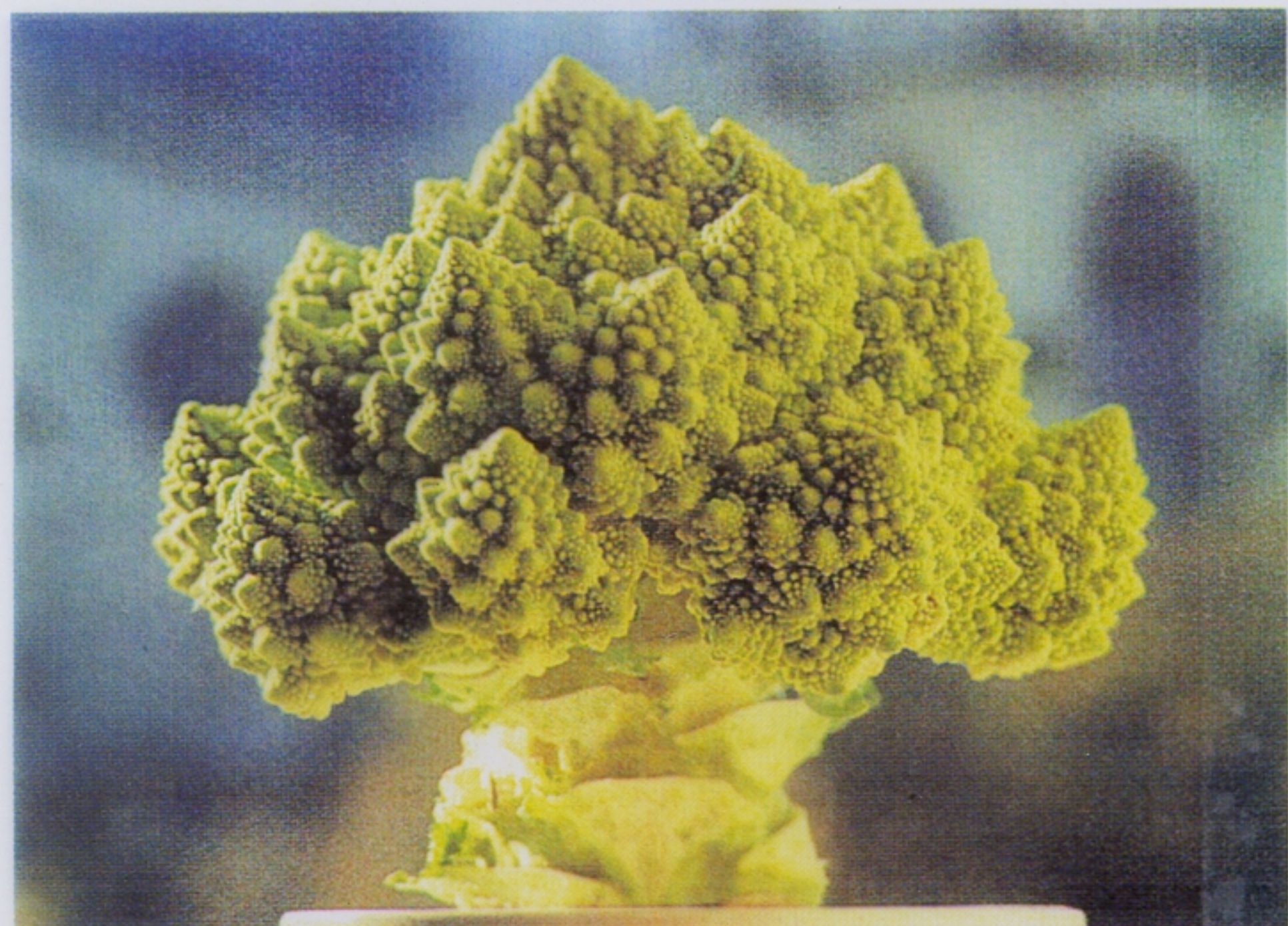


Plate 3: Broccoli Romanesco.

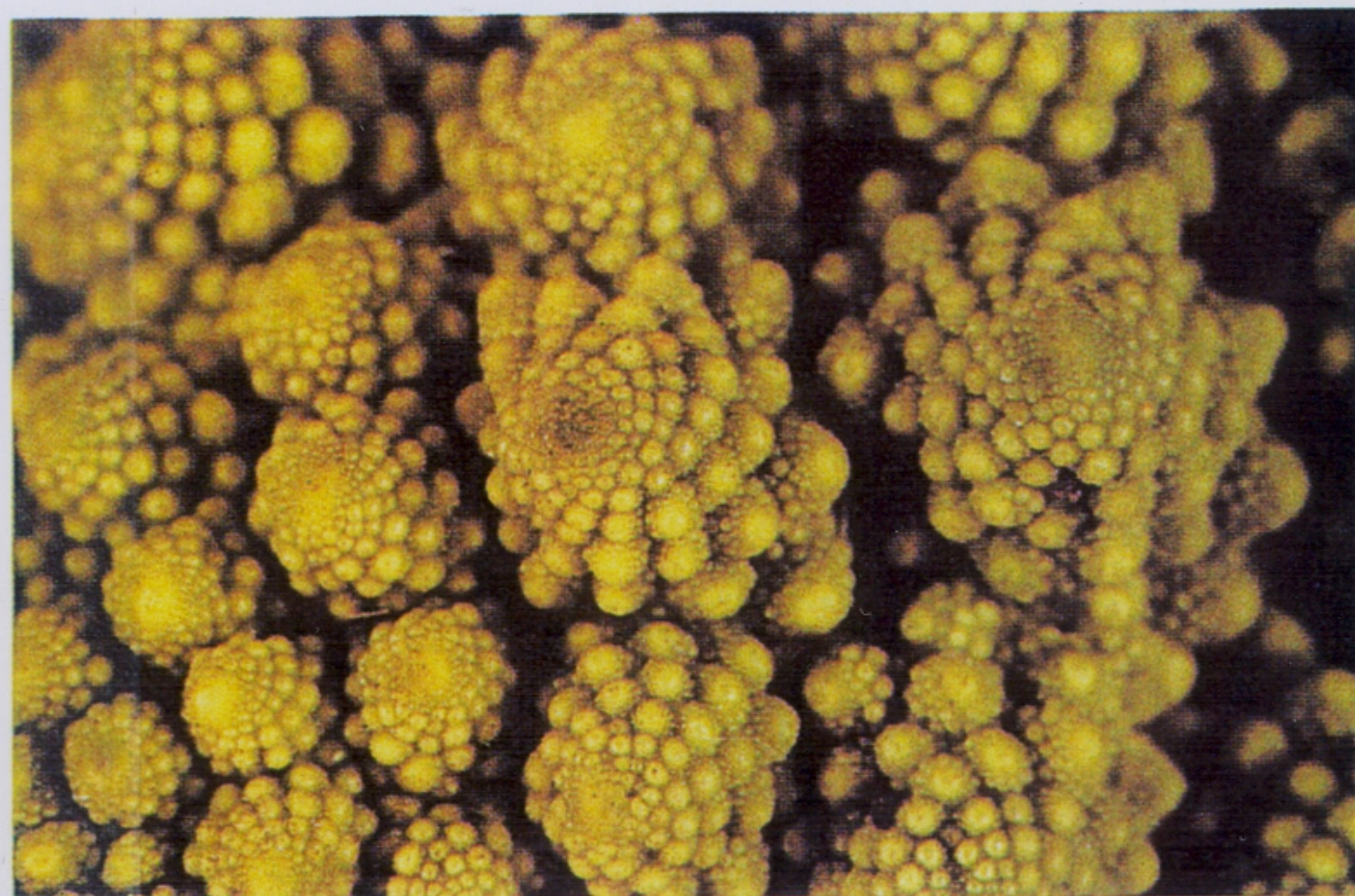
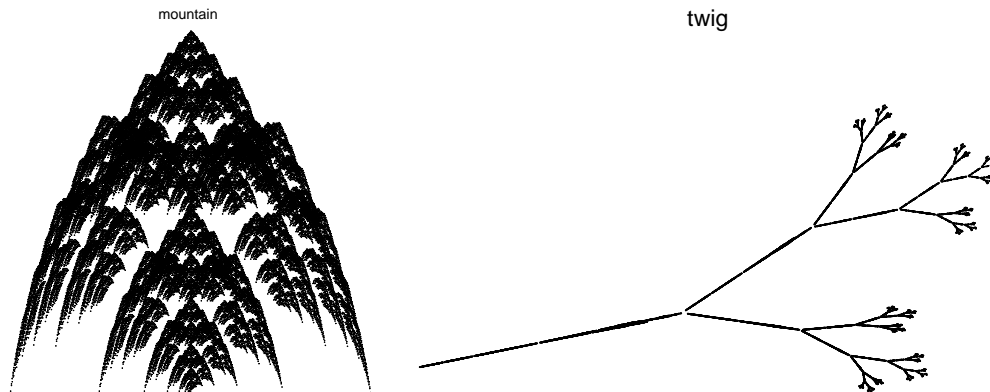
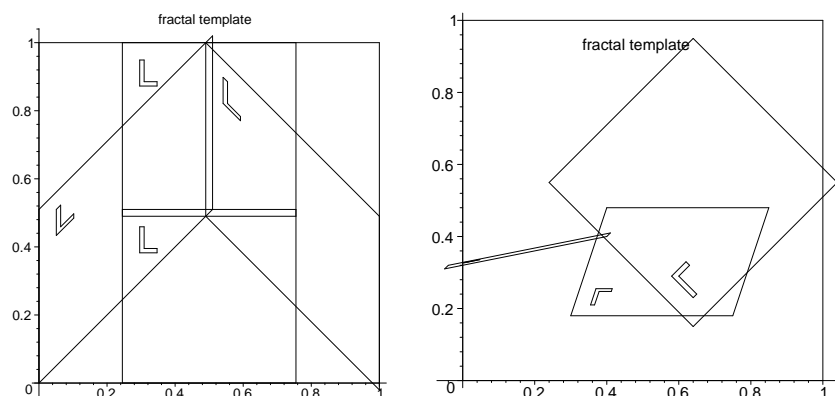


Plate 5: Broccoli Romanesco, detail.

The fact that real objects are too complicated to model with regular geometric objects motivated some people to create more complicated geometric objects, like the ones pictured below.



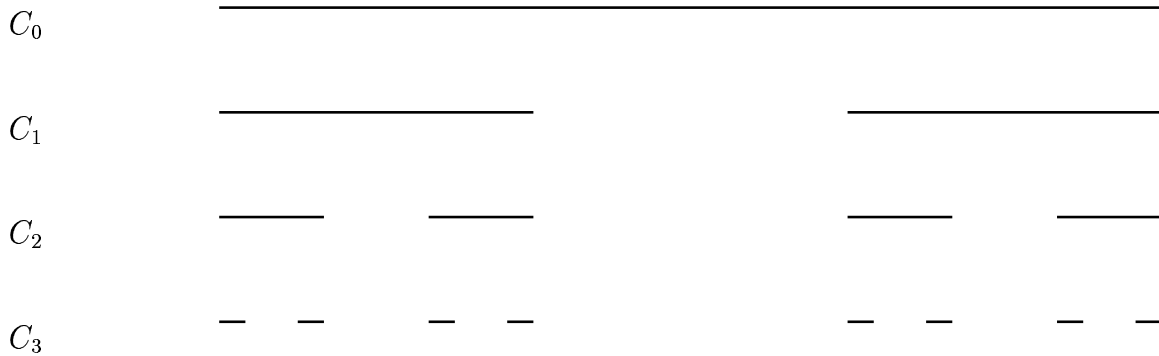
These pictures are examples of **fractals**. For our purposes, a fractal is anything which is self-similar. In other words, one can always write a fractal as some number of rescaled copies of itself. For instance, maybe one can write a fractal as 4 copies of itself scaled down by a factor of 3. Or maybe one can write a fractal as 3 copies of itself, 2 of which are scaled down by a factor of 4 and one scaled down by a factor of 2. In fact, the rescaling does not have to be uniform; they can rescale the sets by different amounts in different directions. Below we have a geometric representation of the different scalings for each of the fractals above. The large square represents the original image and the other smaller parallelograms represent the various rescaled images of that original square.



2 Some examples of fractals

The first fractal we'll make is called the middle thirds Cantor set. It was invented by Georg Cantor, who was quite a character (read about him at <http://www-history.mcs.st-and.ac.uk/~history/BiogIndex.html>). We will denote the middle thirds Cantor set C , and make it using an iterative process.

Start with C_0 being the unit interval, $[0, 1]$. Then remove the middle third and call what's left over C_1 . Now C_1 is the union of two intervals: $C_1 = [0, 1/3] \cup [2/3, 1]$. We next create C_2 by removing the middle third of each of those intervals. If we keep doing this for all natural numbers (i.e. removing the middle third of all the intervals in C_n), what's left over is the middle thirds Cantor set C . Here is a picture of the first several iterations.

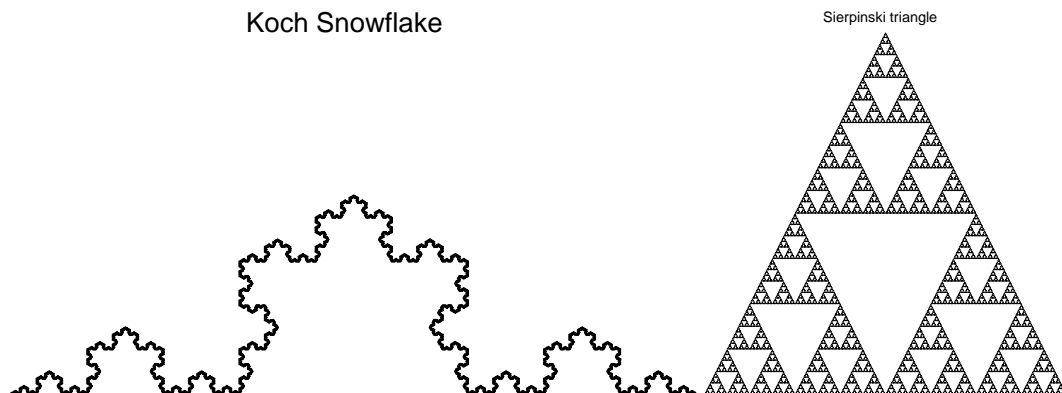


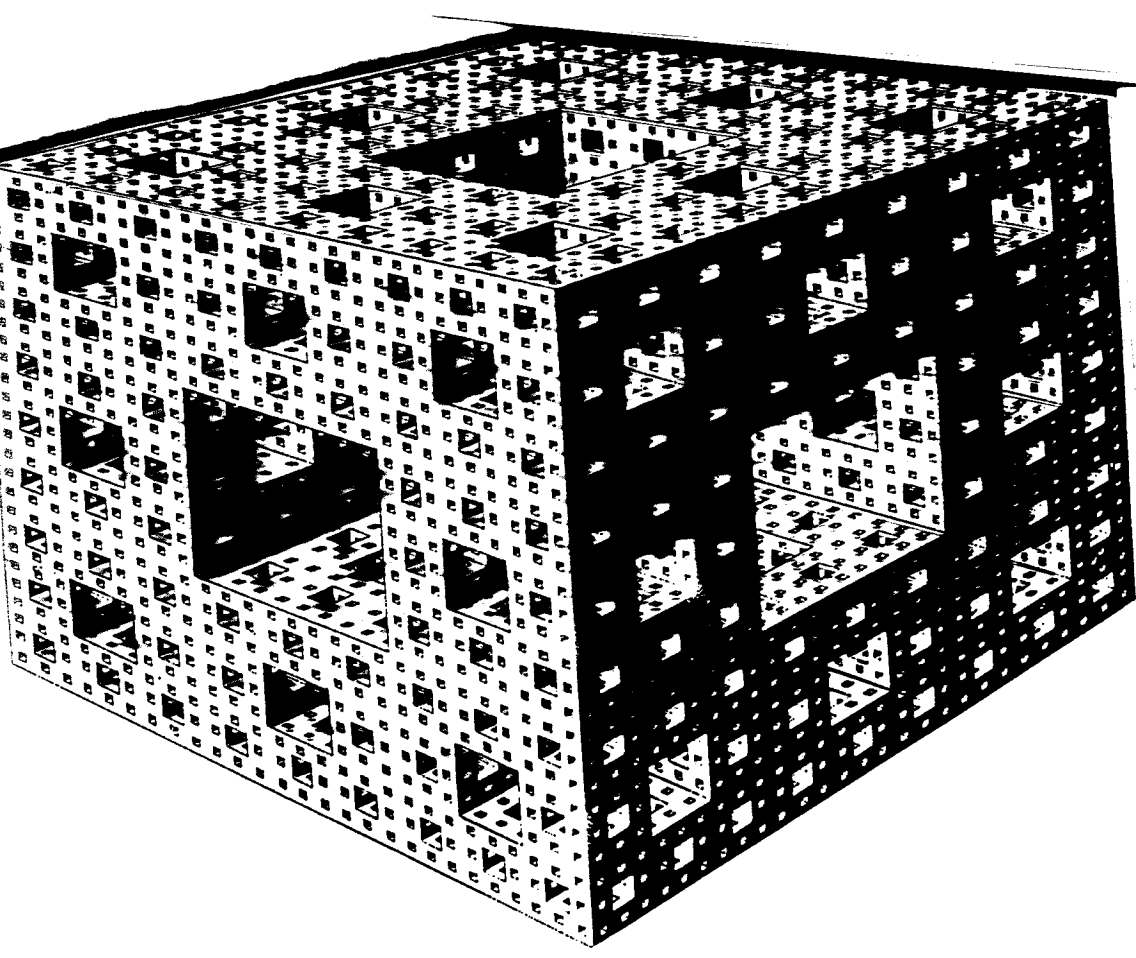
Questions:

1. Does C have any points left after we're done removing all those intervals?
2. Is C a fractal? In other words, is it self-similar?
3. Can you characterize all the points in C ? (You might want to look at appendix A first. This question is actually quite involved.)

We can compute the length of C as follows. First, notice that $C \subset C_n$ for $n = 0, 1, 2, \dots$, so the length of C is less than or equal to the length of C_n . Also, C_n is the disjoint union of 2^n intervals, each of length 3^{-n} . Therefore, the length of C_n is $(2/3)^n \rightarrow 0$ as $n \rightarrow \infty$. We have just shown that the length of C is 0.

Following this procedure, we can create some more fractals. Here are pictures of the von Koch snowflake, the Sierpinski triangle and the Menger sponge.



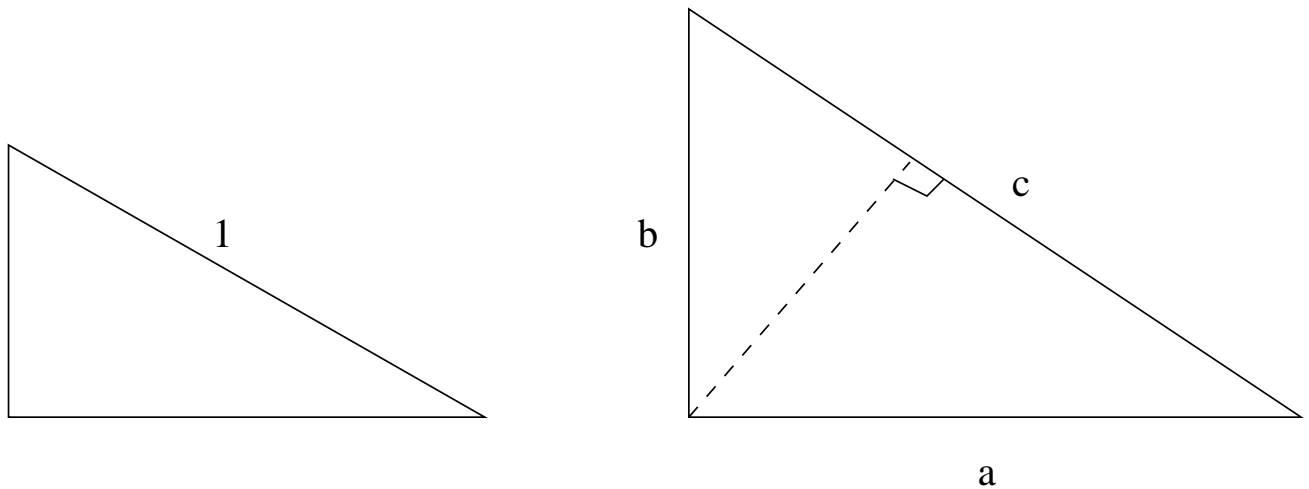


One can make each of these fractals by a deletion process similar to the one we used to make the middle thirds Cantor set (except the von Koch snowflake, which is a little different). Can you draw the first three or four iterations in the deletion process? Then can you compute the length of the von Koch snowflake, the area of the Sierpinski triangle and the volume of the Menger sponge? After you do that, can you think of a way to make the Cantor set or the Sierpinski triangle by adding material to your original image (possibly after rescaling), instead of deleting material?

3 Length, area and volume

One important property of lengths, areas and volumes is that they scale. For instance, if you rescale a line segment by a factor of 3 then you multiply its length by a factor of 3. Also, if you rescale a square by a factor of 3 then you multiply its area by $3^2 = 9$.

This idea of scaling actually leads us to a nice proof of the Pythagorean theorem. Consider the following picture:



1. What is the area of a right triangle in terms of the lengths of its legs?
2. Can you show that all three triangles on the right are similar?
3. Suppose the triangle on the left is also similar to the triangles on the right. What are the side lengths of the triangle on the left?
4. Can you use this to prove the Pythagorean theorem: $a^2 + b^2 = c^2$?

We have seen that the von Koch snowflake has infinite length, the Sierpinski triangle has 0 area and the Menger sponge has 0 volume. This seems quite odd; usually bounded curves have finite length. What's going on?

4 Scaling dimension

The answer is that in some sense the von Koch snowflake has dimension greater than 1! To investigate this phenomenon we will need to talk a little bit about how one measures the dimension of an object. There are several notions of dimension we can use; we will use the notion of *scaling* dimension.

What does it mean to say that a curve is 1-dimensional? First consider something simple, like a line segment.

1. If you rescale a line segment by a factor of 3, how many copies of the original line segment do you need to cover the new line segment? What if you rescale the line segment by a factor of n ?
2. If you rescale a square by a factor of 3, how many copies of the original square do you need to cover the new square? What if you rescale the square by a factor of n ?
3. If you rescale a cube by a factor of 3, how many copies of the original cube do you need to cover the new cube? What if you rescale the cube by a factor of n ?

You should have found that you need n , or n^1 copies of the new interval to cover the old one; similarly, you need n^2 copies of the old square to cover the new one and n^3 copies of the old cube to cover the new one. The important thing here is the exponent: when you scale a d -dimensional object by a factor of n , you need n^d copies of the original to cover the new object. We can rewrite this as follows: let n be the scaling factor and m be the number of copies of the original object one needs to cover the new object. Then this object has dimension d where $m = n^d$, or

$$d = \frac{\log m}{\log n}.$$

1. What is the scaling dimension of the middle thirds Cantor set C ?
2. What is the scaling dimension of the von Koch snowflake K ?
3. What is the scaling dimension of the Sierpinski triangle S ?
4. What is the scaling dimension of the Menger sponge M ?

We have a sizeable caveat regarding the notion of scaling dimension. Namely, this definition only makes sense if the object we're looking at scales. In other words, if we scale the object by a factor of n , we need to be able to exactly cover this new object with some number of copies of the original object. Many everyday objects *do not* scale (circles, for instance), and scaling dimension of these objects makes no sense. However, scaling dimension is very closely related to another

notion called *Hausdorff dimension*. The idea of Hausdorff dimension makes sense for most objects one would want to measure and it agrees with scaling dimension for objects that scale. Unfortunately, the ideas behind Hausdorff are a bit beyond the scope of these discussions.

5 Another way to make fractals

So far, we have talked about making fractals by starting with some set and deleting portions. Our definition of fractal and scaling dimension suggests another approach. Namely, we might want to create our fractals by replacing our starting object with some number of copies of itself, scaled down by some factor.

For example, we can create the middle thirds Cantor set C by

- starting with the unit interval $[0, 1]$ and
- repeatedly replacing it with 2 copies of itself, scaled by a factor of $1/3$. One copy sits with its left endpoint at 0 and the other sits with its right endpoint at 1.

We can do a similar thing with the Von Koch snowflake, the Sierpinski triangle and the Menger sponge. For each of these fractals answer the following questions.

1. How many copies do you need?
2. What are the scaling factors? (The scaling factors do not all need to be the same.)
3. Where do you put each copy?

In fact, with this process, it doesn't matter what you start with. No matter what your original image is, the iteration will always converge to the right fractal!

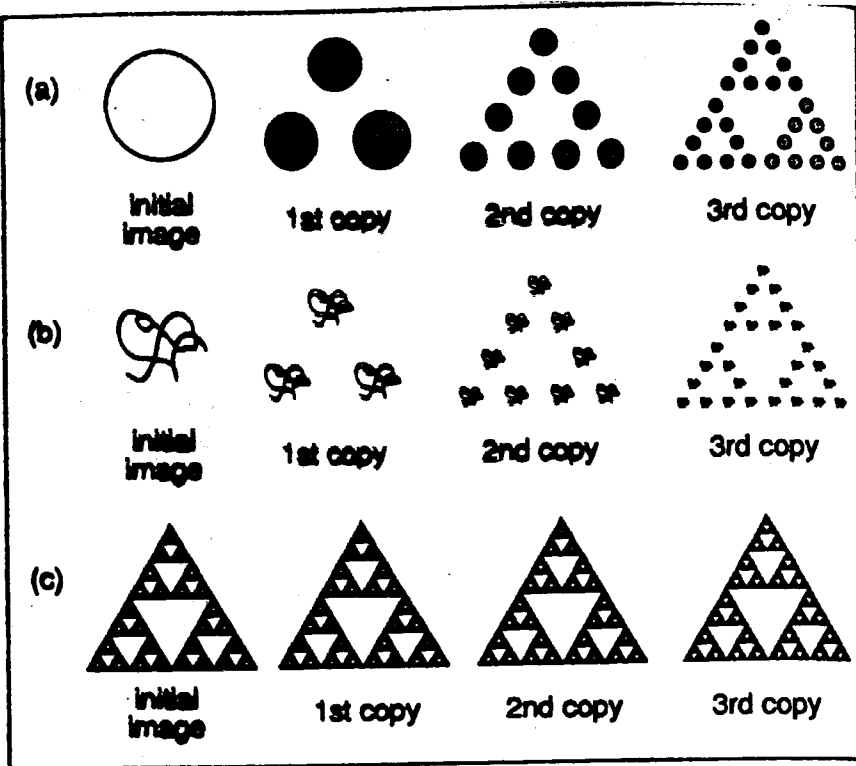


Figure 5.1 : Three iterations of an MRCM with three different initial images.

A The Cantor set and the unit interval

A.1 Binary and trinary digits

We have seen that the middle thirds Cantor set C has 0 length and infinitely many points (e.g. all the numbers 3^{-n} , for $n = 0, 1, 2, \dots$). This shouldn't completely shock you, because the integers also have zero length and infinitely many points. However, it turns out that C has many more points than even the integers, in a quantifiable way.

Recall that we can write any number $x \in [0, 1]$ in binary expansion; i.e. one can write any number $x \in [0, 1]$ in the form

$$x = a_1 2^{-1} + a_2 2^{-2} + a_3 2^{-3} + \dots \quad a_n = 0, 1.$$

Along the same lines we can also write $x \in [0, 1]$ in *trinary* expansion, that is we can write x as

$$x = a_1 3^{-1} + a_2 3^{-2} + a_3 3^{-3} + \dots \quad a_n = 0, 1, 2.$$

1. If $x \in C_n$, what can you say about the about the first n digits in the trinary expansion of x ? You might want to start by thinking about C_1 and C_2 , and then using an induction argument.
2. If $x \in C$, what can you say about the trinary expansion of x ?
3. Can you say anything about the converse? In other words, if $x \in [0, 1]$ satisfies the condition you found in the previous part, is x in the Cantor set?

A.2 Cardinality

You should have found that the numbers in the Cantor set are precisely those whose trinary expansions have only 0 and 2. Why is this important? This result is important because we can use it to show that the Cantor set has the same number of points as the unit interval. To explore this idea, we need to talk about the **cardinality** (i.e. size) of a set, an idea which is related to dimension.

We will denote the cardinality of a set A by $\#(A)$. Given two sets A and B we say $\#(A) \leq \#(B)$ if there is a map $\phi : A \rightarrow B$ such that

$$\phi(a_1) = \phi(a_2) \quad \text{if and only if} \quad a_1 = a_2.$$

Such a map is called **one-to-one**. If $\#(A) \leq \#(B)$ and $\#(B) \leq \#(A)$ then we say $\#(A) = \#(B)$.

1. Show that $\#(C) \leq \#([0, 1])$.
2. If \mathbb{Q} is the set of rational numbers and \mathbb{R} is the set of real numbers show that $\#(\mathbb{Q}) \leq \#(\mathbb{R})$.

3. Show that $\#([0, 1]) = \#(\mathbb{R})$. (Hint: $\phi(t) = \frac{1}{1+e^t}$.)

We can define a map $\phi : [0, 1] \rightarrow C$ by

$$x = a_1 2^{-1} + a_2 2^{-2} + a_3 2^{-3} + \cdots \mapsto \phi(x) = (2a_1)3^{-1} + (2a_2)3^{-2} + (2a_3)3^{-3} + \cdots$$

1. Show that $\phi(x) \in C$.
2. Show that ϕ is one-to-one.
3. What does this say about the cardinality of C ?

We call the set A **countably infinite** if $\#(A) = \#(\mathbb{N})$, where $\mathbb{N} = \{1, 2, 3, \dots\}$ are the natural numbers.

1. Show that the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ are countably infinite. (Hint: Start from 0 and work your way both forwards and backwards.)
2. Show that the set of positive rational numbers \mathbb{Q}^+ are countably infinite. (Hint: Think of a rational number as a pair of integers and arrange these pairs in a grid. Then start with $(0, 0)$ and work diagonally. This trick is called Cantor diagonalization.)

It is remarkable that the unit interval $[0, 1]$ is *not* countably infinite; it is much larger. One proof assumes $[0, 1]$ is countably infinite and derives a contradiction. Suppose $[0, 1]$ is countably infinite. Then there is a map $\phi : [0, 1] \rightarrow \mathbb{N}$ which is one-to-one. Arrange the numbers in $[0, 1]$ in the order they get from ϕ (i.e. label $x < y$ if and only if $\phi(x) < \phi(y)$). Now write down the binary expansions for this list of numbers. This list will look something like

100101110...
10101101110...
0010111101... etc.

Now we will find a new number in $[0, 1]$ which is not on this list, contradicting the fact that we have *all* the numbers in $[0, 1]$ on this list. We can describe the new number as follows: the first digit of the first number on our list is 1, so the first digit of the new number will be 0. The second digit of the second number on our list is 0, so the second digit of the new number will be 1. The third digit of the third number on our list is 1, so the third digit of the new number will be 0. In this way, we can determine each digit of the binary expansion of the new number, which uniquely determines the number. Also, the new number differs from each number on our list by at least one digit, so it can't be on the list. Therefore, we have arrived at a contradiction and shown that $[0, 1]$ is not countably infinite.