Analysis II: The Implicit and Inverse Function Theorems

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Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be C^1. When is the zero set

$$Z = \{x \in \mathbb{R}^n : f(x) = 0\}$$

the graph of another function? When is Z "nicely behaved?" We first consider some examples.

- Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ be linear, and take $n > m$. By the dimension theorem, A has a null space N, and N is a linear subspace of $\mathbb{R}^n$. If the rank of A is m' then $\dim(N) = n - m' \geq n - m$. It is also straight-forward to write N as the graph of a linear function $g : \mathbb{R}^{n-m'} \to \mathbb{R}^{m'}$.
- Let $f(x, y) = x^2 + y^2 - a$. If $a > 0$ the zero set is a circle of radius $\sqrt{a}$ centered at the origin, which is a nice, smooth object. However, you can't write a circle as a graph. When $a = 0$ the zero set collapses to a point. If $a < 0$ then the zero set is empty.
- Let $f(x, y) = x^2 - y^2 - a$. If $a > 0$ the zero set is the union of two graphs $\{x = g_\pm(y) = \pm\sqrt{y^2 + a}\}$. If $a < 0$ the zero set is the union of two graphs $\{y = g_\pm(x) = \pm\sqrt{x^2 - a}\}$. In either case, the zero set is nice. If $a = 0$ the zero set is the union of the two lines $\{x = \pm y\}$, which isn't nice, at least near the origin.

Notice that, in the last two examples, the zero set ceases to have nice properties where the derivative Df vanishes. The inverse function theorem and implicit function theorem both give criterion, in terms of Df, that the zero set $\{f(x) = 0\}$ behaves nicely.

We'll begin with the implicit function theorem. We write $n = m + k$, and consider a C^1 function

$$f : \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m.$$

Denote a point in $\mathbb{R}^k \times \mathbb{R}^m$ as (x, y) where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^m$. We want to write the zero set $Z = \{f(x, y) = 0\}$ (at least locally) as the graph of a function

$g : \mathbb{R}^k \to \mathbb{R}^m$, so that $Z = \{f(x, y) = 0\} = \{y = g(x)\}$. The function g is given implicitly by this relation; our goal it to prove it exists, and that it is C^1.

To simplify notation a little, we suppose $f(0, 0) = 0$, and carry out computations based around $(0, 0)$. The derivative $Df|_{(0,0)}$ is a linear map from $\mathbb{R}^k \times \mathbb{R}^m$ to $\mathbb{R}^m$, which we write as

$$Df|_{(0,0)}(x, y) = A(x) + B(y), \qquad A \in L(\mathbb{R}^k, \mathbb{R}^m), \qquad B \in L(\mathbb{R}^m, \mathbb{R}^m).$$

In this case, our first order Taylor expansion, based at $(0, 0)$ reads

$$f(x, y) = f(0, 0) + Df|_{(0,0)}(x, y) + r(x, y) = A(x) + B(y) + r(x, y). \tag{1}$$

To find our implicit function g we want to solve the equation $0 = A(x) + B(y) + r(x, y)$ for y.

It's worthwhile to briefly consider the case where f is linear. Then $f(x, y) = A(x) + B(y)$, and the condition we need to find the implicit function g is that B is invertible. In this case, we can multiply the equation $0 = A(x) + B(y)$ on the left by B^{-1} to get

$$0 = B^{-1} \circ A(x) + y \Leftrightarrow y = g(x) = -B^{-1} \circ A(x).$$

In the nonlinear case, we can sometimes get lucky, and the remainder term r might be a function of x alone. Again, we need to assume $B = \left.\frac{\partial f}{\partial y}\right|_{(0,0)}$ is invertible, and in this case we can again multiply on the left by B^{-1} to get

$$Z = \{0 = A(x) + B(y) + r(x)\} = \{y = g(x) = -B^{-1}(A(x) + r(x))\}.$$

Most of the time we don't get lucky, and the remainder term depends on both x and y. However, we can use the fact that the remainder term is small to prove that we can ignore it, at least if both x and y are also small. This is the key idea of the proof.

One key tool we will use is the Contraction Mapping Principle, which we quickly review here. Let $\Phi : X \to X$ be a mapping between complete, separable, normed vector spaces. If there is $K < 1$ such that

$$|\Phi(x) - \Phi(y)| \leq K|x - y|$$

then Φ has a unique fixed point. The proof is quite simple. Choose any $x_0 \in X$, and define the sequence of iterates by $x_{k+1} = \Phi(x_k)$. Then by induction

$$|x_{k+1} - x_k| \leq K^k |x_1 - x_0|.$$

Given $\epsilon > 0$ choose N such that

$$K^N \frac{|x_1 - x_0|}{1 - K} < \epsilon.$$

(Note: this is where we need $K < 1$.) If $n \geq m \geq N$ then

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\ &\leq (K^{n-1} + \cdots + K^m)|x_1 - x_0| \leq K^m(1 + K + \cdots + K^{n-m-1})|x_1 - x_0| \\ &= \frac{K^m|x_1 - x_0|}{1-K}(1 - K^{n-m}) \leq K^N \frac{|x_1 - x_0|}{1-K} < \epsilon \end{aligned}$$

Therefore the sequence $\{x_k\}$ is a Cauchy sequence and by completeness it has a limit $\bar{x}$. We claim $\bar{x}$ is the unique fixed point of the contraction Φ. First,

$$\begin{aligned} |\bar{x} - \Phi(\bar{x})| &\leq |\bar{x} - x_n| + |x_n - \Phi(x_n)| + |\Phi(x_n) - \Phi(\bar{x})| \\ &\leq |\bar{x} - x_n| + K^n|x_1 - x_0| + K|x_n - \bar{x}| \to 0 \end{aligned}$$

as $n \to \infty$, proving $\Phi(\bar{x}) = \bar{x}$ as claimed. Now suppose there is some other fixed point y. Then

$$|\bar{x} - y| = |\Phi(\bar{x}) - \Phi(y)| \leq K|\bar{x} - y| < |\bar{x} - y|.$$

This is a contradiction unless $\bar{x} = y$.

Theorem 1. The Implicit Function Theorem. *Let $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$ be C^1 and let $f(0,0) = 0$. Write the derivative at the origin as $Df|_{(0,0)}(x,y) = A(x) + B(y)$ where*

$$A = \left.\frac{\partial f}{\partial x}\right|_{(0,0)} \in L(\mathbb{R}^k, \mathbb{R}^m), \qquad B = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} \in L(\mathbb{R}^m, \mathbb{R}^m),$$

and suppose that B is invertible. Then there is a neighborhood U of $0 \in \mathbb{R}^k$ such that, restricted to $x \in U$, the implicit function $g : U \to \mathbb{R}^m$ exists, is unique, and is C^1.

Proof. We want to solve the equation

$$0 = A(x) + B(y) + r(x,y) \tag{2}$$

for y. Given x, this is equivalent to finding a fixed point of the mapping

$$K_x(y) = -B^{-1}(A(x) + r(x,y)). \tag{3}$$

That is, we want to solve the equation $y = K_x(y) = -B^{-1}(A(x) + r(x,y))$. By the Contraction Mapping Principle, we're done if we can show K_x is a contraction; in this case, K_x will have a unique fixed point, giving us the function $y = g(x)$.

By equation (1), we see

$$r(x,y) = f(x,y) - A(x) - B(y),$$

and so (because now A and B are fixed linear transformations) $r \in C^1$. We compute its derivative at $x = 0, y = 0$:

$$Dr|_{(0,0)}(u,v) = Df|_{(0,0)}(u,v) - A(u) - B(v) = 0.$$

By continuity, there is $\rho > 0$ such that if $\sqrt{|x|^2 + |y^2|} \leq \rho$ then

$$\|B^{-1}\| \left\| \left.\frac{\partial r}{\partial y}\right|_{(x,y)} \right\| \leq \frac{1}{2}.$$

Now use the mean value theorem on K_x to conclude that if $|x|$, $|y_1|$ and $|y_2|$ are all less than ρ then

$$\begin{aligned} |K_x(y_1) - K_x(y_2)| &\leq \|B^{-1}\||r(x,y_1) - r(x,y_2)| \\ &\leq \|B^{-1}\|\|\frac{\partial r}{\partial y}\||y_1 - y_2| \\ &\leq \frac{1}{2}|y_1 - y_2|. \end{aligned} \tag{4}$$

Thus, provided x and y are sufficiently small, K_x is indeed a contraction and has a unique fixed point. We can now define the implicit function $y = g(x)$ as the unique fixed point of K_x.

We still need to prove that g is C^1. We begin with a Lipschitz estimate:

$$\begin{aligned} |g(x)| &\leq |K_x(g(x)) - K_x(0) + K_x(0)| \leq \mathrm{Lip}(K_x)|g(x) - 0| + |K_x(0)| \\ &\leq \frac{1}{2}|g(x)| + |B^{-1}(A(x) + r(x,0)| \leq \frac{1}{2}|g(x)| + 2\|B^{-1}\|\|A\||x| \end{aligned}$$

Here $\mathrm{Lip}(K_x)$ is the Lipschitz constant associated to K_x, so that $|K_x(y_1) - K_x(y_2)| \leq \mathrm{Lip}(K_x)|y_1 - y_2|$; we've just shown in equation (4) that this Lipschitz constant is at most $1/2$. We conclude

$$|g(x)| \leq 4\|B^{-1}\|\|A\||x|. \tag{5}$$

If the $Dg|_{x=0}$ exists, then by the chain rule it must satisfy

$$A + B \circ Dg|_0 = 0 \Rightarrow Dg|_0 = -B^{-1} \circ A.$$

Let's try that in the first order Taylor expansion and see what happens. In this case,

$$\begin{aligned} |g(x) - g(0) - (-B^{-1}(A(x))| &= |B^{-1}(r(x,y))| \leq \|B^{-1}\||r(x,g(x))| \\ &\leq \|B^{-1}\|\bar{r}(\sqrt{|x|^2 + |g(x)|^2}) \\ &\leq \|B^{-1}\|\bar{r}(\sqrt{|x|^2 + 16\|B^{-1}\|^2\|A\|^2|x|^2}). \end{aligned}$$

This last term is sublinear, and so

$$\lim_{x \to 0} \left(\frac{|g(x) - g(0) + B^{-1}(A(x))|}{|x|} \right) = 0$$

as we required, proving $Dg|_0 = B^{-1} \circ A$.

None of the computations above require the basepoint of our computations to be $(0, 0)$, and so we can conclude the implicit function $g : U \to \mathbb{R}^m$ exists and is differentiable on some open set $U \subset \mathbb{R}^k$ containing 0. Moreover, if we write $Df|_{(x,g(x))} = [A\ B]$ for $(x, g(x))$ in the zero set Z, we can write

$$Dg|_x = (B(x, g(x)))^{-1} \circ A(x, g(x)) = \left(\left. \frac{\partial f}{\partial y} \right|_{(x,g(x))} \right)^{-1} \left(\left. \frac{\partial f}{\partial x} \right|_{(x,g(x))} \right).$$

We started with $f \in C^1$, and so this expression for Dg is continuous, completing our proof. □

Corollary 2. *Let $f \in C^k$ satisfy all the other conditions listed above in the implicit function theorem. Then the implicit function g is also C^k.*

Proof. We have just proved the corollary for $k = 1$, and we complete the proof using induction. Thus, we assume the corollary holds for C^{k-1} functions and prove it for C^k functions. In particular, given $f \in C^k$, we can assume the implicit function g is C^{k-1}. The derivative Dg solves the equation

$$A_x + B_x \circ Dg|_x = 0, \qquad A_x = \left. \frac{\partial f}{\partial x} \right|_{(x,g(x))}, \quad B_x = \left. \frac{\partial f}{\partial y} \right|_{(x,g(x))}.$$

Since $f \in C^k$ its derivative is C^{k-1}, and so we have just defined Dg by an implicit equation of C^{k-1} functions, which means that $Dg \in C^{k-1}$. This in turn implies $g \in C^k$. □

Corollary 3. *Matrix inversion is C^∞.*

Proof. Given $A \in GL(n, \mathbb{R})$, its inverse is determined by the equation

$$f(A, B) = A \circ B - \mathrm{Id} = 0 \Leftrightarrow B = \mathrm{Inv}(A).$$

The defining equation just involves matrix multiplication and addition, and so it is smooth. Also, the derivative of f with respect to B is $\frac{\partial f}{\partial B}(V) = AV$, which is invertible because $A \in GL(n, \mathbb{R})$. Now we can apply the implicit function theorem, which tells us that the zero set is a smooth graph, of the function $A \mapsto A^{-1}$. □

We have previously proved this result by first showing Inv is C^1 and then bootstrapping, but once we know the implicit function theorem we can get all the regularity we please at once. It turns out that it's useful to know both proofs.

Next we examine the inverse function theorem, which gives us conditions under which we can find a local inverse of a given function $f : \mathbb{R}^n \to \mathbb{R}^n$. Before we do that, though, we introduce some terminology. Let U and V be open sets in $\mathbb{R}^n$, and suppose $f : U \to V$ is a bijection, *i.e.* f is one-to-one and onto. We say that f is a C^k diffeomorphism if $f \in C^k$ and also $f^{-1} \in C^k$. Notice that regularity of f **does not** imply regularity for f^{-1}. Indeed, this is easy to see even for functions $f : \mathbb{R} \to \mathbb{R}$, as we can see from $f(x) = x^3$. This function is a bijection from $\mathbb{R}$ to itself, and it is infinitely differentiable. However, $f^{-1}(x) = x^{1/3}$ is not differentiable at $x = 0$.

Intuitively, one should think of diffeomorphisms as the mappings which preserve any property depending on derivatives. For example, a sphere and an ellipsoid are diffeomorphic. A sphere and a cube are homeomorphic (they have the same topology), but they are not diffeomorphic (a cube has edges and corners, while a sphere does not).

Theorem 4. The Inverse Function Theorem. *Let $f : U \to \mathbb{R}^n$ be C^k on some open set $U \subset \mathbb{R}^n$, and suppose, for some $p \in U$, the derivative $Df|_p$ is invertible. Then there is some neighborhood V of $f(p)$ and a C^k inverse function $g : V \to U$ such that*

$$x = g(f(x)), \qquad y = f(g(y))$$

for $y \in V$ and $x \in g(V)$.

Remark 1. *It is straight-forward to extend the proof below and show that, under the hypotheses of the inverse function theorem, f is a local diffeomorphism. More precisely, there are small neighborhoods U_0 and V_0 of p and q (respectively) such that $f : U_0 \to V_0$ is a C^k diffeomorphism, and its inverse is also C^k.*

Remark 2. *We will use the implicit function theorem to prove the inverse function theorem. The text by Rudin (which is the canonical reference for the material we're covering) proves the inverse function theorem first, and then uses it to prove the implicit function theorem. In fact, the two theorems are equivalent.*

Proof. Define the map

$$F : U \times \mathbb{R}^n \to \mathbb{R}^n, \qquad F(x, y) = f(x) - y.$$

This is a C^k function, and its zero set is

$$Z = \{(x, y) : F(x, y) = 0\} = \{y = f(x)\},$$

which is the graph of f. We compute the derivative

$$DF|_{(p,f(p))}(u, v) = Df|_p(u) + v.$$

By hypothesis, $Df|_p$ is invertible, so we can apply the implicit function theorem (with x and y interchanged) to get an implicit function $g : V \to U$. More precisely, there is a possibly smaller neighborhood $U_0 \subset U$ such that $g : V \to U_0$ is a bijection and for $(x, y) \in U_0 \times V$

$$y = f(x) \Leftrightarrow F(x, y) = 0 \Leftrightarrow x = g(y).$$

We just need to verify that if $y \in V$ then $f(g(y)) = y$, and if $x \in U_0$ then $g(f(x)) = x$. If $y \in V$, then $x = g(y)$ is the unique point in U_0 such that $f(x) = f(g(y)) = y$, and so $f \circ g = \mathrm{Id}$. Similarly, if $x \in U_0$ then $y = f(x) \in V$, and (by the implicit function theorem) there is a unique point $g(f(x))$ such that

$$F(x, f(x)) = f(x) - f(x) = 0.$$

However, x itself satisfies this last equation, and so (by uniqueness) we must have $g \circ f = \mathrm{Id}$. □

The final theorem we will treat here is the rank theorem. Recall that, if $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then the rank of A is the dimension of the image. In terms of matrices, the rank is the size of the largest square minor with a nonzero determinant. If $A : \mathbb{R}^n \to \mathbb{R}^m$ has rank k, then we can reorder the variables in the domain $\mathbb{R}^n$ and the target $\mathbb{R}^m$, and change basis in the first k variables, so that

$$A(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0).$$

That is, after a linear change of basis (which is particular to A), a rank k linear transformation will look like orthogonal projection P onto the first k variables.

The rank theorem says that, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, and (in a neighborhood) Df has constant rank k, then there is a C^1 change of coordinates to make f into the projection P. Before we prove this theorem, we will need some preliminary results.

Definition 1. *Two maps $f : A \to B$ and $g : C \to D$ are equivalent if there are bijections $\alpha : A \to C$ and $\beta : B \to D$ such that $g = \beta \circ f \circ \alpha^{-1}$.*

If $f : U \to V$ is C^k, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets, we can refine the notion of equivalence a little. In this case, we want α and β to be C^k diffeomorphisms. In this case the composition $\beta \circ f \circ \alpha^{-1}$ is still a C^k function

Lemma 5. *C^k equivalence is an equivalence relation.*

Proof. We start by showing f is always equivalent to itself. Taking α and β to be the identity transformations, we see $f = \mathrm{Id} \circ f \circ \mathrm{Id}^{-1}$. Next, suppose f is equivalent to g, which we can write as $f = \alpha \circ g \circ \beta^{-1}$. Then $g = \beta \circ f \circ \alpha^{-1}$, and so g is equivalent to f. Finally, suppose f is equivalent to g and g is equivalent to h. We can write these relations as $f = \beta \circ g \circ \alpha^{-1}$ and $g = \gamma \circ h \circ \delta^{-1}$, and so

$$f = \beta \circ \gamma \circ h \circ \delta^{-1} \circ \alpha^{-1} = (\beta \circ \gamma) \circ h \circ (\alpha \circ \delta)^{-1} = \psi \circ h \circ \phi^{-1},$$

where $\psi = \beta \circ \gamma$ and $\phi = \alpha \circ \delta$. This shows f is equivalent to h. □

How should you think of C^k equivalence? Let $f : U_1 \to V_1$ and $g : U_2 \to V_2$ be C^k, where $U_i \subset \mathbb{R}^n$ and $V_i \subset \mathbb{R}^m$ are open sets. Then f is equivalent to g means that f and g look the same after changing variables.

Definition 2. *Let $f : U \to V$ be C^1, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, and let $x \in U$. The rank of f at x is the rank of $Df|_x$.*

Notice that the rank function is lower semi-continuous. In other words, the rank can suddenly jump up, but it cannot suddenly jump down. The easiest way to see this is to remember that, if f has rank k at x, then the derivative $Df|_x$ has a $k \times k$ minor with a nonzero determinant. By continuity, if x' is near x the determinant of the same $k \times k$ minor will still be nonzero, and so the rank of f at x' is at least k.

Lemma 6. *Rank is unchanged by C^1 equivalence.*

Proof. Let $f : U_1 \to V_1$ and $g : U_2 \to V_2$ be C^1 maps, with $U_i \subset \mathbb{R}^n$ and $V_i \subset \mathbb{R}^m$ open, be equivalent. This means there are C^1 diffeomorphisms $\alpha : U_2 \to U_1$ and $\beta : V_2 \to V_1$ such that $f = \beta \circ g \circ \alpha^{-1}$. We need to show that, for any $x_1 \in U_1$, the rank of f at x_1 is the same as the rank of g at $x_2 = \alpha^{-1}(x_1)$. However, by the chain rule,

$$Dg|_{x_2} = D\beta|_{\beta(f(x_1))} \circ Df|_{x_1} \circ (D\alpha|_{x_1})^{-1}.$$

By hypothesis, the two maps $D\alpha$ and $D\beta$ are linear isomorphisms, and so they preserve the rank of Df. □

Before stating the rank theorem, we will establish some notation. Fix positive integers n, m, k, with $k < n$ and $k < m$, and let $f : U \times V \to \mathbb{R}^m$ be a C^1 function with $U \times V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ where U and V are open sets. Write coordinates in the domain as $z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. We write a product open set in the target space as $U' \times V' \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$, with coordinates $z' = (x', y') \in \mathbb{R}^k \times \mathbb{R}^{m-k}$.

Theorem 7. The Rank Theorem. *With the notation as in the paragraph above, suppose the rank of f is identically k in the open set $U \times V$. Then, in a possible smaller open set $\tilde{U} \times \tilde{V}$, the function f is C^1 equivalent to the orthogonal projection $P_k(x, y) = (x, 0)$.*

Proof. Pick a basepoint $z_0 = (x_0, y_0) \in U \times V$, and let $z'_0 = (x'_0, y'_0) = f(x_0, y_0)$. Define the two translations

$$\tau_1 : \mathbb{R}^n \to \mathbb{R}^n, \qquad \tau_1(x, y) = (x - x_0, y - y_0)$$

$$\tau_2 : \mathbb{R}^m \to \mathbb{R}^m, \qquad \tau_2(x', y') = (x' - x'_0, y' - y'_0);$$

these translations are C^∞ (in fact, analytic) diffeomorphisms. Now, changing variables by

$$f_1(x, y) = \tau_2(f(\tau_1(x, y))) = f(x - x_0, y - y_0) - (x'_0, y'_0),$$

we get an equivalent map such that $f(0,0) = (0,0)$. Thus, without loss of generality, we can assume $f(0,0) = (0,0)$, and we proceed with this normalization, and do all our local computations at the origin.

Next we normalize the derivative $Df|_{(0,0)}$. Let $\mathcal{O} \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a rotation sending $\{(0,y)\} = \{0\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$ onto the kernel of $Df|_{(0,0)}$, and let $\mathcal{O}' \in L(\mathbb{R}^m, \mathbb{R}^m)$ be a rotation sending the image of $Df|_{(0,0)}$ onto $\{(x',0)\} = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^m$. Such rotations exist because the corresponding linear subspaces have the same dimensions. Change variables by

$$f_1(x,y) = \mathcal{O}' \circ f \circ \mathcal{O}(x,y),$$

which is another C^∞ equivalence. By the chain rule,

$$Df_1|_{(0,0)} = \mathcal{O}' \circ Df|_{(0,0)}\, \mathcal{O} : \mathbb{R}^k \times \{0\} \to \mathbb{R}^k \times \{0\}$$

is invertible, and is the zero map on the complimentary $\mathbb{R}^{n-k}$ factor. In other words, without loss of generality, we can assume

$$Df|_{(0,0)} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

where $A \in L(\mathbb{R}^k, \mathbb{R}^k)$ is invertible. Our goal now is to find a C^l change of coordinates, such that in these coordinates $f(x,y) = (x,0) \in U' \times V' \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$.

Let $i : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ be the inclusion map and let $\pi : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^k$ be orthogonal projection onto the first k coordinates. Then the composition

$$g : U \to U', \qquad g(x) = \pi(f(i(x))),$$

where $U \subset \mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $U' \subset \mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$ are open sets satisfies $g(0) = 0$ and

$$Dg|_0 = \pi \circ \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = A$$

is an invertible linear map from $\mathbb{R}^k$ to itself. By the inverse function theorem, g has a local C^1 inverse function

$$h : U' \to U, \qquad h(0) = 0, \qquad Dh|_0 = A^{-1},$$

so that $x' = g(x)$ precisely when $x = h(x')$, provided $x \in U$ and $x' \in U'$. Rewriting this in terms of the original function f, we see that, given $x' \in U'$ there is a unique $x \in U$ satisfying $\pi(f(x,0)) = x'$. Associated to this x' there is also a unique $y' \in V' \subset \mathbb{R}^{m-k}$ such that $(x',y') \in f(i(U))$. Moreover, this function $x' \mapsto y'$ is C^1 by construction. Thus, we can write the image set $f(i(U))$ as the graph of a function

$$\phi : U' \to V', \qquad \phi(x') = f(h(x'),0) - \pi(f(h(x'),0)),$$

and this function ϕ, being the composition of C^1 functions, is itself C^1.

Now define the map

$$\Psi : U' \times V' \to U \times V', \qquad \Psi(x', y') = (h(x'), y' - \phi(x')).$$

We can see this map is a local diffeomorphism two ways. First we can compute its derivative at $(0, 0)$ to get

$$D\Psi|_{(0,0)} = \begin{bmatrix} A^{-1} & 0 \\ * & \mathrm{Id} \end{bmatrix},$$

where Id is the identity map from $\mathbb{R}^{m-k}$ to itself. This matrix is invertible, whatever the entries in the lower left-hand corner are. In this particular case, we can write down the inverse map for Ψ explicitly: it is

$$\Psi^{-1}(x, y') = (g(x), y' + \phi(g(x))).$$

If $x \in U$ then

$$f(x, 0) = (x', \phi(x')) \Rightarrow \Psi \circ f(x, 0) = \Psi(x', \phi(x')) = (h(x'), \phi(x') - \phi(x')) = (x, 0),$$

and so, restricted to the slice $U \times \{0\}$, the composition $\Psi \circ f$ is the correct projection.

Finally, we find a diffeomorphism γ so that $\Psi \circ f \circ \gamma = P_k$. Define

$$F : U \times U \times V \to U', \qquad F(\zeta, x, y) = \pi(f(\zeta, y)) - x.$$

Observe that

$$F(0, 0, 0) = 0, \qquad \left.\frac{\partial F}{\partial \zeta}\right|_{(0,0,0)} = \mathrm{Id} \in L(\mathbb{R}^k, \mathbb{R}^k).$$

By the implicit function theorem, there is a C^1 function $\zeta = \zeta(x, y)$ describing the zero set as its graph, so $F(\zeta(x, y), x, y) = 0$. Now define

$$\gamma(x, y) = (\zeta(x, y), y), \qquad G = \Psi \circ f \circ \gamma.$$

We claim that $G = P_k$, completing the proof. By the chain rule, using the fact that $f(\zeta(x, y), x, y) = 0$, we have

$$0 = \left.\frac{\partial}{\partial x}\right|_{(0,0)} (F(\zeta(x, y), x, y)) = \frac{\partial F}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial F}{\partial x} = \mathrm{Id}\frac{\partial \zeta}{\partial x} - \mathrm{Id},$$

and so $\left.\frac{\partial \zeta}{\partial x}\right|_{(0,0)} = \mathrm{Id} \in L(\mathbb{R}^k, \mathbb{R}^k)$. Thus,

$$D\gamma|_{(0,0)} = \begin{bmatrix} \mathrm{Id} & * \\ 0 & \mathrm{Id} \end{bmatrix},$$

which is invertible. By construction, we also have $\gamma(0,0) = (0,0)$, and so by the inverse function theorem, γ is a diffeomorphism from a neighborhood of the origin to itself.

Recall that we defined ζ as the implicit function for the zero set of F, and so $\zeta(x,0) = 0$. Plugging this in, we see

$$\pi(G(x,y) = \pi(\Psi(f(\gamma(x,y)))) = \pi(\gamma(x,0)) = x.$$

Notice that this is true for all suffiently small y, not just for $y = 0$ (which is what we proved above). Let $\pi^\perp$ be the orthogonal projection onto $\{0\} \times \mathbb{R}^{m-k} \subset \mathbb{R}^m$, so that $\pi^\perp(x', y') = (0, y') = (x', y') - \pi(x', y')$. Also let $G_{y'} = \pi^\perp \circ G$. Then, near $(0,0)$,

$$DG = \begin{bmatrix} \mathrm{Id} & 0 \\ * & \frac{\partial G_{y'}}{\partial y} \end{bmatrix}.$$

Up to now, we have only used the fact that the rank of f is *at least* k in the open neighborhood $U \times V$; now we use the fact that its rank is equal to k. We know that G is C^1 equivalent to f, so it must also have rank k near $(0,0)$. Given the expression of DG above, this is only possible if, in this neighborhood,

$$\frac{\partial G_{y'}}{\partial y} = 0 \Leftrightarrow G_{y'} = \text{constant}.$$

Evaluating at $(0,0)$, we see that $G_{y'} = 0$, and so $G(x,y) = (x,0)$, which is precisely what we wanted to show.

□

As part of the proof of the Rank Theorem, we actually proved the following result.

Corollary 8. *Suppose $f : U \to \mathbb{R}^m$ is C^l on the open set $U \subset \mathbb{R}^n$, and that f has rank k on all of U. Then f is C^l equivalent to a map of the form $G(x,y) = (x, g(x,y))$, where $g : U \to \mathbb{R}^{m-k}$ is also C^l.*

Corollary 9. *If $f : U \to \mathbb{R}$ is C^l and has rank 1 at x, then in a neighborhood of x the level sets $\{x \in U : f(x) = constant\}$ are C^l-equivalent to $(n-1)$-dimensional discs.*

Proof. Near x the rank cannot decrease, but by the dimension theorem the rank also cannot increase. Thus in a neighborhood of x the rank must always be 1, and so f is locally equivalent to projection onto a line. The level sets of an orthogonal projection onto a line form a stack of parallel $(n-1)$-dimensional hyperplanes, and the corollary follows. □

Corollary 10. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^l and suppose the rank of f at x is n. Then the image of a sufficiently small neighborhood U of x under f is C^l equivalent to an n-dimensional disc.*

Proof. The rank of f cannot suddenly decrease, but the rank is also at most n. Thus, in a small enough neighborhood U of x, the rank of f is always n. By the rank theorem, in a (possibly smaller) neighborhood f is C^l equivalent to $x \mapsto (x, 0)$, and so the image of this neighborhood under f is C^l equivalent to a disc. $\square$