# Partial Differential Equations Notes II

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In this set of notes we closely examine the Laplace operator $\Delta(u) = \operatorname{div}(\nabla(u)) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. As the Laplacian will serve as our model operator for second order, elliptic, differential operators, we will be well-served to understand it as well as we can.

**Some basic definitions:** We fix some bounded domain $D \subset \mathbf{R}^n$, and suppose that the boundary $\partial D$ of $D$ is at least $C^1$. For most purposes, it will suffice to work on the ball $B_R(0)$ of radius $R$, centered at 0. A **harmonic** function $u$ on $D$ satisfies $\Delta u(x) = 0$ for all $x \in D$. More generally, we are interested in solving the **Poisson equation**:

$$\Delta u = \phi(x) \tag{1}$$

for some continuous function $\phi$. In this equation, the given information is the domain $D$ and the function $\phi$ on the right-hand-side, and the unknown (*i.e.* the thing we're solving for) is the function $u$. In order to specify a solution, we still need to prescribe boundary data for $u$. The two most common boundary data are Dirichlet boundary values and Neumann boundary values, which we define now.

**Definition 1.** *The function $u \in C^2(D) \cap C^0(\bar{D})$ solves the Dirichlet boundary value problem for* (1) *if*

$$\Delta u = \phi, \qquad u|_{\partial D} = f, \tag{2}$$

*where $f \in C^0(\partial D)$ is given. Similarly, we say $u \in C^2(D) \cap C^0(\bar{D})$ solves the Neumann boundary value problem for* (1) *if*

$$\Delta u = \phi, \qquad \left.\frac{\partial u}{\partial N}\right|_{\partial D} = \langle \nabla u, N \rangle|_{\partial D} = g \tag{3}$$

*for some given $g \in C^0(\partial D)$. Here $N$ is the outward unit normal vector for $\partial D$.*

In general, if $u$ solves the Poisson equation (1), then we call $u|_{\partial D}$ the **Dirichlet data** of $u$, and we call $\frac{\partial u}{\partial N}\big|_{\partial D}$ the **Neumann data** of $u$. Either of these boundary data will determine $u$, so we can only prescribe one the Dirichlet data or the Neumann data, but not both. One can look at more general boundary value problems than (2) and (3), but most of the time these two will give you all the information you could need.

**Some properties of harmonic functions:** We begin with the equation $\Delta u = 0$.

**Theorem 1.** *Let $D$ be bounded and let $\partial D$ be $C^1$. If $u$ is harmonic with zero Dirichlet data (that is, $u|_{\partial D} = 0$) then $u(x) = 0$ for all $x \in D$. If $u$ is harmonic with zero Neumann data ($\frac{\partial u}{\partial N}\big|_{\partial D} = 0$), then $u$ is constant.*

*Proof.* We need to recall the product rule and the divergence theorem. If $X$ is any vector field and $f$ is a function, then the product rule states

$$\operatorname{div}(fX) = \langle \nabla f, X \rangle + f \operatorname{div} X.$$

Also, the divergence theorem says

$$\int_{\partial D} \langle X, N \rangle dA = \int_D \operatorname{div}(X) dV.$$

Now apply the divergence theorem to the vector field $u\nabla u$ and use the product rule to get

$$\int_{\partial D} u \frac{\partial u}{\partial N} dA = \int_D |\nabla u|^2 + u\Delta u dV = \int_D |\nabla u|^2 dV.$$

If $u$ has either Dirichlet or Neumann boundary data, this integral is zero, and so $|\nabla u|(x) = 0$ for all $x \in D$, which in turn (because $D$ is connected) forces $u$ to be constant. In the case that $u$ has Dirichlet boundary data, this constant must be zero. $\square$

Recall that $\omega_n$ is the volume of the unit ball in $\mathbf{R}^n$, so that for any $x_0 \in \mathbf{R}^n$ and $r > 0$ we have $\text{Vol}(B_r(x_0)) = \omega_n r^n$ and $\text{Area}(\partial B_r(x_0)) = n\omega_n r^{n-1}$.

**Theorem 2.** *Let $u \in C^2(D)$ be harmonic, let $x_0 \in D$, and let $R > 0$ be small enough so that $B_R(x_0) \subset D$. Then*

$$u(x_0) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x) dA(x) = \frac{1}{\omega_n R^n} \int_{B_R(x_0)} u(x) dV(x).$$

*Proof.* For any domain $D$ with $C^1$ boundary, the divergence theorem tells us

$$0 = \int_{\partial D} \frac{\partial u}{\partial N} dA = \int_D \Delta u dV = 0.$$

We use this fact to conclude that, for $0 < r < R$,

$$\begin{aligned} 0 &= \int_{\partial B_r(x_0)} \frac{\partial u}{\partial N}(x) dA(x) = r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(x_0 + ry) dA(y) \\ &= r^{n-1} \frac{\partial}{\partial r} \int_{S^{n-1}} u(x_0 + ry) dA(y), \end{aligned}$$

and so $\int_{S^{n-1}} u(x_0 + ry) dA(y)$ is a constant function of $r$. Now integrate the derivative of this function from $r = 0$ to $r = R$ and interchange the order of integration to get

$$\begin{aligned} 0 &= \int_0^R \frac{\partial}{\partial r} \left( \int_{S^{n-1}} u(x_0 + ry) dA(y) \right) dr = \int_{S^{n-1}} \int_0^R \frac{\partial}{\partial r} u(x_0 + ry) dr dA(y) \\ &= \int_{S^{n-1}} u(x_0 + Ry) - u(x_0) dA(y) \\ &= \int_{S^{n-1}} u(x_0 + Ry) dA(y) - n\omega_n u(x_0). \end{aligned}$$

It follows that

$$u(x_0) = \frac{1}{n\omega_n} \int_{S^{n-1}} u(x_0 + Ry) dA(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x) dA(x).$$

This last equation is in fact valid for all $r \in (0, R)$, so we can integrate it to obtain the solid mean value theorem:

$$u(x_0) = \frac{1}{\omega_n R^n} \int_{B_R(x_0)} u(x) dV(x).$$

□

We'll see later on that a continuous function is harmonic if and only if it satisfies the mean value property. First, let's see why the mean value property might be useful.

**Theorem 3.** *Let $D \subset \mathbf{R}^n$ be a bounded domain with $C^1$ boundary and let $u \in C^2(D) \cap C^0(\bar{D})$ be harmonic. Then*

$$\sup_{x \in D} u(x) = \sup_{x \in \partial D} u(x), \qquad \inf_{x \in D} u(x) = \inf_{x \in \partial D} u(x).$$

*Moreover, if there is an interior point $x_0 \in D$ such that either $u(x_0) = \sup_{x \in D} u(x)$ or $u(x_0) = \inf_{x \in D} u(x)$ then $u$ is constant.*

The part of this theorem regarding the supremum of $u$ is called the maximum principle for harmonic functions, and the part regarding the infimum is called the minimum principle. It actually holds for solutions to a wide class of elliptic differential equations; we will give a different proof of this in a later set of notes.

*Proof.* We prove the maximum principle. Suppose there is an interior point $x_0 \in D$ such that $u(x_0) = \sup_{x\in D} u(x)$ and choose $R > 0$ small enough so that $B_R(x_0) \subset D$. Then for $0 < r < R$ we have

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(x) dA(x)$$

by the mean value property. On the other hand, the fact that $u(x_0)$ is the largest possible value for $u$ tells us that $u(x) = u(x_0)$ for all $x \in \partial B_r(x_0)$. This holds for all $r \in (0, R)$, so $u$ is constant on the ball $B_R(x_0)$.

Now suppose that there is some $x_*$ with $u(x_*) < u(x_0)$. (Notice we can't have $u(x_*) > u(x_0)$ by definition.) Join $x_0$ to $x_*$ with a continuous path $\gamma$, say with $\gamma(0) = x_0$ and $\gamma(1) = x_*$. There is some $\tau \in (0, 1)$ such that $u(\gamma(t)) < u(x_0)$ for $t > \tau$ but $u(\gamma(\tau)) = u(x_0)$. However, we can then draw a small sphere about $x_{**} = \gamma(\tau)$ , and the average value of $u$ over this sphere will be strictly less than the value at the center, which contradicts the average value property of harmonic functions. Therefore, such an $x_*$ cannot exist, and we conclude that $u$ is constant over $D$. The fact that $\sup_{x\in D} u(x) = \sup_{x\in\partial D} u(x)$ follows, because otherwise $u$ would have an interior maximum.

The minimum principle follows by applying the maximum principle to $-u$, which is still harmonic. □

The main application of the maximum principle is the following comparison principle.

**Corollary 4.** *Let $u$ and $v$ be harmonic on the domain $D$, and suppose $u|_{\partial D} \leq v|_{\partial D}$. Then $u \leq v$ on all of the domain $D$.*

*Proof.* Apply the maximum principle to $w = u - v$. □

We can use the maximum principle to prove uniqueness of harmonic functions with given boundary values.

**Theorem 5.** *Let $D \subset \mathbf{R}^n$ be a bounded domain with $\partial D \in C^1$, and let $f \in C^0(\partial D)$. If $u, v \in C^2(D) \cap C^0(\bar{D})$ both solve*

$$\Delta u = \phi(x) = \Delta v, \qquad u|_{\partial D} = f = v|_{\partial D}$$

*then $u(x) = v(x)$ for all $x \in D$.*

*Proof.* If we apply the maximum principle to $w_+ = u - v$ then we see $u \leq v$ in $D$. Conversely, if we apply the maximum principle to $w_- = v - u$ then we see $u \geq v$ in $D$. □

In fact, we can improve this last theorem by giving estimates.

**Theorem 6.** *Let $D \subset B_R(0) \subset \mathbf{R}^n$ be a domain in with $C^1$ boundary and let $u, v \in C^2(D) \cap C^0(\bar{D})$ satisfy*

$$|\Delta u - \Delta v| \leq \epsilon_1, \qquad |u|_{\partial D} - v|_{\partial D}| \leq \epsilon_2.$$

*Then for all $x \in D$ we have the estimate*

$$|u(x) - v(x)| \leq \frac{\epsilon_1 R^2}{2n} + \epsilon_2.$$

*Proof.* If we apply the maximum principle to

$$u(x) - v(x) - \epsilon_2 - \frac{\epsilon_1}{2n}(R^2 - |x|^2)$$

we get the estimate

$$u(x) \leq v(x) + \epsilon_2 + \frac{\epsilon_1 R^2}{2n},$$

while if we apply the maximum principle to

$$v(x) - u(x) - \epsilon_2 - \frac{\epsilon_1}{2n}(R^2 - |x|^2)$$

we get

$$v(x) \le u(x) + \epsilon_2 + \frac{\epsilon_1 R^2}{2n}.$$

□

As another application of the mean value property we prove some interior gradient estimates for harmonic functions. Let $u$ be harmonic in $D$, let $y \in D$, and choose $R > 0$ so that $B_R(y) \subset D$. We can interchange derivatives so see that $\frac{\partial u}{\partial x_i}$ is also harmonic for $i = 1, 2, \ldots, n$. Then the mean value property and the divergence theorem tells us

$$\nabla u(y) = \frac{1}{\omega_n R^n} \int_{B_R(y)} u(x) dV(x) = \frac{1}{\omega_n R^n} \int_{\partial B_R(y)} u(x) dA(x).$$

Taking the length of the vectors on both sides of this inequality we have

$$|\nabla u(y)| \le \frac{n}{R} \sup_{x \in \partial B_R(y)} |u(x)| \Rightarrow |\nabla u(y)| \le \frac{n}{\mathrm{dist}(y, \partial D)} \sup_{x \in D} |u(x)|. \tag{4}$$

The first application of our gradient estimate is Liouville's theorem. (We'll strengthen this theorem later.)

**Theorem 7.** *A bounded, harmonic function defined on all of* $\mathbf{R}^n$ *is constant.*

*Proof.* Let $u \in C^2(\mathbf{R}^n)$ be bounded and harmonic, and choose $M > 0$ such that $|u(x)| \le M$ for all $x$. Then, for any $y$ we have

$$|\nabla u(y)| \le \frac{nM}{\mathrm{dist}(x, y)}.$$

The theorem follows if we let $\mathrm{dist}(x, y) \to \infty$. □

We can iterate (4) to get estimates for the higher order derivatives of $u$. To write this out, it is useful to use multi-index notation. A multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a list of $n$ non-negative numbers, and we write $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then we can write the higher order partial derivatives of $u$ as

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} = D^\alpha u.$$

**Theorem 8.** *Let $u$ be harmonic in $D$ and let $D' \subset \bar{D}' \subset D$ with $\bar{D}'$ compact. Then*

$$\sup_{y \in D'} |D^\alpha u(y)| \le \left( \frac{n|\alpha|}{d} \right)^{|\alpha|} \sup_{x \in D} |u(x)|, \quad d = \mathrm{dist}(D', \partial D) = \inf\{\mathrm{dist}(x, y) : y \in D', x \notin D\}, \tag{5}$$

*where $\alpha$ is any multi-index.*

We can combine (5) and the Arzela-Ascoli theorem to prove the following convergence theorem.

**Theorem 9.** *A bounded sequence of harmonic functions on a domain $D$ has a subsequence which converges uniformly on compact subsets.*

We will see later that the limit function is also harmonic. This sort of theorem is usually called a **pre-compactness** result. It says that the space of harmonic functions is pre-compact, *i.e* bounded subsets of the space of harmonic functions are compact in the appropriate topology.

The last theorem we prove in this section is a version of the Harnack inequality. We will prove more general versions later.

**Theorem 10.** *Let $R > 0$ and $u \in C^2(B_{2R}(0)) \cap C^0(B_{2R}(0))$ be a non-negative, harmonic function. Then there is a constant $c = c(n)$ depending only on the dimension $n$ such that*

$$\sup_{x \in B_R(0)} u(x) \leq c(n) \inf_{x \in B_R(0)} u(x).$$

*Proof.* Let $x, y \in B_R(0)$. We aim to show that $u(y) \leq cu(x)$, and to that end we pick two intermediate points $w, z$ to compare. More precisely,

$$w = \frac{2}{3}x + \frac{1}{3}y, \qquad z = \frac{1}{3}x + \frac{2}{3}y;$$

geometrically, $w$ and $z$ are the points $1/3$ and $2/3$ of the way along the line segment joining $x$ and $y$, respectively. Now, $B_{r/3}(w) \subset B_r(x)$, so by the mean value theorem we have

$$\begin{aligned} u(w) &= \frac{1}{\mathrm{Vol}(B_{r/3}(w))} \int_{B_{r/3}(w)} u = \frac{3^n}{\mathrm{Vol}(B_r(x))} \int_{B_{r/3}(w)} u \\ &\leq \frac{3^n}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} u = 3^n u(x) \end{aligned}$$

We use a similar calculation to get $u(z) \leq 3^n u(w)$ and $u(y) \leq 3^n u(z)$, so that in the end we have

$$u(y) \leq (3^n)^3 u(x) = 3^{3n} u(x).$$

□

**Remark 1.** *The constant in the proof is clearly not optimal, but in practice this never matters. All we really care about is the fact that the constant depends only on the dimension.*

**Exercise:** Where in the proof did we use the fact that $u \geq 0$?

Harnack's inequality is very useful, and it says that harmonic functions can't oscillate too much.

**Exercise:** Prove the following strengthening of Harnack's inequality: if $u \in C^2(B_R(0))$ is harmonic, with $u \geq 0$ then for any $r \in (0, R)$ there is a constant $c = c(n, r, R)$ such that $\sup_{x \in B_r(0)} u(x) \leq c \inf_{x \in B_r(0)} u(x)$. (Hint: this is basically the same argument as in the proof of Harnack's inequality above, but comparing $u$ at more points.)

In fact, we can use the same technique to prove the following generalized version of the Harnack inequality for the Laplacian.

**Theorem 11.** *Let $D \subset \mathbf{R}^n$ and let $D' \subset \bar{D}' \subset D$, with $\bar{D}'$ compact, and let $u \in C^2(D)$ be a non-negative harmonic function. Then there is a constant $C = C(n, D, D')$ such that*

$$\sup_{x \in D'} u(x) \leq C \inf_{x \in D'} u(x).$$

Again, the value of the constant $C$ is unimportant; it is only important that $C$ is independent of $u$.

*Proof.* Let

$$d_1 = \mathrm{dist}(D', \mathbf{R}^n \backslash D) = \inf\{\mathrm{dist}(x, y) : x \in D', y \notin D\}$$

and let

$$d_2 = \mathrm{diam}(D') = \sup_{x,y \in D'} (\mathrm{dist}(x, y)).$$

These are both finite, positive numbers because $\bar{D}' \subset D$ is compact. Now choose $x, y \in D'$; as before, we want to show that $u(x) \leq cu(y)$ for some constant $c$ which does not depend on $u$. Connect $x$ to $y$ with a curve $\gamma$, such that

$$\gamma(0) = x, \qquad \gamma(1) = y, \qquad \mathrm{length}(\gamma) \leq d_2 + 1.$$

Notice that for any $z \in \gamma$ the ball $B_{d_1}(z) \subset D$, so we can apply Theorem 10 with $2R = d_1$. We complete the proof with a chaining argument, by choosing a sequence of points $z_0, z_1, \ldots, z_k$ along $\gamma$, with

$$z_0 = x, \qquad z_k = y, \qquad \frac{d_1}{2} \leq \operatorname{dist}(z_j, z_{j+1}) \leq d_1,$$

and repeatedly applying Theorem 10 to get $u(z_j) \leq c_1 u(z_{j+1})$, where $c_1$ depends only on $n$ and $d_1$. The number $k$ of points we have to choose is bounded above by $\frac{2(d_2+1)}{d_1}$, which is independent of $u$, and so we get

$$u(x) \leq c_1^{2(d_2+1)/d_1} u(y).$$

□

**The Greens function and the Poisson integral:** In this section we solve the boundary value problem (2):

$$\Delta u = 0, \qquad u|_{\partial D} = f$$

for some continuous function $f$ on $\partial D$. It will be helpful to define the function

$$\Gamma_n : (0, \infty) \to \mathbf{R}, \qquad \Gamma_n(r) = \begin{cases} \frac{1}{2\pi} \log(r) & n = 2 \\ -\frac{1}{n(n-2)\omega_n} r^{2-n} & n \geq 3. \end{cases}$$

**Exercise:** Verify by direction computation that $\Delta(\Gamma_n(|x|)) = 0$ on the domain $\mathbf{R}^n \backslash \{0\}$. It will be useful to recall the form of $\Delta$ in polar coordinates, which is $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \Delta_\theta$, where $\Delta_\theta$ collects all the terms in $\Delta$ involving angular derivatives. In particular, $\Delta(\Gamma_n) = \Gamma_n'' + \frac{n-1}{r}\Gamma_n'$ because $\Gamma_n$ is radially symmetric.

We'll also need to recall some integration by parts formulas. The divergence theorem (or Green's formula, or the Gauss divergence theorem, or.....) says that

$$\int_D u \Delta v dV(x) = \int_{\partial D} u \frac{\partial v}{\partial N} dA(x) - \int_D \langle \nabla u, \nabla v \rangle dV(x).$$

Taking the difference $\int_D (u\Delta v - v \Delta u) dV$ we have

$$\int_D (u \Delta v - v \Delta u) dV(x) = \int_{\partial D} u \frac{\partial v}{\partial N} - v \frac{\partial u}{\partial N} dA(x).$$

**Theorem 12.** *If $n \geq 3$ and $u \in C^2(D) \cap C^1(\bar{D})$ then for $y \in D$ we have*

$$u(y) = \int_D \Delta u(x) \Gamma_n(|x-y|) dV(x) + \int_{\partial D} \left( u(x) \frac{\partial}{\partial N} \Gamma_n(|x-y|) - \Gamma_n(|x-y|) \frac{\partial u}{\partial N}(x) \right) dA(x).$$

*Proof.* Let $r > 0$ be small enough so that $B_r(y) \subset D$ and apply the second Green's formula to $u$ and $v = \Gamma_n(|x-y|)$ on the domain $D \backslash B_r(y)$. We obtain

$$\begin{aligned} \int_{D \backslash B_r(y)} \Gamma_n(|x-y|) \Delta u dV(x) &= \int_{\partial D} \Gamma_n(|x-y|) \frac{\partial u}{\partial N} - u \frac{\partial}{\partial N} \Gamma_n(|x-y|) dA(x) \\ &\quad - \int_{\partial B_r(y)} \Gamma_n(|x-y|) \frac{\partial u}{\partial N} - u \frac{\partial}{\partial N} \Gamma_n(|x-y|) dA(x); \end{aligned}$$

we want to keep the first term of this expression as is, but we need to analyze the second term as $r \to 0$. First, we realize that $\Gamma_n(|x-y|)$ is a radial function on $B_r(y)$, so

$$\frac{\partial}{\partial N} \Gamma_n(|x-y|) = \frac{d}{dr} \Gamma_n(r) = \Gamma_n'(r) = \frac{r^{1-n}}{n \omega_n}.$$

This means

$$\int_{\partial B_r(y)} u(x)\frac{\partial}{\partial N}\Gamma_n(|x-y|) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(y)} u(x)dA(x) \to u(y)$$

as $r \to 0$, because the latter integral is just the average value of $u$ over the ball $B_r(y)$. Finally, we estimate the remaining term as

$$\begin{aligned}\left|\int_{\partial B_r(y)} \Gamma_n(|x-y|)\frac{\partial u}{\partial N} dA\right| &\leq |\Gamma_n(r)| n\omega_n r^{n-1} \sup_{B_r(y)} |\nabla u| \\ &= \frac{r}{n-2}\sup_{B_r(y)} |\nabla u| \to 0.\end{aligned}$$

When we take the limit as $r \to 0$ we put all these estimates together to read

$$\int_D \Gamma_n(|x-y|)\Delta u dV(x) = \int_{\partial D} \Gamma_n(|x-y|)\frac{\partial u}{\partial N} - u\frac{\partial}{\partial N}\Gamma_n(|x-y|)dA(x) + u(y).$$

The theorem follows. □

**Exercise:** Check that the theorem also holds for $n = 2$. (In fact, the proof is almost the same.)

Notice that if $\Delta u = 0$ then we have a formula for $u(y)$ entirely in terms of the boundary data of $u$. We will derive a better integral formula for harmonic functions soon. Also, as a function of $y \in \partial D$, the function $\Gamma_n(|x-y|)$ is analytic, and so we get the following corollary.

**Corollary 13.** *Let $D \subset \mathbf{R}^n$ be a bounded domain with $C^1$ boundary, and let $u \in C^2(D) \cap C^1(\bar{D})$ be a harmonic. Then $u$ is analytic on the interior of $D$.*

We're led at this point to define the Green's function of the Laplace operator on the domain $D$.

**Definition 2.** *Let $D \subset \mathbf{R}^n$ be a bounded domain with $\partial D \in C^1$. For $x, y \in D$ we define the function $G(x,y)$ by $G(x,y) = \Gamma_n(|x-y|) + h_y(x)$, where $h_y$ solves the boundary value problem*

$$\Delta h_y = 0, \qquad h_y|_{\partial D} = -\,\Gamma_n(|x-y|)|_{\partial D}.$$

We list some properties of the Green's function $G$.

**Proposition 14.** *If $D \subset \mathbf{R}^n$ is a bounded domain with $C^1$ boundary then*

- *For $x \neq y$ the function $x \mapsto G(x,y)$ is harmonic.*
- *For $y \in D$ and $x \in \partial D$, we have $G(x,y) = 0$.*
- *The Green's function in symmetric: $G(x,y) = G(y,x)$.*
- *If $u \in C^2(D) \cap C^0(\bar{D})$ then*

$$u(y) = \int_{\partial D} u(x)\frac{\partial G}{\partial N}(x,y)dA(x) + \int_D G(x,y)\Delta u(x)dV(x).$$

The first two of these properties is immediate from the definition of $G(x,y)$, we will prove the symmetry property later, and the last property follows from Theorem 12.

**Proposition 15.** *The Green's function on the ball $B_R(0)$ is given by*

$$G(x,y) = \Gamma_n(|x-y|) - \Gamma_n\left(\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right).$$

*Proof.* First, it is easy to check that

$$\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|^2=R^2+\frac{|x|^2|y|^2}{R^2}-2\langle x,y\rangle,$$

and so

$$\lim_{x\to 0}\Gamma_N\left(\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|\right)=\Gamma_n(R).$$

Next, notice that

$$\frac{R}{|x|}x=\frac{|x|}{R}y\Leftrightarrow y=\frac{R^2}{|x|^2}x\Rightarrow |y|=\frac{R^2}{|x|}>R\Rightarrow y\notin B_R(0),$$

so the function

$$\Gamma_n\left(\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|\right)$$

is actually smooth for $x,y\in B_R(0)$. Next, for fixed $y$ and $x\neq y$,

$$G(x,y)-\Gamma_n(|x-y|)=-\Gamma_n\left(\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|\right)$$

is a harmonic function of $x$, so we only need to check that $x\mapsto G(x,y)$ has the correct boundary values. Indeed, if $x\in\partial B_R(0)$ then $\frac{|x|}{R}=\frac{R}{|x|}=1$, so that in this case $G(x,y)=0$. □

**Lemma 16.** *On the ball, $G(x,y)=G(y,x)$.*

*Proof.* We have two terms to evaluate. First, $\Gamma_n(|x-y|)=\Gamma_n(|y-x|)$ by the properties of the absolute value function. Next we compute

$$\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|^2=R^2+\frac{|x|^2|y|^2}{R^2}-2\langle x,y\rangle=\left|\frac{R}{|y|}y-\frac{|y|}{R}x\right|^2,$$

so that

$$\Gamma_n\left(\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|\right)=\Gamma_n\left(\left|\frac{R}{|y|}y-\frac{|y|}{R}x\right|\right).$$

□

**Lemma 17.** *On the ball, fix $y\in B_R(0)$ and take the normal derivative of $G(x,y)$ with respect to $x$. The result is*

$$\frac{\partial G}{\partial N_x}=\frac{R^2-|y|^2}{n\omega_n R}|x-y|^{-n}.$$

*Proof.* We use the symmetry of the Greens function, which we've just proved, so that

$$\frac{\partial}{\partial x_i}G(x,y)=\frac{\partial}{\partial x_i}\left(\Gamma_n(|x-y|)-\Gamma_n\left(\left|\frac{R}{|y|}y-\frac{|y|}{R}x\right|\right)\right)=\frac{1}{n\omega_n}\left(\frac{x_i-y_i}{|x-y|^n}-\frac{\left(\frac{R}{|y|}y_i-\frac{|y|}{R}x_0\right)\left(\frac{-|y|}{R}\right)}{|x-y|^n}\right).$$

Now we evaluate

$$\begin{aligned}\frac{\partial G}{\partial N_x}&=\langle\nabla G,\frac{x}{|x|}\rangle=\frac{1}{n\omega_n|x-y|^n}\cdot\frac{1}{|x|}\left(|x|^2-\langle x,y\rangle+\langle x,y\rangle-\frac{|x|^2|y|^2}{R^2}\right)\\&=\frac{R^2-|y|^2}{n\omega_n|x-y|^n}.\end{aligned}$$

□

As a corollary of the lemma immediately above and the last point in Proposition 14, we have a version of the Poisson integral formula.

**Theorem 18.** *If $f \in C^1(\partial D)$ then the solution to* (2) *on $B_R(0)$, which is*

$$\Delta u = 0, \qquad u|_{\partial B_R(0)} = f,$$

*is given by*

$$u(y) = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{f(x) dA(x)}{|x-y|^n}.$$

We will prove later that in fact we only need the boundary data $f$ to be continuous, in order that the Poisson integral formula holds.

It will be convenient to define the Poisson integral kernel

$$K(x,y) = \frac{R^2 - |y|^2}{n\omega_n R} \cdot |x-y|^{-n}.$$

We can rewrite this expression as follows. First rescale the Poisson integral so that we integrate over a unit sphere instead of a sphere of radius $R$. To do this we write $x = R\eta$ and $y = r\xi$, and let $\theta$ be the angle between $\eta$ and $\xi$, so that the Poisson integral becomes

$$\begin{aligned} u(y) &= u(r\xi) = \frac{R^{n-2}(R^2 - r^2)}{n\omega_n} \int_{S^{n-1}} \frac{f(R\eta)}{(R^2 + r^2 - 2Rr\cos\theta)^{n/2}} dA(\eta) \\ &= \int_{S^{n-1}} f(R\eta)\hat{K}(R\eta, r\xi) dA(\eta), \end{aligned}$$

where

$$\hat{K}(x,y) = \hat{K}(R\eta, r\xi) = \frac{R^{n-2}(R^2 - r^2)}{n\omega_n (R^2 + r^2 - 2Rr\cos\theta)^{n/2}}.$$

This normalized Poisson kernel has the advantage of being homogeneous of degree zero: for any $\rho > 0$ we have $\hat{K}(\rho R\eta, \rho r\xi) = \hat{K}(R\eta, r\xi)$. It is also symmetric: $\hat{K}(R\eta, r\xi) = \hat{K}(R\xi, r\eta)$. We use these two properties to prove that harmonic functions satisfy the unique continuation property.

**Theorem 19.** *Let $u, v$ be harmonic in the domain $D$ and suppose that $u \equiv v$ on a nonempty open set. Then $u \equiv v$ on all of $D$.*

There are a number of different proofs of this fact; the proof you'd see in a typical complex analysis course relies on the power series representation of a harmonic function. This proof is due to W. Gustin, and was published in the American Journal of Mathematics (Vol. 70, No. 1, pg. 212–220).

To prove the unique continuation we need some technical results. Let $u_1$ be harmonic in the domain $D_1$ and $u_2$ harmonic in the domain $D_2$. If $p_1 \in B_{r_1}(p_1) \subset D_1$ and $p_2 \in B_{r_2}(p_2) \subset D_2$ we define the bilinear form

$$\Psi(r_1, r_2) = \int_{S^{n-1}} u_1(p_1 + r_1\xi) u_2(p_2 + r_2\xi) dA(\eta).$$

**Lemma 20.** *Let $u_1, u_2, p_1, p_2, r_1, r_2$ be as above, and suppose that $\rho_1$ and $\rho_2$ satisfy $p_1 \in B_{\rho_1}(p_1) \subset D_1$ and $p_2 \in B_{\rho_2}(p_2) \subset D_2$ and $r_1 r_2 = \rho_1 \rho_2$. Then $\Psi(r_1, r_2) = \Psi(\rho_1, \rho_2)$.*

*Proof.* We can translate $D_1$ by $-p_1$ and $D_2$ by $-p_2$, so without loss of generality we can suppose that both $p_1$ and $p_2$ are the origin. We will also assume without loss of generality that $0 < r_1 < \rho_1$, which implies $0 < \rho_2 < r_2$. Define $k = \frac{r_1}{\rho_2} = \frac{\rho_1}{r_2}$ and observe that

$$\hat{K}(\rho_1\eta, r_1\xi) = \hat{K}(kr_2\eta, k\rho_2\xi) = \hat{K}(r_2\eta, \rho_2\xi) = \hat{K}(r_2\xi, \rho_2\eta).$$

Now use the Poisson integral formula to write

$$u_1(r_1\xi) = \int_{S^{n-1}} \hat{K}(\rho_1\eta, r_1\xi)u_1(\rho_1\eta)dA(\eta), \quad u_2(\rho_2\eta) = \int_{S^n-1} \hat{K}(r_2\xi, \rho_2\eta)u_2(r_2\xi)dA(\xi)$$

so that

$$\begin{aligned} \Psi(r_1, r_2) &= \int_{S^{n-1}} u_1(r_1\xi)u_2(r_2\xi)dA(\xi) = \int_{S^{n-1}} \left( \int_{S^{n-1}} \hat{K}(\rho_1\eta, r_1\xi)u_1(\rho_1\eta)dA(\eta) \right) u_2(r_2\xi)dA(\xi) \\ &= \int_{S^{n-1}} \left( \int_{S^{n-1}} \hat{K}(r_2\xi, \rho_2\eta)u_2(r_2\xi)dA(\xi) \right) u_1(\rho_1\eta dA(\eta) \\ &= \int_{S^{n-1}} u_2(\rho_2\eta)u_1(\rho_1\eta)dA(\eta) = \Psi(\rho_1, \rho_2). \end{aligned}$$

□

As an immediate consequence of this lemma, we find the following proposition.

**Proposition 21.** *Let $u$ be harmonic in $D$, let $p \in D$ and suppose $0 \leq a < b < c$ with $ac = b^2$ and $B_c(p) \subset D$. Then*

$$\int_{S^{n-1}} u(p + a\xi)u(p + c\xi)dA(\xi) = \int_{S^{n-1}} u^2(p + b\xi)dA(\xi). \tag{6}$$

*Proof.* Apply the lemma with $u_1 = u_2 = u$, $p_1 = p_2 = p$, $r_1 = a$, $r_2 = c$ and $\rho_1 = \rho_2 = b$. □

**Exercise:** With the notation as in the proposition above, prove that if $u$ vanishes on $B_a(p)$ then $u$ also vanishes on $B_b(p)$.

**Exercise:** Use the fact you've just proven and a chaining argument to prove the unique continuation property of harmonic functions: if $u$ and $v$ are harmonic functions on a domain $D$ and $u \equiv v$ on a nonempty open set then $u \equiv v$ on all of $D$.

**Exercise:** Show that if $u$ satisfies (6) then it is harmonic. (Hint: take $a = 0$ and show $u$ satisfies the mean value property on small enough spheres.)

In fact, we can wring more information out of our integral representations of functions.

**Theorem 22.** *If $f \in C^1(\partial B_R(0))$ and $\phi \in C^0(B_R(0))$ then the solution to* (2)

$$\Delta u = \phi, \qquad u|_{\partial B_R(0)} = f$$

*is given by*

$$u(y) = \int_{B_R(0)} \phi(x) \left( \Gamma_n(|x-y|) - \Gamma_n \left( \left| \frac{R}{|x|}x - \frac{|x|}{R}y \right| \right) \right) dV(x) + \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{f(x)dA(x)}{|x-y|^n}.$$

*Proof.* We break this up into two separate problems: write $u = u_0 + u_1$, where $u_0$ is harmonic with $u_0|_{\partial B_R(0)} = f$, and $u_1$ satisfies $\Delta u_1 = \phi$ with $u_1|_{\partial B_R(0)} = 0$. By Theorem 18 we have

$$u_0 = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{f(x)dA(x)}{|x-y|^n},$$

and by the last point in Proposition 14

$$u_1 = \int_{B_R(0)} \phi(x)G(x,y)dV(x) = \int_{B_R(0)} \phi(x) \left( \Gamma_n(|x-y|) - \Gamma_n \left( \left| \frac{R}{|x|}x - \frac{|x|}{R}y \right| \right) \right) dV(x).$$

□

**Regularity and convergence:** In this section we sharpen some of the regularity statements of our theorems above. For instance, we will prove that we only need $f \in C^0(\partial B_R(0))$, rather than $f \in C^1(\partial B_R(0))$ in the Poisson integral formula of Theorem 18. As a basic application, we will prove some convergence theorems for families of harmonic functions.

**Proposition 23.** *Let $f \in C^0(\partial B_R(0))$ and let*

$$u(y) = \begin{cases} \frac{R^2-|y|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{f(x)dA(x)}{|x-y|^n} & |y| < R \\ f(y) & |y| = R. \end{cases}$$

*Then $u \in C^2(B_R(0)) \cap C^0(\bar{B}_R(0))$ and $u$ is harmonic inside $B_R(0)$.*

*Proof.* For $y \in B_R(0)$, we can differentiate underneath the integral sign to conclude $u \in C^2(B_R(0))$. Similarly, we can take the Laplacian of $u$ with respect to $y$ to get

$$\Delta_y u = \Delta_y \left( \int_{\partial B_R(0)} f(x) \frac{\partial G}{\partial N}(x,y) dA(x) \right) = \int_{\partial B_R(0)} f(x) \left\langle \nabla_x \Delta_y G(x,y), \frac{\partial}{\partial r} \right\rangle dA(x) = 0.$$

It remains to verify that $u \in C^0(\bar{B}_R(0))$. First, observe that by Theorem 18 we have that for all $y \in \bar{B}_R(0)$

$$1 = \frac{R^2 - |y|^2}{n\omega_n R} \int_{\partial B_R(0)} \frac{dA(x)}{|x-y|^n}.$$

Now let $M = \sup_{x \in \partial B_R(0)} |f(x)|$ (this is finite because $f$ is a continuous function on a compact set) and for $\epsilon > 0$ choose $\delta > 0$ so that $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0| < \delta$. Now let $y \in B_R(0)$ with $|y - x_0| < \frac{\delta}{2}$. Notice that $y$ is **inside** the ball $B_R(0)$, so $u(y)$ is given by the Poisson integral, and that

$$u(x_0) = u(x_0) \cdot 1 = u(x_0) \int_{\partial B_R(0)} K(x,y) dA(x).$$

Then

$$\begin{aligned} |u(y) - u(x_0)| &\leq \int_{\partial B_R(0)} K(x,y)|f(x) - f(x_0)| dA(x) \\ &\leq \int_{|x-x_0|<\delta} K(x,y)|f(x) - f(x_0)| dA(x) + \int_{|x-x_0|>\delta} K(x,y)|f(x) - f(x_0)| dA(x) \\ &\leq \epsilon + \frac{2M(R^2 - |y|^2)R^{n-2}}{(\delta/2)^n}. \end{aligned}$$

We conclude that if $|y - x_0|$ is sufficiently small then $|u(y) - u(x_0)| < 2\epsilon$, and so $u \in C^0(\partial B_R(0))$. □

A consequence of the Poisson integral formula is another way the characterize harmonic functions.

**Corollary 24.** *Let $D \subset \mathbf{R}^n$ be a domain, and let $u \in C^0(\bar{D})$. Then $u$ is harmonic in $D$ if and only if $u$ satisfies the mean value property: for every ball $B_R(x_0)$ with $\bar{B}_R(x_0) \subset D$ we have*

$$u(x_0) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x) dA(x).$$

*Proof.* We've already shown that if $u$ is harmonic then it satisfies the mean value property. Conversely, suppose $u$ is continuous and satisfies the mean value property in $D$, let $x_0 \in D$ and choose $R > 0$ so that $\bar{B}_R(x_0) \subset D$. On this ball, let $v$ be given by the Poisson integral, that is

$$v(x) = \begin{cases} \frac{R^2-|x-x_0|^2}{n\omega_n R} \int_{\partial B_R(x_0)} \frac{u(y)dA(y)}{|y-x-x_0|^n} & |x - x_0| < R \\ u(x) & |x - x_0| = R. \end{cases}$$

By Proposition 23, $v$ is harmonic. Moreover, evaluating at $x = x_0$ we have

$$v(x_0) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x)dA(x) = u(x_0),$$

because $u$ satisfies the mean value property. Thus, for any $x_0 \in D$, we have that $u(x_0) = v(x_0)$, where $v$ is harmonic, and so we conclude that $u$ must be harmonic. □

**Corollary 25.** *Let $\{u_n\}$ be a sequence of harmonic functions in a domain $D$ with $u_n \to u$ uniformly. Then $u$ is also harmonic.*

*Proof.* The limit $u$ is continuous and satisfies the mean value property, so it is harmonic. □

As an application of this last result, we can prove Harnack's convergence theorem.

**Theorem 26.** *Let $\{u_n\}$ be a monotone nondecreasing sequence of harmonic functions (that is, $u_n - u_m \geq 0$ for $n \geq m$) in a domain $D$, and suppose there is some $y \in D$ such that $\{u_n(y)\}$ converges. Then $\{u_n\}$ converges uniformly to a harmonic function on any subdomain $D' \subset D$, when $\bar{D}'$ is compact.*

*Proof.* Choose $\epsilon > 0$. Because $\{u_n(y)\}$ converges, there is $N$ such that if $n \geq m \geq N$ we have

$$0 \leq u_n(y) - u_m(y) < \epsilon.$$

By the generalized Harnack inequality (Theorem 11), there is a constant $C = C(n, D, D')$ such that

$$\begin{aligned} \sup_{x\in D'} |u_n(x) - u_m(x)| &= \sup_{x\in D'} (u_n(x) - u_m(x)) \leq C \inf_{x\in D'} (u_n(x) - u_m(x)) \\ &\leq C(u_n(y) - u_m(y)) < C\epsilon. \end{aligned}$$

Thus $\{u_n\}$ converges uniformly on $D'$ and, by the previous theorem, the limit function is also harmonic. □

One of the best features of the Laplace operator is that it is smoothing: for instance, if we start by knowing that $u$ is continuous and satisfies enought of the properties of a harmonic function, then it is in fact $C^\infty$ and harmonic. To prove these regularity results, we will need some tools.

**Exercise:** Show that the function

$$\chi(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is smooth (that is it has derivatives to all orders), and evaluate the $k$th derivative of $\chi$ at $t = 0$. (Hint: evaluate the first derivative using L'Hospital's rule, and then apply induction to compute the higher order derivatives.)

**Exercise:** Show that the function $\rho(x) = c_n\chi(1-|x|^2)$, where we choose $c_n$ so that $\int_{\mathbf{R}^n} \rho(x)dV(x) = 1$, is smooth, $\rho(0) = c_n > 0$, and $\mathrm{spt}(\rho_n) \subset B_1(0)$. Why is $\rho$ called a "bump function?"

**Exercise:** For $r > 0$ let $\rho_r(x) = r^{-n}\rho(x/r)$, and show that for all $r > 0$ we have $\int_{R^n} \rho_r(x)dV(x) = 1$. What is the support fo $\rho_r$? If $x \neq 0$, what is $\lim_{r\to 0} \rho_r(x)$? What is $\lim_{r\to 0} \rho_r(0)$?

For small choices of $r > 0$, the function $\rho_r$ is called an **approximate identity**, or a **mollifier**. This is because, for any $u \in L^1(\mathbf{R}^n)$, convolution with $\rho_r$ produces a smooth function

$$u_r(x) = \rho_r * u(x) = \int_{\mathbf{R}^n} \rho_r(x - y)u(y)dV(y).$$

which is very close to $u$, both in the $L^1(\mathbf{R}^n)$ norm and pointwise. In fact, when we take the limit as $r \to 0$ we obtain something which is not exactly a function (it's a distribution), but is commonly called the Dirac delta function $\delta_0$, which you may have seen in a physics class. You probably didn't construct the Dirca delta function in your physics class, but instead you listed some of its properties (such as $\int_{\mathbf{R}^n} f(x)\delta_0(x)dV(x) = f(0)$ for all continuous functions $f$). Now we see that it's not too hard to properly construct this "function." In fact, there are a number of other choices we can make for the starting function $\rho$, such as $c_n e^{-\frac{1}{2}|x|^2}$, and still end up with the same thing in the limit.

Next we need to define the notion of a weak derivative.

**Definition 3.** *Let $u \in L^1(\mathbf{R}^n)$. If it exists, the weak derivative $D_i u$ in the ith direction is a function $D_i u \in L^1(\mathbf{R}^n)$ such that*

$$\int_{\mathbf{R}^n} D_i u \phi dV(x) = -\int_{\mathbf{R}^n} u \frac{\partial \phi}{\partial x_i} dV(x), \qquad \forall \phi \in C^1(\mathbf{R}^n), \quad \operatorname{spt}(\phi) \textit{ compact.}$$

Notice that if $u \in C^1(\mathbf{R}^n)$ then (after integrating by parts) we see $D_i u = \frac{\partial u}{\partial x_i}$. Moreover, the weak derivative is unique (if it exists) by basic properties of the Lebesque integral.

**Definition 4.** *We say a function $u \in L^1(\mathbf{R}^n)$ is weakly harmonic if for all compactly supported $\phi \in C^2(\mathbf{R}^n)$ we have $\int_{\mathbf{R}^n} u \Delta \phi dV(x) = 0$.*

Notice that if $u \in C^2(\mathbf{R}^n)$ is in fact harmonic, we can integrate by parts twice to get, for all $\phi \in C^2(\mathbf{R}^n)$ with compact support,

$$0 = \int_{\mathbf{R}^n} \phi \Delta u dV(x) = \int_{\mathbf{R}^n} u \Delta \phi dV(x),$$

so that any $C^2$ harmonic function is also weakly harmonic.

**Lemma 27.** *The weak Laplacian of $\Gamma_n(|x|)$ is the Dirac delta function $\delta_0$. In other words, if $\phi \in C^2(\mathbf{R}^n)$ with $\operatorname{spt}(\phi)$ compact, then we have*

$$\int_{\mathbf{R}^n} \Gamma_n(|x|) \Delta \phi(x) dV(x) = \phi(0)$$

*Proof.* Choose $R > 0$ so that $\operatorname{spt}(\phi) \subset B_R(0)$ and choose $r > 0$ sufficiently small. Then by (12) we have

$$\int_{\mathbf{R}^n \setminus B_r(0)} \Gamma_n(|x|) \Delta \phi(x) dV(x) = \int_{\partial B_r(0)} \left( \Gamma_n(|x|) \frac{\partial \phi}{\partial r} - \phi(x) \frac{\partial}{\partial r} \Gamma_n(|x|) \right) dA(x).$$

Letting $r \to 0$ we have

$$\int_{\mathbf{R}^n \setminus B_r(0)} \Gamma_n(|x|) \Delta \phi(x) dV(x) \to \int_{\mathbf{R}^n} \Gamma_n(|x|) \Delta \phi(x) dV(x).$$

On the other hand, we have

$$\left| \int_{\partial B_r(0)} \Gamma_n(|x|) \frac{\partial \phi}{\partial r} dA(x) \right| \leq \frac{r^{n-1}}{(n-2) r^{n-2}} \sup |\nabla \phi| \to 0,$$

and

$$-\int_{\partial B_r(0)} \phi(x) \frac{\partial}{\partial r} \Gamma_n(|x|) dA(x) = \frac{1}{n \omega_n r^{n-1}} \int_{\partial B_r(0)} \phi(x) dA(x) \to \phi(0).$$

□

**Corollary 28.** *For a general bounded domain $D \subset \mathbf{R}^n$, with $C^1$ boundary, the Green's function $G$ is symmetric: $G(x, y) = G(y, x)$.*

*Proof.*

$$\begin{aligned}
G(x,y) - G(y,x) &= \int_D (G(x,z)\delta_0(y-z) - G(y,z)\delta_0(x-z))dV(z) \\
&= \int_D (G(x,z)\Delta_z\Gamma_n(|y-z|) - G(y,z)\Delta_z\Gamma_n(|x-z|))dV(z) \\
&= \int_{\partial D} (G(x,z)\frac{\partial}{\partial N}\Gamma_n(|y-z|) - G(y,z)\frac{\partial}{\partial N}\Gamma_n(|x-z|))dA(z) \\
&= 0.
\end{aligned}$$

Here we have integrated by parts, and used the fact that $G$ is harmonic inside $D$ and 0 on its boundary. □

We will now prove Weyl's lemma, which says that any weakly harmonic function is in fact harmonic.

**Theorem 29.** *If $D \subset \mathbf{R}^n$ is a domain with a $C^1$ boundary and $u \in L^1(D)$ satisfies $\int_D u\Delta\phi dV(x) = 0$ for all compactly supported function $\phi \in C^2(D)$ then $u$ is in fact smooth and harmonic.*

*Proof.* Recall that

$$u_r(x) = \rho_r * u(x) = r^{-n}\int_D \rho\left(\frac{|x-y|}{r}\right)u(y)dV(y)$$

is a smooth, compactly supported function (at least for $r > 0$ sufficiently small). We begin by showing that for all $f, g \in C^2(D)$ we have

$$\begin{aligned}
\Delta_y \int_D f(x-y)g(x)dV(x) &= \Delta_y \int_D f(z)g(y-z)DV(z) \qquad (7)\\
&= \int_D f(z)\Delta_y g(y-z)dV(z) \\
&= \int_D f(y-x)\Delta g(x)dV(x).
\end{aligned}$$

We apply (7) to $u_r$ and $\phi$ to get

$$\begin{aligned}
\int_D u_r(x)\Delta\phi(x)dV(x) &= \frac{1}{r^n}\int_D\left(\int_D \rho\left(\frac{|x-y|}{r}\right)u(y)\Delta\phi(x)dV(y)\right)dV(x) \qquad (8)\\
&= \int_D u(y)\left(\int_D \frac{1}{r^n}\rho\left(\frac{|x-y|}{r}\right)\Delta\phi(x)dV(x)\right)dV(y) \\
&= \int_D u(y)\Delta_y\left(r^{-n}\int_D \rho\left(\frac{|x-y|}{r}\right)\phi(x)dV(x)\right)dV(y) \\
&= \int_D u(y)\Delta_y\phi_r(y)dV(y).
\end{aligned}$$

We can now check by that $u_r$ is harmonic by choosing $\phi \in C^2(D)$ with $\mathrm{spt}(\phi) \subset D$. The function $u$ is weakly harmonic, so, using (8)

$$0 = \int_D u(y)\Delta_y\phi_r(y)dV(y) = \int_D u_r(y)\Delta\phi(y)dV(y).$$

However, $u_r \in C^\infty(D)$ so we can integrate by parts twice to get

$$\int_D \phi(y)\Delta u_r(y)dV(y) = 0.$$

This is true for all compactly supported $\phi$, so we must have $\Delta u_r \equiv 0$.

Next we look at the family $\{u_r\}_{r>0}$. We'd like to show this family is bounded and equicontinuous, so that we can apply the Arzela-Ascoli theorem to extract a convergent subsequence as $r \to 0$. Now, because $u_r = \rho_r * u$ and $\|\rho\|_{L^1(\mathbf{R}^n)} = 1$ we have

$$\|u_r\|_{L^1(D)} \leq \|\rho_r\|_{L^1(D)}\|u\|_{L^1(D)} \leq \|u\|_{L^1(D)}.$$

However, $u_r$ is harmonic, so for any $R > 0$ we have

$$|u_r(y)| = \frac{1}{\omega_n R^n}\left|\int_{B_R(y)} u_r(x)dV(x)\right| \leq \frac{\|u_r\|_{L^1(D)}}{\omega_n R^n} \leq \frac{\|u\|_{L^1(D)}}{\omega_n R^n},$$

which shows $\{u_r\}$ is uniformly bounded. To show $\{u_r\}$ is equicontinuous, it suffices (by the Fundamental Theorem of Calculus) to find a uniform bound on $|\nabla u_r|$. If we do have $|\nabla u_r| \leq L$ independent of $r$, then we can integrate this out to obtain $|u_r(x) - u_r(y)| \leq L|x - y|$ where $L$ is independent of $r$, which shows $\{u_r\}$ is equicontinuous. Because $u_r$ is harmonic, for every $\Omega$ with $\bar{\Omega}$ a compact subset of $D$ we have

$$\sup_{\Omega} |\nabla u_r| \leq c_1 \sup_{D} |u_r| \leq c_2\|u\|_{L^1(D)}.$$

Now that we have $\{u_r\}$ a bounded, equicontinuous family of functions, we extract a subsequence $u_{r_i}$ such that $u_{r_i} \to v \in C^\infty$ uniformly on every compact subset $\Omega \subset D$. However, $u_{r_i} = \rho_{r_i} * u \to u$ in $L^1(D)$, and $L^1(D)$ is Hausdorff, so we must have $u = v$. Thus $u \in C^\infty(D)$, and we can integrate by parts properly to see

$$0 = \int_D u\Delta\phi dV = \int_D \phi\Delta u dV$$

for all compactly supported $\phi \in C^2(D)$, which implies $\Delta u = 0$. □

Our final theorem in this section is a removable singularities theorem. Recall that we say $f = o(g)$ near $x_0$ if

$$\lim_{x\to x_0}\left(\frac{|f(x)|}{|g(x)|}\right) = 0,$$

and that $f = O(g)$ near $x_0$ if there is a constant $M > 0$ and an open set $U$ containing $x_0$ such that $|f(x)| < M|g(x)|$ for all $x \in U$.

**Theorem 30.** *Suppose $u \in C^2(B_R(0)\backslash\{0\})$ and $u$ is harmonic on this domain. Furthermore, suppose that*

$$u(x) = \begin{cases} o(\log(|x|)) & n = 2 \\ o(|x|^{2-n}) & n \geq 3. \end{cases}$$

*Then $u$ extends to a harmonic function on $B_R(0)$; in other words, there if a harmonic function $v \in C^2(B_R(0))$ such that $v(x) = u(x)$ for $x \neq 0$.*

*Proof.* On the ball $B_{R/2}(0)$ we define the function

$$v(y) = \frac{(R/2)^2 - |y|^2}{n\omega_n(R/2)}\int_{\partial B_{R/2}(0)} \frac{u(x)dA(x)}{|x-y|^n}.$$

By Theorem 18 we have $v \in C^\infty(\bar{B}_{R/2}(0))$ and $\Delta v = 0$. Now choose $\epsilon > 0$ and let

$$w(x) = \begin{cases} u(x) - v(x) + \epsilon\log\left(\frac{2|x|}{R}\right) & n = 2 \\ u(x) - v(x) - \epsilon\left(|x|^{2-n} - \frac{R^{2-n}}{2^{2-n}}\right) & n \geq 3. \end{cases}$$

We know that for any $\delta \in (0, R/2)$ we have $\Delta w = 0$ on $B_{R/2}(0) \backslash B_\delta(0)$, because $w$ is the sum of harmonic functions. Also, $w|_{\partial B_{R/2}(0)} = 0$.

Now we need to examine the behavior of $w$ on $\partial B_\delta(0)$. If $\delta > 0$ is sufficently small, then the term $-\epsilon|x|^{2-n}$ (or $\epsilon \log |x|$, for $n = 2$) dominates everything else, and this term is negative. Thus, by the maximum principle,

$$w \leq 0 \text{ on } \partial B_{R/2}(0) \cup \partial B_\delta(0) \Rightarrow w \leq 0 \text{ on } B_{R/2}(0) \backslash B_\delta(0),$$

which we can rearrange to read

$$u(x) \leq v(x) + \epsilon \left( |x|^{2-n} - \left(\frac{R}{2}\right)^{2-n} \right), \qquad \forall x \in B_{R/2}(0) \backslash B_\delta(0)$$

for $n \geq 3$. (The corresponding statement holds with a logarithm replacing the power of $|x|$ for $n = 2$.) Now, for any fixed $\delta$ we can let $\epsilon \to 0$ to get $u(x) \leq v(x)$ for all $x \in B_{R/2}(0) \backslash B_\delta(0)$ for all $\delta > 0$ sufficiently small, which in turn tells us

$$u(x) \leq v(x) \qquad \forall x \in B_{R/2}(0) \backslash \{0\}.$$

We can swap $u$ and $v$ and use the exact same argument on the function

$$\tilde{w} = \begin{cases} v(x) - u(x) + \epsilon \log\left(\frac{2|x|}{R}\right) & n = 2 \\ v(x) - u(x) - \epsilon\left(|x|^{2-n} - \frac{R^{2-n}}{2^{2-n}}\right) & n \geq 3. \end{cases}$$

We conclude

$$u(x) \geq v(x) \qquad \forall x \in B_{R/2}(0) \backslash \{0\},$$

and so we must have $u(x) = v(x)$ for all $x \in B_{R/2}(0) \backslash \{0\}$. □

**Subharmonic and superharmonic functions:** It will turn out to be useful to consider functions which satisfy a differential inequality involving the Laplace operator. As an application, we will use Perron's method to prove that one can solve the Dirichlet problem (2) on a very large class of domains.

**Definition 5.** *Let $D \subset \mathbf{R}^n$ be a domain with $\partial D \in C^1$. A function $u \in C^2(D) \cap C^0(\bar{D})$ is subharmonic if $\Delta u \geq 0$, and $u$ is superharmonic if $\Delta u \leq 0$.*

This definition might seem counter-intuitive: a subharmonic function satisfies that $\Delta u$ is larger than what you get for a harmonic function. We will see in a moment that this is actually a very sensible definition.

**Theorem 31.** *If $u \in C^2(D)$ is subharmonic in $D$, and $x_0 \in D$, and $R > 0$ small enough so that $B_R(x_0) \subset D$ then $u$ satisfies a sub-mean value property:*

$$u(x_0) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x) dA(x).$$

*Proof.* We can still use the divergence theorem, which in this case tells us that for $0 < r < R$

$$\begin{aligned} 0 &\leq \int_{B_r(x_0)} \Delta u dV(x) = \int_{\partial B_r(x_0)} \frac{\partial u}{\partial N}(x) dA(x) = r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(x_0 + ry) dA(y) \\ &= r^{n-1} \frac{\partial}{\partial r} \int_{S^{n-1}} u(x_0 + y) dA(y), \end{aligned}$$

and so $\int_{S^{n-1}} u(x_0 + ry)dA(y)$ is an increasing function of $r$. Now integrate the derivative of this function from $r = 0$ to $r = R$ and interchange the order of integration to get

$$\begin{aligned} 0 &\leq \int_0^R \frac{\partial}{\partial r}\left(\int_{S^{n-1}} u(x_0 + ry)dA(y)\right) dr = \int_{S^{n-1}} \int_0^R \frac{\partial}{\partial r} u(x_0 + ry) dr dA(y) \\ &= \int_{S^{n-1}} u(x_0 + Ry) - u(x_0) dA(y) \\ &= \int_{S^{n-1}} u(x_0 + Ry)dA(y) - n\omega_n u(x_0). \end{aligned}$$

It follows that

$$u(x_0) \leq \frac{1}{n\omega_n} \int_{S^{n-1}} u(x_0 + ry)dA(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(x_0)} u(x)dA(x).$$

This last equation is in fact valid for all $r \in (0, R)$, so we can integrate it to obtain the solid mean value theorem:

$$u(x_0) \leq \frac{1}{\omega_n R^n} \int_{B_R(x_0)} u(x)dV(x).$$

□

**Theorem 32.** *If $u \in C^2(D) \cap C^0(\bar{D})$ is subharmonic then $\sup_{x\in D} u(x) = \sup_{x\in\partial D} u(x)$. Moreover, if there is an interior point $x_0 \in D$ such that $u(x_0) = \sup_{x\in D} u(x)$ then $u$ is constant.*

**Remark 2.** *Notice that this theorem does* **not** *say that the minimum of a subharmonic function will occur on the boundary of D; usually the minimum occurs at an interior point.*

**Exercise:** Prove the maximum principle for subharmonic functions, mimicking the proof for harmonic functions.

We'll give a different, independent proof of the maximum principle in the case that $D$ is bounded now, because sometimes it's useful to know two different ways to prove something.

*Proof.* We first assume the easy case, that $\Delta u > 0$. Suppose that $u$ achieves its maximum at an interior point $x_0 \in D$. Then, by the second derivative test, we have $\Delta u(x_0) \leq 0$, which contradicts the fact that $\Delta u > 0$.

In the case we only have $\Delta u \geq 0$ we choose $\epsilon > 0$ and define $u_\epsilon(x) = u(x) + \epsilon|x|^2$. Then

$$\Delta u_\epsilon = \Delta u + 2n\epsilon > 0,$$

so $u_\epsilon$ cannot have an interior maximum by the version of the maximum principle we've just proved. Also, because $D$ is bounded, we have $D \subset B_R(0)$ for some $R > 0$. Thus

$$\sup_{x\in D} u(x) \leq \sup_{x\in D} u_\epsilon(x) = \sup_{x\in\partial D} u_\epsilon(x) \leq \sup_{x\in\partial D} u(x) + \epsilon R^2.$$

Letting $\epsilon \to 0$ we have $\sup_{x\in D} u(x) = \sup_{x\in\partial D} u(x)$.

Finally, if $u$ has an interior maximum we violate the sub-mean value property of subharmonic functions. □

**Corollary 33.** *If $u \in C^2(D) \cap C^0(\bar{D})$ is superharmonic then $\inf_{x\in D} u(x) = \inf_{x\in\partial D} u(x)$. Moreover, if there is an interior point $x_0 \in D$ such that $u(x_0) = \inf_{x\in D} u(x)$ then $u$ is constant.*

*Proof.* Apply the maximum principle for subharmonic functions to $-u$. □

**Corollary 34.** *Suppose $u, v \in C^2(D) \cap C^0(D)$ satisfy*

$$\Delta u \geq 0, \qquad \Delta v = 0, \qquad u|_{\partial D} \leq v|_{\partial D}.$$

*Then $u \leq v$ on all of $D$.*

*Proof.* Apply the maximum principle for subharmonic functions to $w = u - v$. □

This last corollary explains why we call a function $u$ with $\Delta u \geq 0$ subharmonic. Indeed, the subharmonic functions in $D$ are all the functions which are less than the harmonic function in $D$ with the same boundary values. In other words, if we fix the boundary values, the graph of a subharmonic function will always lie below the graph of a harmonic function.

We next prove a version of the Hadamard three circles theorem and use it to generalize Liouville's theorem to subharmonic functions.

**Theorem 35.** *Let $u$ be subharmonic in a domain $D \subset \mathbf{R}^2$ and suppose $\bar{B}_{r_2}(0) \backslash B_{r_1}(0) \subset D$ for some $0 < r_1 < r_2$. Then the maximum modulus function*

$$M(r) = \sup_{|x|=r} u(x)$$

*satisfies*

$$M(r) \leq \frac{M(r_1)\log(r_2/r) + M(r_2)\log(r/r_1)}{\log(r_2/r_1)}. \tag{9}$$

*Moreover, equality occurs if and only if $u$ is the function on the right hand side of* (9).

The content of this theorem is the statement that $M(r)$ is a convex function of $\log(r)$.

*Proof.* We can check that the function

$$\phi(r) = \frac{M(r_1)\log(r_2/r) + M(r_2)\log(r_2/r)}{\log(r_2/r_1)}$$

satisfies

$$\Delta\phi = 0, \qquad \phi(r_1) = M(r_1), \qquad \phi(r_2) = M(r_2)$$

in the annulus $B_{r_2}(0) \backslash B_{r_1}(0)$. Then the function $v(x) = u(x) - \phi(|x|)$ satisfies

$$\Delta v \geq 0 \text{ on } B_{r_2}(0) \backslash B_{r_1}(0), \quad v|_{\partial(B_{r_2}(0)\backslash B_{r_1}(0))} \leq 0.$$

The theorem now follows from the maximum principle. □

**Theorem 36.** *Let $u$ be subharmonic on $\mathbf{R}^2 \backslash \{0\}$ and suppose there is a constant $M$ such that $u(x) \leq M$ for all $x$. Then $u$ is constant.*

*Proof.* For a fixed $r_1$ and $r$, let $r_2 \to \infty$ in (9):

$$M(r) \leq M(r_1) \lim_{r_2\to\infty} \left( \frac{\log(r_2) - \log(r)}{\log(r_2) - \log(r_1)} \right) + \lim_{r_2\to\infty} \left( M(r_2) \frac{\log(r) - \log(r_1)}{\log(r_2) - \log(r_1)} \right) = M(r_1),$$

so that $M(r) \leq M(r_1)$ for $r > r_1$. (Here we have used $M(r_2) \leq M$ to take a limit.) On the other hand, we can let $r_1 \to 0$ to get $M(r) \leq M(r_2)$ for $r < r_2$. These inequalities hold for any $r_1 < r_2$, so we conclude $M(r)$ is constant. The maximum principle them implies $u$ is constant. □

There is a version of Hadamard's theorem for higher dimensions.

**Theorem 37.** *Let $u$ be subharmonic in $D \subset R^n$, with $\bar{B}_{r_2}(0) \backslash B_{r_1}(0) \subset D$, and (as above) let*

$$M(r) = \sup_{|x|=r} u(x).$$

*Then*

$$M(r) \leq \frac{M(r_1)(r^{2-n} - r_2^{2-n}) + M(r_2)(r_1^{2-n} - r^{2-n})}{r_1^{2-n} - r_2^{2-n}}, \tag{10}$$

*with equality if and only if $u(x) = a + b|x|^{2-n}$ for some $a$ and $b$.*

**Exercise:** Mimic the proof of Theorem 35 to prove Theorem 37.

**Example:** Theorem 37 does not imply a Liouville theorem for subharmonic functions in higher dimensions. In fact, the function

$$u(x) = \begin{cases} -\frac{1}{8}(15 - 10|x|^2 + 3|x|^4), & |x| \leq 1 \\ -\frac{1}{|x|}, & |x| > 1 \end{cases}$$

is a bounded, subharmonic functions which is non-constant.

Recalling that harmonic functions are exactly the continuous functions which satisfy the mean value property, we make the following definition.

**Definition 6.** *A continuous function $v \in C^0(D)$ is subharmonic if it satisfies the sub-mean value property: for all $p \in D$ and $r > 0$ with $B_r(p) \subset D$ we have*

$$v(p) \leq \frac{1}{\omega_n r^n} \int_{B_r(p)} v(x) dV(x).$$

**Exercise:** Show that these two definitions of subharmonic functions agree for functions $v \in C^2(D)$.

**Exercise:** Show that in fact it suffices to check the sub-mean-value property on small balls.

We close this set of notes with Perron's method of solving boundary value problems by finding barriers. This method is not constructive; it tells you that a solution to the problem exists, but doesn't actually hand you that solution. To do this we need some tools.

**Definition 7.** *Let $D \subset \mathbf{R}^n$ be a domain and let $y \in \partial D$. A barrier at $y$ is a function $Q_y$ which is subharmonic on $D$, and satisfies*

$$Q_y(y) = 0, \qquad Q_y(x) < 0 \text{ for all } x \in \partial D \backslash \{y\}.$$

Observe that, as a consequence of the maximum principle, $Q_y < 0$ inside $D$.

Our main theorem is the following.

**Theorem 38.** *Let $D \subset \mathbf{R}^n$ be a bounded domain. The classical boundary value problem*

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = f$$

*is solvable for all $f \in C^0(\partial D)$ if and only if $D$ has a barrier $Q_y$ for each of its boundary points $y \in \partial D$. Moreover, in this case the solution is unique.*

Before we prove this theorem, it might be useful to see which domains have barriers for each of their boundary points.

**Proposition 39.** *A domain $D \subset \mathbf{R}^n$ has a barrier at $y \in \partial D$ if it satisfies an exterior sphere condition at $y$: there is some ball $B = B_R(z)$ such that $\bar{B} \cap \bar{D} = \{y\}$.*

*Proof.* Let $B = B_R(y)$ be an exterior ball contacting $\partial D$ at $y$ as above. Then

$$Q_y(x) = \begin{cases} \log(|x - y|/R) & n = 2 \\ R^{2-n} - |x - y|^{2-n} & n \geq 3 \end{cases}$$

is a barrier at $y$. $\square$

**Exercise:** Show that if $\partial D$ is $C^2$ then $D$ has a barrier at each of its boundary points.

We finally prove Theorem 38.

*Proof.* To start, we suppose that one can solve the boundary value problem for all $f \in C^0(\partial D)$. Choose $y \in \partial D$ and let $u$ be the solution to the boundary value problem

$$\Delta u = 0, \qquad u|_{\partial D} = -\operatorname{dist}(x, y).$$

Then $u$ is a barrier for $D$ at $y$.

To prove the converse, we suppose that $D$ admits a barrier at each of its boundary points, and choose $f \in C^0(\partial D)$. Define the family of functions

$$\sigma_f = \{v \in C^0(\bar{D}) : \Delta u \geq 0, v|_{\partial D} \leq f\}.$$

Notice that $f$ is a continuous function on the compact set $\partial D$, so there are the extrema

$$m = \inf_{x \in \partial D} f(x), \qquad M = \sup_{x \in \partial D} f(x)$$

are both finite and achieved. By the maximum principle, we have $v \leq M$ for every $v \in \sigma_f$.

We will show that the function

$$u(x) = \sup\{v(x) : v \in \sigma_f\} \tag{11}$$

is the solution we're looking for. The uniqueness of the solution follows from the maximum principle. It is a little involved to prove that $u$ solves the boundary value problem, so we break the proof up into a series of steps. It will also be convenient to define the average value of a function: if $p \in D$ with $B_r(p) \subset D$ we let

$$\operatorname{avg}_v(p, r) = \frac{1}{\omega_n r^n} \int_{B_r(p)} v(x) dV(x).$$

Step 1: If $v, \tilde{v} \in \sigma_f$ then so is $w = \max(v, \tilde{v})$. Indeed, $w(x) = \max(v(x), \tilde{v}(x))$ is a continuous function and, since $v|_{\partial D} \leq f$ and $\tilde{v}|_{\partial D} \leq f$, we also have $w|_{\partial D} \leq f$. If $p \in D$ and $r > 0$ is small enough so that $B_r(p) \subset D$ then

$$w(p) = \max(v(p), \tilde{v}(p)) \leq \max(\operatorname{avg}_v(p, r), \operatorname{avg}_{\tilde{v}}(p, r)) \leq \operatorname{avg}_w(p, r),$$

which implies $w$ is subharmonic.

Step 2: In this step we define what is called the **harmonic lift** of a subharmonic function. Let $v \in \sigma_f$ a choose $B_r(p) \subset D$. Now let $u_{p,r}$ solve the boundary value problem

$$\Delta u_{p,r} = 0 \text{ in } B_r(p), \qquad u_{p,r}|_{\partial B_r(p)} = v$$

and define

$$v_{p,r}(x) = \begin{cases} u_{p,r}(x) & x \in B_r(p) \\ v(x) & x \in D \backslash B_r(p). \end{cases}$$

We want to show $v_{p,r} \in \sigma_f$. Indeed, $v_{p,r} \in C^0(D)$ because the function values of $u_{p,r}$ and $v$ agree on $\partial B_r(p)$, and $v_{p,r}|_{\partial D} = v|_{\partial D} \leq f$. Also observe that $v_{p,r} \geq v$ by the maximum principle. (We'll use this fact later). Finally, we show $v_{p,r}$ is subharmonic by showing for all $q \in D$ there is a $\rho_0$ such that if $\rho < \rho_0$ then $v_{p,r}(q) \leq \operatorname{avg}_{v_{p,r}}(q, \rho)$. If $q \in B_r(p)$ we can choose $\rho_0 = r - \operatorname{dist}(q, p)$. In this case, because $v_{p,r}$ is harmonic in $B_r(p)$, which contains $B_{\rho_0}(q)$, we have

$$v_{p,r}(q) = \operatorname{avg}_{v_{p,r}}(q, \rho).$$

If $q \notin B_r(p)$ we choose

$$\rho_0 = \min(\operatorname{dist}(q, \partial D), \operatorname{dist}(q, B_p(r)));$$

in this ball we have $v_{p,r} = v$, and the estimate follows because $v$ is subharmonic.

Step 3: Let $\bar{B}_r(p) \subset D$ and choose a countable set $X \subset B_r(p)$. Then there is a harmonic function $h \in C^2(B_r(p))$ such that $u(x) = h(x)$ for all $x \in X$. We construct $h$ by taking a diagonal limit as follows. Enumerate $X = \{x_1, x_2, \ldots,\}$ and fix $\rho$ so that $r < \rho < \text{dist}(p, \partial D)$. Now, for each $i$ choose a sequence $v_i^j \in \sigma_f$ such that

$$\lim_{j\to\infty} v_i^j(x_i) = u(x_i)$$

and define the function

$$v^j(x) = \max(m, v_1^j, v_2^j, \ldots, v_j^j).$$

Observe that (by Step 1) $v^j \in \sigma_f$ for each $j$, and (by the maximum principle) $m \leq v^j \leq M$. By Step 2, the harmonic function $u^j = (v^j)_{p,r}$ is in $\sigma_f$, and we still have the bounds $m \leq u^j \leq M$ by the maximum principle. Thus we have a bounded sequence of harmonic functions, and we can extract a subsequence (which we still denote as $u^j$) which converges uniformly to some $h$. However, $h$ is the uniform limit of a sequence of harmonic functions, so it is harmonic. To evaluate $h(x_i)$ we use (11) to see that, for $j \geq i$,

$$v_i^j(x_i) \leq v^j(x_i) \leq (v^j)_{p,r}(x_i) \leq u(x_i).$$

We then conclude

$$h(x_i) = \lim_{j\to\infty} (v^j)_{p,r}(x_i) = \lim_{j\to\infty} v_i^j(x_i) = u(x_i).$$

Step 4: The function $u$ defined by (11) is continuous. Choose $p \in D$ and $r > 0$ with $r < \text{dist}(p, \partial D)$. Then we let $X = \{p, x_2, x_3, \ldots,\}$ where $x_i \to p$. Apply Step 3 with this choice of $X$ to see (by the continuity of the harmonic function $h$)

$$u(p) = h(p) = \lim_{i\to\infty} h(x_i) = \lim_{i\to\infty} u(x_i).$$

Step 5: The function $u$ is harmonic. This time we apply Step 3 with $X$ being a countable dense subset of $B_r(p)$, where $\bar{B}_r(p) \subset D$. After applying this step we get a harmonic function $\hat{h}$ on $B_r(p)$ such that $\hat{h} = u$ on $X$. However, both $u$ and $\hat{h}$ are continuous, so $u \equiv \hat{h}$ on $B_r(p)$. Thus $u$ is harmonic on $B_r(p)$. Since this holds for all balls $B_r(p)$ compactly contained in $D$ our function $u$ must be harmonic.

Step 6: We next prove that if $y \in \partial D$ then $\liminf_{x\to y} u(x) \geq f(y)$. Choose positive numbers $\epsilon$ and $K$, and let

$$v(x) = f(y) - \epsilon + KQ_y(x).$$

This is a continuous, subharmonic function. Now choose $\delta = \delta(\epsilon)$ so that

$$f(x) > f(y) - \epsilon \text{ on } \partial D \cap B_\delta(y),$$

which implies $v(x) \leq f(x)$ on $\partial D \cap B_\delta(y)$ (because $Q_y \leq 0$). Moreover, $Q_y$ is a negative function on the compact set $\partial D \backslash B_\delta(y)$, so we can find $K = K(\epsilon)$ large enough so that $v(x) \leq g(x)$ on $\partial D \backslash B_\delta(y)$. Then we have $v|_{\partial D} \leq f$, and so $v \in \sigma_f$. By definition, $u \geq v$, so

$$g(y) - \epsilon = \lim_{x\to y} v(x) \leq \liminf_{x\to y} u(x).$$

Step 7: We prove that for each $y \in \partial D$ we have $f(y) \geq \limsup_{x\to y} u(x)$. In fact, we can recycle the proof of Step 6 with some minus signs. Define

$$\tilde{u}(x) = \sup_{-w\in\sigma_{-f}} (-w(x));$$

this is another harmonic function in $D$, and by Step 6 we have

$$\liminf_{x\to y} \tilde{u}(x) \geq -f(y).$$

If $v \in \sigma_f$ and $-w \in \sigma_{-f}$ then

$$(v - w)|_{\partial D} \leq 0 \Rightarrow v - w \leq 0 \text{ on } D.$$

Take the supremum of $v - w$ to see that $u + \tilde{u} \leq 0$, so

$$\limsup_{x \to y} u(x) \leq \limsup_{x \to y} (-\tilde{u}(x)) = -\liminf_{x \to y} \tilde{u}(x) \leq f(y).$$

These last two steps combine to show that $u = f$ on $\partial D$, and so $u$ is a continuous, weakly harmonic function with the correct boundary values. Weyl's lemma then tells us that $u$ is in fact $C^\infty$ and harmonic, completing the proof. □

**Remark 3.**

- *This method of constructing solutions by taking harmonic lifts of subharmonic functions is called the* **Perron method***.*
- *Notice that we only used the barriers to force the solution u to have the prescribed boundary values. Without barriers we can still attempt the Perron method, but we'd have no control over the boundary values.*
- *This is really the starting point to look at viscosity solutions of elliptic PDE.*
- *Also notice that the key step in our proof was an application of the Cantor diagonalization trick, very similar to our proof of the Arzela-Ascoli theorem.*