Counting

Introduction: You all know how to count, and you know it's fairly easy. However, sometimes it can be a little harder to count the number of items in a very large set, or the chances of having a particular outcome among very many possibilities. Here are some examples.

- Cards in a deck of cards come in four suits: spades and clubs (which are black) and hearts and diamonds (which are white). How many of the possible poker hands (which have five cards) contain exactly three red cards? Or at least three red cards? Or two clubs and three diamonds? How many hands contain at least one card from each suit?
- Suppose you go to order a dozen donuts for a group of friends. There are five different kinds of donuts to choose from. How many different ways can you select your twelve donuts?
- Information in a computer is stored as a sequence of on or off switches, which you can represent as as string of 0's and 1's (with the 0 representing an off switch). Such a string is called a binary string (or sometimes a binary word). How many 16 digit binary strings are there with exactly four 1's? Or at most four 1's?
- How many different passwords are eight characters long? How many of them have at least one lower-case letter, one upper-case letter, and one number? How many of them have at least one letter, one number, and one special character (like an &).

It can be very useful to count large collections efficiently, or at least to give a good estimate, particularly when you want to figure out problems in data storage. In this set of notes we'll go through some techniques for counting. Mostly, when you want to count something it's a good idea to compare it to something you've already counted. This is the strategey we'll use.

Onto, one-to-one, and bijective functions: Before we begin, we review some basics you already know. A set X is a collection of things, each of which is called an element. We write $x \in X$ to mean that x is an element of X. If X has finitely many elements, we write $|X|$ to mean the number of elements in X; in general, we use $|X|$ to denote the **cardinality** of X, which you can think of as roughly the size of X.

If we have two sets X and Y, a function $f : X \to Y$ is a rule which assigns some element $y = f(x) \in Y$ to each element $x \in X$. For instance, X could be the days of this week, and Y could be the two element set of sunny or rainy days. Now we have a function $f : X \to Y$ which assigns rain to each day of this week in which it's rained, and sunny otherwise. This is a well-defined, perfectly reasonable function. Some other functions you're used to are things like

$$f(x) = 1 + x^2, \qquad f(x) = e^x - 1, \qquad f(x) = 1 - \sin(x),$$

where $x \in \mathbb{R}$ is a real number. In this case, both the domain and target spaces are the real numbers. But functions make sense whenever you have sets, that is whenever you can any two collections of things.

A function $f : X \to Y$ is **one-to-one** if whenever $x_1 \neq x_2$ we have$f(x_1) \neq f(x_2)$. That is, f doesn't ever squish two different elements of X to the same element of Y. Also, $f : X \to Y$ is **onto** if for every $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$; in this case there may be many. Sometimes we'll call a one-to-one function **injective** and sometimes we'll call an onto function **surjective**. A function which is both onto and one-to-one is a **bijection**, or it is called **bijective**. We have the following rules fro counting the number of elements in sets, which are pretty self-evident.

Rule 1. *Let X and Y be sets with finitely many elements, and let $f : X \to Y$ be a function.*

- *If f is one-to-one then $|X| \leq |Y|$.*
- *If f is onto then $|X| \geq |Y|$.*

- *Putting these last two rules together, we see that if f is a bijection then $|X| = |Y|$.*

In fact, we can generalize this rule to any pair of sets X and Y, even if they have infinitely many elements. We say $|X| \leq |Y|$ if there is a one-to-one function $f : X \to Y$, and we say $|X| \geq |Y|$ if there is an onto function $f : X \to Y$.

Exercise: Is it true that $|X| \geq |Y|$ if and only if there is a one-to-one map $f : Y \to X$? Please explain your answer.

We might sometime over-count by some constant factor. For instance, we might end up counts the number of ears in a room by counting the number of heads in a room, in which case we'd be over-counting the number of ears in a room by a factor of 2 (we hope!). In general, we say a function $f : X \to Y$ is k-to-1 if for every $y \in Y$ there are exactly k distinct $x \in X$ such that $f(x) = y$.

Rule 2. *If $f : X \to Y$ is k-to-1 then $|X| = k|Y|$.*

Counting by an exact correspondence: Let's begin by comparing the problems of counting 16 digit binary strings with exactly four 1's and the problem of counting donuts. These two problems seem completely unrelated, but in fact they're the same. To see why that is, let's look at a possible list of all the donuts we'd get in a dozen, chosen from chocolate, glazed, apple, lemon, and plain donuts. We could have three chocolate, three glazed, four apple, two plain, and no lemon donuts. Writing this out as a list, with a 0 for each donut, we see

$$c = 000, \quad g = 000, \quad a = 0000, \quad l = \{\}, \quad p = 00.$$

If we insert a 1 into each of those gaps, we have a 16 digit binary string with exactly four 1's, namely

$$0001000100001100.$$

Now we have an identification between the 16 digit binary strings with exactly four 1's and the choices of a dozen donuts. We define a bijective function between these two sets as follows. Let X be the set of 16 digit binary strings with exactly four 1's and let Y be the set of dozens of donuts, which we can represent as five lists of 0's, one for each flavor. We can also represent elements of Y as five non-negat ive integers c, g, a, l, p, representing the number of chocolate, glazed, apple, lemon, and plain donuts. Now define $f : X \to Y$ by letting c be the number of 0's before the first 1, g the number of 0's between the first and second 1, a the number of 0's between the second and third 1, l the number of 0's between the third and fourth 1, and p the number of 0's after the fourth 1. For instance, the binary string 1000100100010000 gives us no chocolate donuts, 3 glazed donuts, 2 apple donuts, 3 lemon donuts, and 4 plain donuts.

Exercise: Verify that the function described above is a bijection.

Counting sequences: This is going to be our general strategey: we'll become very skilled at counting some things, sequences in particular. Then we will compare everything we want to count to sequences. Before we go into more details, it will be useful to write out some notation. Let X be a set with n elements $x_1, \ldots, x_n$, which we can write as $X = \{x_1, x_2, \ldots, x_n\}$. Notice that the order of this listing doesn't matter. For instance, your sock drawer might have 4 red socks, 6 blue socks and 10 polka-dot socks. When you go to pull these socks out of the drawer to check if they're all clean, it doesn't matter whether you pull out the red socks or the blue socks first, or which of the polka-dot socks you pull out first. Also notice that all the elements in the list of elements in X are distinct. For instance, you might have one lucky pair of polka-dot socks, but listing it twice on your list of socks doesn't miraculously give you two lucky pairs of socks.

A sequence is an **ordered** list of elements in X, which we write as $(x_1, x_2, \ldots, x_k)$. This is a sequence of k elements in X. This time, the order is important, so that the sequences (a, b, c) and (b, a, c) are different. Also, the same element can sometime appear in the sequence twice, so that we can have the sequence (a, b, a).

Alright, sets and sequences of socks is a little silly, so we'll spend the rest of this time looking at sets and sequences of numbers. Our basic set will be the positive integers from 1 to n, that is

$$X_n = \{1, 2, \ldots, n\}, \qquad |X_n| = n.$$

Now a sequence with k elements is just an ordered list of numbers from 0 to n. How many sequences with k elements are there?

Rule 3. *We can count the number of sequences of k positive numbers from 1 to n as follows.*

- *The number of k-element sequences with positive numbers between 1 and n is n^k.*
- *Provided $n \geq k$, the number of k-element sequences with* **distinct** *positive numbers between 1 and n is $\frac{n!}{(n-k)!}$.*

The number $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ is called n-factorial. It's a good exercise to verify that $2! = 2, 3! = 6, 4! = 24, 5! = 120, \ldots$, and that $n!$ becomes very very large as n increases.

Proof. First we count the number of sequences of k positive integers from 1 to n. We have n choices for each of the k integers in the sequence, and none of these choices depends on the others. This gives a total of $n \cdot n \cdot n \cdots n = n^k$ total choices for our sequence.

Now suppose we're not allowed to choose the same integer twice in our sequence. The we have n choices for the first integer, $(n-1)$ choices for the second integer, and so on. None of these choices depends on the others, so multiplying everything together we see that we have

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+2) \cdot (n-k+1) = \frac{n \cdot (n-1) \cdots 2 \cdot 1}{(n-k) \cdot (n-k-1) \cdots 2 \cdot 1} = \frac{n!}{(n-k)!}$$

total sequences to choose from. □

We can refine the last part of this proof a little.

Rule 4. *Let X be a set with n elements. Then for every positive integer $k \leq n$, the number of subsets of X with exactly k elemens is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

The number $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a very important number in counting, in is called n choose k. You can think of it as counting the number of ways to choose k pebbles out of a bag of n pebbles.

Proof. We've just seen that there are $\frac{n!}{k!}$ sequences of k elements with distinct elements of X. That is, there are $\frac{n!}{k!}$ ways to choose a sequence of elements $(x_1, x_2, \ldots, x_k)$, where all the elements of the sequence are distinct. We can then get a subset $\{x_1, \ldots, x_k\}$ by just looking at the collection of terms in this sequence. However, in this last step, we've forgotten the order the elements were listed in the sequence. For instance, (x_1, x_2) and (x_2, x_1) give different sequences but the same subset. So we need to count the number of ways to reorder a sequence of k positive integers between 1 and n. We have k choices of where to put the first term in the sequence, $k-1$ choices for the second, and so on. Multiplying these all together we get

$$k \cdot (k-1) \cdots 2 \cdot 1 = k!,$$

and so the number of k-element subsets of X is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. □

Notice that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{(n-k)}.$$

We can also see this by counting, because we can identify each k-element subset X with its complement, which has exactly $n-k$ elements.

Now we're finally able to count the number of 16 digit binary strings with exactly four 1's. Indeed, we can place the four 1's in any of 16 places, so the number of 16digit binary strings with exactly four 1's is

$$\binom{16}{4} = \frac{16!}{4! \cdot 12!} = 1820.$$

Counting combinations of sets: Suppose we have two sets X_1 and X_2, and we know $|X_1|$ and $|X_2|$. Can we count $|X_1 \times X_2|$? How about $|X_1 \cup X_2|$?

Rule 5. *If we know* $|X_1|$ *and* $|X_2|$ *we can count* $|X_1 \times X_2|$ *and* $|X_1 \cup X_2|$ *as follows*

- $|X_1 \times X_2| = |X_1| \cdot |X_2|$
- $|X_1 \cup X_2| = |X_1| + |X_2| - |X_1 \cap X_2|$.

Here are some examples. The PIN for a bank card here is a five digit number, with each digit being an integer between 1 and 10. How many different PIN's are there? You have 10 choices for each digit, and 5 digits to choose, so the total number of PINs is

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5 = 100,000,$$

which actually is not a terribly large number. On the other hand, a computer password has 8 characters, and you have at least 72 choices for each character (upper case letters, lower case letters, numbers, and special characters like #), so there are at least

$$72^8 = \simeq 7.2 \times 10^14$$

possible computer passowrds. Which is an awful lot.

Let's count the number of maths and physics majors at UCT. Suppose there are 80 second year maths majors and 115 second year physics majors. (The real number of these majors proved difficult to find.) We might estimate that the total number of maths and physics majors is $80 + 115 = 195$, but this is over-counting because we're ignoring the fact that there are some students who major in both maths and physics. So let's suppose that there are 23 double majors, which means we need to subtract 23 from our sum to find the the combined number of maths and physics majors is

$$80 + 115 - 23 = 172.$$

The pidgeonhole principle Sometimes you don't need to count exactly, you might only need to count a lower bound. For instance, if you're having three friends over for dinner, you need to have at least four plates to serve food. In this instance, you don't need to count the number of plates you have exactly, you only need to count up to four. The pidgeonhole principle is the most common tool to get this sort of estimate.

Rule 6. *Let* X *and* Y *be sets with finitely many elements.*

- *If* $|X| > |Y|$ *then no function* $f : X \to Y$ *can be one-to-one. Thus, for every* $f : X \to Y$ *there exists an element* $y \in Y$ *such that at least two different* $x_1 \neq x_2 \in X$ *satisfy* $f(x_1) = y = f(x_2)$.
- *If* $|X| > k|Y|$ *for some positive integer* k*, then for any* $f : X \to Y$ *there is some* $y \in Y$ *such that at least* $(k+1)$ *different elements* $x_1, \ldots, x_{k+1}$ *satisfy* $f(x_1) = f(x_2) = \cdots = f(x_{k+1}) = y$.

It's slightly frustrating that the pidgeonhole principle is non-constructive. That is, we know there exists such a y and $x_1, \ldots, x_{k+1}$ satisfying

$$f(x_1) = f(x_2) = \cdots = f(x_{k+1}) = y,$$

but we don't have a way to figure out which element y is.

Here's an example. A normal human being has at most $200,000$ hairs on his or her head. The current population of Cape Town is a little more than 3.5 million people, and let's say that more than 3.2 million of them have hair on their heads. Now we let X be all the people in Cape Town who have hair on their heads, and Y be the numbers from 1 to $200,000$, and let $f : X \to Y$ count the number of hairs on a person's head. Then, because $\frac{3,200,000}{200,000} = 16$, the pidgeonhole principle tells us that some set of 17 people have exactly the **same** number of hairs on each of their heads. Now, we don't have any way to figure out which 17 people these are, but they are out there, somewhere in Cape Town.

Permutations, products, and a division rule: We consider first the set of sequences of length k.

Rule 7. *Let S be the set of sequences of length k, with n_1 choices for the first term in the sequence, n_2 choices for the second term, and so on. Then*

$$|S| = n_1 \cdot n_2 \cdots n_k.$$

Every SA bank note has a string of 7 digits on it; what are the chances that none of those digits is repeated? To figure this out, we have to count the number of all 7 digit strings of numbers from 0 to 9, and then the number so such strings with no repeated digits. The number of 7 digit strings is 10^7, because we're just choosing 7 numbers from 0 to 9. To figure out the number of strings with no repeated digits, we count: we have 10 choices for the first digit, 9 choices for the second, and so on. Thus

$$\frac{\#(\text{7 digit numbers with no repeated digits})}{\#(\text{7 digit numbers})} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{10^7} = \frac{9!}{3! \cdot 10^6} \simeq .0605,$$

or about 6%.

We might want to re-order the terms of a sequence; how many ways are there to do this? Suppose we have k elements in our sequence; we can represent all the possible ways to re-order this sequence another sequence of k distinct numbers from 1 to k. For instance, the sequence $(2, 1, 3)$ tells us we're re-ordering a sequence of three elements, and we're moving the first term into the second slot, the second term into the first slot, and leaving the third term where it is. Or, the sequence $(3, 2, 1, 5, 4)$ tells us we're re-ordering a sequence with 5 terms as follows: the first term moves to the third slot, the second term stays where it was, the third term moves to the third slot, and the last to terms swap places. This is called a **permutation** of the numbers $\{1, 2, \ldots, k\}$ How many permutations of $\{1, 2, \ldots, k\}$ are there? We have k choices for where to put the first term in the sequence, $k-1$ choices for the second term, and so on. Multiplying these all together we have

$$\#\text{permutations of } \{1, 2, \ldots, k\} = k \cdot (k-1) \cdots 2 \cdot 1 = k!.$$

Suppose you invite 7 friends over to dinner, and you all sit around your round table. How many different ways are there for the eight of you to sit around the table? You can write out a seating arrangement by writing out a permuations of 8 numbers. For instance, the permutation $(2, 4, 3, 1, 7, 5, 8, 6)$ says person 2 sits in the first seat, person 4 sits in the second seat, and so on. This means we might count the number of seating arrangments as $8!$, but that's not quite right. In fact, we can see we're overcounting by noticing we have a round table, and so there is no first seat. This means the seating arrangement $(3, 1, 8, 5, 8, 6, 2, 4)$ is equivalent to the first arrrangement we listed. We from one arrangement to another by rotating the table (it might help to draw a picture here), and there are 8 possible rotations. This means the total number of possible seatings is really $\frac{8!}{8} = 7!$.

Exercise: Can you figure out how many different seating arrangements you have for n friends seated around a round table?

Bookkeeper's Rule: We have one final rule for counting to talk about.

Rule 8. *Provided $l_1, l_2, \ldots, l_k$ as distinct, the number of sequences with n_1 copies of l_1, n_2 copies of l_2, and so on, up through n_k copies of l_k is*

$$\frac{(n_1 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!}.$$

Let's illustrate this last rule with a simple example. I want to go on a 20 km walk, which will include 5 km northward, 5 km southward, 5 km eastward, and 5 km westward. How many possible walks can I do like this? We can write down such a walk as a sequence of N, S, E, W, with each N indicating 1 northward km, and so on. One such sequence is

$$NNEESSWWNNNEEESSSWWW,$$

and (by the rule we jsut discussed), the total number of such sequences is

$$\frac{(5+5+5+5)!}{5! \cdot 5! \cdot 5! \cdot 5!} = \frac{20!}{(5!)^4},$$

which is about 11.7 billion.

Poker hands: As a final applications, we'll compute the probability of some poker hands. Recall that a standard deck of cards has four suits: hearts, spades, diamond, and clubs. Each card can either be a number from 2 to 10, or an ace (A), or a jack (J), or queen (Q), or a king (K). There are one of each of these possible cards, making a deck of 52 cards in total. A hand in poker has 5 cards, and so there are

$$\binom{52}{5} = \frac{52!}{5!47!} = 2,598,960$$

possible poker hands.

How many hands have four of a kind, *i.e.* four cards with the same numerical value? For instance, you could have four 3's or four kings or four 7's. Such a hand is determined by

- the value of the four of a kind (13 choices)
- the suit of the fifth card (4 choices) and
- the numerical value of the fifth card ($13 - 1 = 12$ choices),

and so there are $13 \cdot 4 \cdot 12 = 624$ possible four of a kind hands. Considering that there are almost 2.6 million possible poker hands, a hand with four of a kind is very rare!

How many poker hands have two pairs? For instance, you could have a pair of 10's and a pair of 8's, or you could have a pair of 3's and a pair of jacks. A hand with two pair is determined by

- the numerical value of the of the first pair (13 choices)
- the two suits of the cards in the first pair ($\binom{4}{2}$ choices)
- the numerical value of the card in the second pair (12 choices)
- the suit of the two cards in the second pair ($\binom{4}{2}$ choices)
- the numerical value of the fifth card (11 choices)
- the suit of the fifth card (4 choices)

However, this over-counts the number of two-pair hands by exactly a factor of 2, because we can swith the first pair with the second pair and have the same poker hand. Thus the total number of two-pair poker hands is

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2} = 123,552.$$

Thus we see that, while a two pair hand is still very good, it is 198 times as common as a hand with four of a kind.