Matrix Operations

We begin by defining some basic operations on matrices. Some of these formulas might seem a little complicated, but we'll see later that they really are natural operations.

First we review some operators on vectors. If v and w are vectors in $\mathbb{R}^n$ then we can add w to w component-wise:

$$v+w=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}+\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}=\begin{bmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{bmatrix}.$$

We can also multiply a vector by a scalar (*i.e.* real number) $c \in \mathbb{R}$ component-wise

$$cv=c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}.$$

Notice that 0 times any vector gives the zero vector, and c times the zero vector is still the zero vector, regardless of what c is. We also have the standard commutative and distributive rules:

$$v+w=w+v, \qquad c(v+w)=cv+cw.$$

Now let A and B both be $m \times n$ matrices. We can add A and B component-wise, much the same way we did with vectors, to produce another $m \times n$ matrix C. More precisely,

$$\begin{aligned} A+B &= \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}+\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}+B_{11} & \cdots & A_{1n}+B_{1n} \\ \vdots & & \vdots \\ A_{m1}+B_{m1} & \cdots & A_{mn}+B_{mn} \end{bmatrix}=C. \end{aligned}$$

Equivalently, we can write the ith jth component of C as $C_{ij}=A_{ij}+B_{ij}$. We can also multiply a matrix by a scalar component-wise, the same as we did with a vector:

$$cA=c\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}=\begin{bmatrix} cA_{11} & \cdots & cA_{1n} \\ \vdots & & \vdots \\ cA_{m1} & \cdots & cA_{mn} \end{bmatrix}$$

As with vectors, we have the usual commutative and distributive rules:

$$A+B=B+A, \qquad c(A+B)=cA+cB.$$

The definition of matrix multiplication is a little trickier. Let A be an $m \times k$ matrix and let B be a $k \times n$ matrix. Then one can multiply A and B **in this order** to get an $m \times n$ matrix C, with components

$$AB=C, \qquad C_{ij}=\sum_{l=1}^{k} A_{il}B_{lj}=A_{i1}B_{1j}+A_{i2}B_{2j}+\cdots+A_{ik}B_{kj}.$$

Notice that the multiplication in the other order, namely BA, will usually not make any sense at all. If you remember what the dot product of two vectors is, there is an easy way to remember the formula above for the matrix product. The entry C_{ij} of the product AB is the dot product of the ith row of A with the j column of B.

It's worthwhile to consider some quick examples. Let's take

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix},$$

so that

$$AB = \begin{bmatrix} 0 & 12 & 13 \\ 6 & 3 & 5 \end{bmatrix},$$

whereas BA is not even defined. Next we take the same A but define

$$C = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}.$$

This time both AC and CA make sense, but AC is a 2×2 matrix while CA is a 3×3 matrix; we have

$$AC = \begin{bmatrix} 12 & -1 \\ 3 & 16 \end{bmatrix}, \qquad CA = \begin{bmatrix} 6 & 9 & -3 \\ 5 & 6 & 2 \\ 4 & 0 & 16 \end{bmatrix}.$$

In fact, even if the multiplication in either order gives matrices of the same horizontal and vertical sizes, the results will usually be different. Let

$$D = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}, \qquad E = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

Then

$$DE = \begin{bmatrix} 10 & 7 \\ 8 & 7 \end{bmatrix}, \qquad ED = \begin{bmatrix} 17 & -2 \\ 7 & 0 \end{bmatrix}.$$

Exercise: Verify all these matrix multiplications.

The definition of matrix multiplication looks a little complicated, doesn't it? We will see, however, that it is completely natural, because it turns multiplication into the composition of some functions.

We will pay particular attention to square matrices, that is matrices with the same number of rows as columns. We can write a general square matrix as

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}.$$

There is a particularly nice $n \times n$ matrix called the identity matrix I, which is

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix};$$

that is, all the entries of I are zero, except those on the diagonal stretching from the top left to the bottom right, which are 1:

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

The identity matrix has the very nice property that for **any** other $n \times n$ matrix A, we have

$$AI = IA = A.$$

Exercise: Can you verify that $AI = IA = A$?

Recall that real numbers have an additive identity 0 and a multiplicative identity 1, so that for any $c \in \mathbb{R}$ we have

$$c + 0 = c, \qquad c \cdot 1 = c.$$

Numbers also have additive inverses:

$$c + (-c) = 0,$$

and, so long as $c \neq 0$, multiplicative inverses:

$$c \cdot \frac{1}{c} = 1.$$

For the most part, the same is true of square matrices. If A is an $n \times n$ matrix then

$$A + 0 = A, \qquad AI = IA = A,$$

where 0 is the $n \times n$ matrix all of whose entries are 0 and I is the $n \times n$ identity matrix listed above. Just as with scalars (*i.e* real numbers) the matrix A will have an additive inverse $-A$, whose entries are just the negative of the entries of A, which we write as $A + (-A) = 0$. Some matrices A will also have multiplicative inverses, which we write as A^{-1}. In this case, we have

$$AA^{-1} = A^{-1}A = I.$$

Later on we will determine some tests to see if A^{-1} exists, and some ways to compute A^{-1} if it does exist.