Partial Differential Equations Notes III

In these notes we prove some versions of the maximum principle for general elliptic operators. After that, we discuss some applications of the maximum principle, particularly using the moving planes argument of Alexandrov [A] (see also [GNN]).

Definitions: It will be useful to recall the following definition. A second order linear differential operator L has the form

$$L(u) = a_{ij}(x)u_{ij} + b_k(x)u_k + c(x)u, \qquad (1)$$

where subscripts denote partial derivatives and we sum over repeated indices. The operator L is elliptic at a point x if the coefficient matrix $[a_{ij}(x)]$ is positive definite, and L is uniformly elliptic on a domain $\Omega \subset \mathbb{R}^n$ if there is $\Lambda > 1$ such that

$$\frac{1}{\Lambda}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad (2)$$

for all $x \in \Omega$.

Let $F = F(x, u, Du, D^2u)$ be a (nonlinear) differential operator which is C^1 in all its arguments, and let w be a C^2 function. The linearization of F about w is the linear differential operator defined by

$$L_w(f) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(w + \epsilon f) = a_{ij}(x)f_{ij} + b_k(x)f_k + c(x)f, \qquad (3)$$

and we say F is uniformly elliptic if there is a number $\Lambda > 1$, which is independent of x and w, such that (2) holds, where a_{ij} are the coefficients of the linearization L_w.

It is worthwhile to consider some examples. The mean curvature operator

$$H(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is elliptic, but not uniformly elliptic. You lose control of Λ is $|\nabla u| \to \infty$, which is precisely what happens when $u(x) = \sqrt{R^2 - |x|^2}$ and $|x| \to R^-$. On the other hand, if $|\nabla u|$ is uniformly bounded then the nonlinear operator $H(u)$ is uniformly elliptic about u. The Monge-Ampere operator

$$M(u) = \det D^2u$$

is elliptic about w if and only if w is convex, that is, if and only if D^2w is positive definite.

Basic Maximum Principles: We start with the weak maximum principle.

Theorem 1. *Let L be a uniformly ellptic, linear operator of the form* (1) *with $c \leq 0$, and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq 0, \qquad u|_{\partial\Omega} \leq 0.$$

Then, unless $u \equiv 0$, for all $x \in \Omega$ we have $u(x) < 0$.

Proof. We suppose the theorem is not true, which means u has a non-negative maximum at some $p \in \Omega$, and derive a contradiction. This is easy if $L(u) > 0$, because

$$u(p) \geq 0, \qquad \nabla u(p) = 0, \qquad \left. D^2u\right|_p (e, e) \leq 0, \qquad (4)$$

where e is any unit vector. Now let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $a_{ij}(p)$, which are all positive, and let e_i be the eigenvector associated to λ_i. Then

$$L(u)(p) = a_{ij}(p)u_{ij}(p) + b_k(p)u_k(p) + c(p)u(p) = \sum_{i=1}^{n} \lambda_i \left. D^2u\right|_p (e_i, e_i) + c(p)u(p) \leq 0, \qquad (5)$$

which contradicts $L(u) > 0$.

For the general case, we build a barrier function as follows. Recall that $[a_{ij}(p)]$ is positive definite, so (after a rotation) we can assume $a_{11}(p) > 0$. We define

$$w(x) = u(x) + \epsilon z(x) = u(x) + \epsilon(e^{\alpha(x_1 - p_1)} - 1),$$

where α and ϵ are constants we choose later. Observe that

$$L(z)(p) = e^{\alpha(x_1-p_1)}(\alpha^2 a_{11}(p) + \alpha b_1(p)) + c(p)(e^{\alpha(x_1-p_1)} - 1),$$

and we can choose $\alpha > 0$ sufficiently large to that $L(z)(p) > 0$. By continuity we also have $L(z) > 0$ is a small neighborhood of p. Because p is a local maximum for u, we can find a nearby $q \in \Omega$ such that $u(q) < u(p)$, and now choose a positive

$$0 < \epsilon < \frac{u(p) - u(q)}{z(q)}.$$

Then

$$w(q) = u(q) + \epsilon z(q) < u(p), \qquad w(p) = u(p),$$

and so w has a positive interior maximum and satisfies $L(w) > 0$, which contradicts (5) as applied to w. □

Taking differences, we immediately obtain the following comparison theorem:

Corollary 2. *Let L be a uniformly ellptic, linear operator of the form* (1) *with $c \leq 0$, and let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq L(v), \qquad u|_{\partial\Omega} \leq v|_{\partial\Omega}.$$

Then, unless $u \equiv v$, for all interior points $x \in \Omega$ we have $u(x) < v(x)$.

Proof. Apply Theorem 1 to $w = u - v$. □

We have a condition on the normal derivative of u at $\partial\Omega$ as well.

Theorem 3. *Let L be uniformly elliptic, and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq 0, \qquad u \leq 0, \qquad u|_{\partial\Omega} = 0.$$

Then, unless $u \equiv 0$, for each $p \in \partial\Omega$ we have

$$\frac{\partial u}{\partial N} > 0,$$

where N is the unit outward normal vector for $\partial\Omega$.

Observe that we do not place a condition on the sign of c here.

Proof. We first prove this result for $c \equiv 0$. Fix $p \in \partial\Omega$, and choose $r_1 > 0$ small enough so that the ball $B_{r_1}(\tilde{x})$ tangent to $\partial\Omega$ at p lies completely inside Ω. Let B_1 be this ball and let $B_2 = B_{r_1/2}(p)$. Now, for some constants α and ϵ we define

$$w = u + \epsilon z = u + \epsilon(e^{-\alpha|x-\tilde{x}|^2} - e^{-\alpha r_1^2}).$$

Observe that

$$z|_{B_1} > 0, \qquad z|_{\partial B_1} = 0, \qquad z < 0 \quad \text{otherwise}.$$

By Theorem 1, we may assume $u < 0$ inside Ω, so in particular $u < 0$ on $\bar{B}_1 \backslash \{p\}$. Now pick $\epsilon > 0$ small enough so that $w = u + \epsilon z \leq 0$ on $(\partial B_2) \cap B_1$, and (as before) pick $\alpha > 0$ large enough so that $L(w) > 0$. Then, applying Theorem 1 to w on $B_1 \cap B_2$, we see w attains its maximum at p, so

$$0 \leq \frac{\partial w}{\partial N}(p) = \frac{\partial u}{\partial N}(p) + \epsilon \frac{\partial z}{\partial N}(p).$$

A quick computation shows

$$\frac{\partial z}{\partial N}(p) = -2\alpha e^{-\alpha r_1^2} \sum_{i=1}^{n} N_i x_i < 0,$$

which implies $\frac{\partial u}{\partial N}(p) > 0$.

Now we use the result above and a barrier to prove the theorem in the general case. Let $v = e^{-\beta x_1} u$, where $\beta > 0$ is a constant we choose later, and as before we can take $a_{11}(p) > 0$. Then

$$0 \leq L(u) = e^{\beta x_1} L'(v) + v L(e^{\beta x_1}),$$

where L' is a uniformly elliptic, linear operator with no zero order term. Rearranging the above inequality we get

$$0 \leq L'(v) + v(a_{11}\beta^2 + b_1\beta + c) = L'(v) + c'v.$$

Choose $\beta > 0$ large enough so that $L'(v) > 0$, at least near p. By what we have just proved,

$$\frac{\partial v}{\partial N}(p) > 0 \Rightarrow \frac{\partial u}{\partial N}(p) = e^{\beta p_1} \frac{\partial v}{\partial N}(p) > 0.$$

□

Finally, we prove the strong maximum principle.

Theorem 4. *Let $F = F(x, u, Du, D^2u)$ be a uniformly elliptic nonlinear differential operator which is C^1 in all its arguments, and non-increasing in u:*

$$u < v \Rightarrow F(x, u, Du, D^2u) \geq F(x, v, Dv, D^2v).$$

Let $u, v, \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$u \geq v, \qquad F(x, Du, D^2u) = F(x, Dv, D^2v).$$

If there is a point $p \in \bar{\Omega}$ such that

$$u(p) = v(p), \qquad Du(p) = Dv(p)$$

then $u \equiv v$.

Proof. As before, we assume $u(p) = v(p)$, $Du(p) = Dv(p)$, and that there are points where $u > v$, and derive a contradiction. There are two possible cases: either $u(q) > v(q)$ for all interior points $q \in \Omega$, or there are some interior points p with $u(p) = v(p)$. In the first case, let B_R be a ball of radius R such that $p \in \partial B_R$ and $\bar{B}_R \backslash \{p\} \subset \Omega$, and then let $A = B_R \backslash B_{\tilde{R}}$ where $\tilde{R} < R$ and $B_{\tilde{R}}$ has the same center as B_R. Also let $\Gamma_0 = \partial B_{\tilde{R}}$ and $\Gamma_1 = \partial B_R$. In the second case we can also choose an annulus A with the same form, provided we make a smart choice of p. We choose $p \in \partial\{u(x) = v(x)\}$ and observe that $\{u(x) \neq v(x)\}$ is a nonempty open set in Ω, so there is a q near p such that $u(q) > v(q)$. Now let q be the center of our annulus, $R = \text{dist}(p, q)$, such that $R < \text{dist}(q, \partial\Omega)$, and construct A as above. In either case, we now have an annulus A where $u \geq v$ on A, $u(p) = v(p)$ and $Du(p) = Dv(p)$ for some $p \in \Gamma_1$ and $u - v \geq \epsilon$ on Γ_0 for some $\epsilon > 0$.

For $0 \le t \le 1$ define

$$\chi(t) = F(x, tDu + (1-t)Dv, tD^2u + (1-t)D^2v).$$

Then by the mean value theorem

$$0 = F(x, Du, D^2u) - F(x, Dv, D^2v) = \chi(1) - \chi(0) = \chi'(t_0) = L(w)$$

for some $t_0 \in (0,1)$, where $w = u - v$ and L is the linearization of F, linearized about $t_0u + (1-t_0)v$. By the hypothesis on F, the linear operator L is a uniformly elliptic operator of the form (1) with $c \le 0$. (This is where we use the fact that F is monotone in the u variable.) In addition to $L(w) = 0$, we also have

$$w|_A \ge 0, \qquad w|_{\Gamma_0} \ge \epsilon, \qquad w(p) = 0.$$

We complete the proof by finding a barrier z with

$$L(z) > 0, \qquad z|_{\Gamma_0} = \epsilon, \qquad z|_{\Gamma_1} = 0, \qquad \frac{\partial z}{\partial r} < 0. \tag{6}$$

Indeed, once we construct z we use Corollary 2 on A to get $w \ge z$ and so

$$\frac{\partial w}{\partial r}(p) \le \frac{\partial z}{\partial r} < 0,$$

which contradicts $Du(p) = Dv(p)$. It is straightforward to check that, for M large enough, the function

$$z(x) = f(|x|^2/2), \qquad f(s) = \frac{\epsilon(e^{-Ms} - e^{-MR^2/2})}{e^{-M\tilde{R}^2/2} - e^{-MR^2/2}}$$

satisfies all the conditions in (6).

The following corollary is a special case of the strong maximum principle.

Corollary 5. *Let $F = F(x, Du, D^2u)$ be a uniformly elliptic, nonlinear differential operator which is homogeneous, i.e. $F(x, 0, \ldots, 0) = 0$ for all x. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

$$u \le 0, \qquad F(x, Du, D^2u) = 0$$

then either $u \equiv 0$ or $u < 0$ on the interior of Ω.

□

Moving planes and constant mean curvature surfaces: We include here Alexandrov's proof [A] that the only compact, embedded, constant mean curvature surface in $\mathbb{R}^3$ without boundary is the round sphere (see also [Ho]). Let $\Sigma \subset \mathbb{R}^3$ be a compact, embedded, constant mean curvature surface without boundary, and let Ω be the 3-dimensional region it bounds. The strategy is to use moving planes to show that Σ has a plane of symmetry perpendicular to any unit vector $\gamma \in S^2$. Once we do this, we still need to show that all these symmetry planes pass through a common point, which is easy. If x_0 is the center of mass of Σ (*i.e.* the average of all the position vectors of Σ), then each symmetry plane must contain x_0. After translation, we can assume $x_0 = 0$, and so Σ is invariant under all reflections through planes passing through the origin, which implies Σ is invariant under all rotations fixing 0. Thus Σ must be a round sphere.

Now fix some direction $\gamma \in S^2$. For $\lambda \in \mathbb{R}$ we let

$$T_\lambda = \{\langle x, \gamma\rangle = \lambda\}, \qquad \Sigma(\lambda) = \{x \in \Sigma : \langle x, \gamma\rangle > \lambda\},$$

and we let $\Sigma'(\lambda)$ be the reflection of $\Sigma(\lambda)$ through T_λ. Also define

$$\lambda_0 = \sup\{\lambda : \Sigma(\lambda) \ne \varnothing\}, \qquad \lambda_1 = \inf\{\lambda : \Sigma'(\tilde{\lambda}) \subset \Omega, \lambda < \tilde{\lambda} < \lambda_0\}.$$

Then $\Sigma'(\lambda_1)$ must contact Σ to first order at some point $p \in \Sigma \backslash \Sigma(\lambda_1)$. Near this point p, we can write Σ as the graph of u and $\Sigma'(\lambda_1)$ as the graph of v such that

$$u \geq v, \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 1 = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right).$$

Here u and v are functions defined on a small neighborhood of p in the the common tangent plane to Σ and $\Sigma'(\lambda_1)$. Moreover, because Σ and $\Sigma'(\lambda_1)$ contact to first order we have

$$u(p) = v(p), \qquad \nabla u(p) = \nabla v(p).$$

The strong maximum principle tells us $u \equiv v$ in a small neighborhood of p. However, soltutions to the equation $H(u) = 1$ are analytic, and so $\Sigma'(\lambda_1) = \Sigma \backslash \Sigma(\lambda_1)$, which means T_{λ_1} is a plane of symmetry for Σ. This completes the proof of Alexandrov's theorem.

The proof above yields the following more general theorem.

Theorem 6. *Let $\Sigma \subset \mathbb{R}^n$ be a compact, embedded hypersurface without boundary, and let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be its principle curvatures. If*

$$F(\kappa_1, \ldots, \kappa_n) = a,$$

where $a \in \mathbb{R}$ and F is a homogeneous, C^1 function, then Σ is a round sphere.

Moving planes and nonlinear equation on bounded domains: We present here some classical results of Gidas, Ni, and Nirenberg about positive solutions to nonlinear partial differential equations. Our model equation is

$$\Delta u + f(u) = 0, \qquad u > 0, \tag{7}$$

where f is a C^1 function. We consider this equation in either a bounded domain Ω with smooth boundary, or in the whole space of $\mathbb{R}^n$.

Theorem 7. *Let $\Omega = B_R = \{|x| < R\}$ and let $u \in C^2(\bar{\Omega})$ be a positive solution to (7) on Ω with the boundary condition $u|_{\partial\Omega} = 0$. Then $u(x) = u(r)$ and for $0 < r < R$ we have $\frac{\partial u}{\partial r} < 0$.*

Theorem 8. *Let $\Omega = B_R \backslash B_{\tilde{R}}$ and let $u \in C^2(\bar{\Omega})$ be a positive solution to (7) with the boundary condition $u|_{|x|=R} = 0$. Then*

$$\frac{R+\tilde{R}}{2} \leq |x| < R \quad \Rightarrow \quad \frac{\partial u}{\partial r} < 0.$$

Notice that in this last theorem we do not place a boundary condition on the inner sphere $|x| = \tilde{R}$.

We use Alexandrov's technique of moving planes. For a fixed direction $\gamma \in S^{n-1}$ and $\lambda \in \mathbb{R}$ we take

$$T_\lambda = \{\langle x, \gamma \rangle = \lambda\}, \qquad \Sigma(\lambda) = \{x \in \Omega : \langle x, \gamma \rangle > \lambda\},$$

and we let $\Sigma'(\lambda)$ be the reflection of $\Sigma(\lambda)$ across T_λ. We also define

$$\lambda_0 = \sup\{\lambda : \Sigma(\lambda) \neq \varnothing\}, \qquad \lambda_2 = \inf\{\tilde{\lambda} : \Sigma'(\lambda) \subset \Omega, \tilde{\lambda} < \lambda < \lambda_0\},$$

and we let λ_1 be the time of first contact of $\partial(\Sigma'(\lambda))$ with $\partial\Omega$. This first contact occurs either at a point $p \in \partial\Omega$ where $\partial(\Sigma'(\lambda_1))$ is tangent to $\partial\Omega$, or at a point $p \in T_{\lambda_1} \cap \partial\Omega$ where $T_{\lambda_1} \perp \partial\Omega$. We call $\Sigma(\lambda_1)$ the maximal cap, and $\Sigma(\lambda_2)$ the optimal cap. Observe that $\lambda_2 \leq \lambda_1 < \lambda_0$, and it is possible to have $\lambda_2 < \lambda_1$. For a point $x \in \Sigma(\lambda)$, we denote the reflection of x across T_λ by x^λ.

For the following technical results we take $\gamma = e_1$, so that

$$\Sigma(\lambda) = \{x \in \Omega : x_1 > \lambda\}, \qquad (x_1, x')^\lambda = (2\lambda - x_1, x'), \qquad \Sigma'(\lambda) = \{x : x^\lambda \in \Sigma(\lambda)\}.$$

We take $u \in C^2(\bar{\Omega})$ with $u > 0$ in Ω and

$$\Delta u + b_1 u_1 + f(u) = 0, \qquad u|_{\partial\Omega \cap \{x_1 > \lambda\}} = 0. \tag{8}$$

Proposition 9. *Let u satisfy* (8)*and assume $b_1 \geq 0$ on $\Sigma(\lambda_1)\cup\Sigma'(\lambda_1)$. Then for any $\lambda \in (\lambda_1, \lambda_0)$ we have, for $x \in \Sigma(\lambda)$,*

$$u_1(x) < 0, \qquad u(x) < u(x^\lambda), \tag{9}$$

and therefore $u_1 < 0$ in $\Sigma(\lambda_1)$. Moreover, if $u_1(p) = 0$ for some $p \in \Omega \cap T_{\lambda_1}$ then $u(x) = u(x^{\lambda_1})$, and

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup (T_{\lambda_1} \cap \Omega),$$

and $b_1 \equiv 0$.

We first use this technical proposition to prove Theorems 7 and 8. In the case of a ball, the proposition tells us $u_1 < 0$ in $\{x : |x| < R, x_1 > 0\}$, and, by continuity, $u_1(x) \leq 0$ for $x_1 = 0$. However, we can also apply the same argument with $\gamma = -e_1$, and so $u_1(x) = 0$ for $x_1 = 0$. The equality case of Proposition 9 tells us $u(x_1, x') = u(-x_1, x')$. Now rotate to obtain this symmetry for any choice of unit vector e.

In the case of the annulus, Proposition 9 shows $u_1 < 0$ in the maximal cap $\{x : |x| < R, x_1 > (R + \tilde{R})/2\}$. Moreover, if $u_1(x) = 0$ for some $x_1 = (R + \tilde{R})/2$ then Ω has to be symmetric about the plane $x_1 = (R + \tilde{R})/2$, which is it not. Thus $u_1 < 0$ on $\{x : |x| < R, x_1 \geq (R + \tilde{R})/2\}$. Again, we can apply this argument for any unit vector e to obtain Theorem 8.

To prove Proposition 9 we need the following lemmas.

Lemma 10. *Let $x_0 \in \partial\Omega$ with $N_1(x_0) > 0$, and let $u \in C^2(\Omega \cap \{|x - x_0| < \epsilon\})$ for some $\epsilon > 0$ with $u > 0$ in Ω and*

$$\Delta u + b_1(x))u_1 + f(u) = 0, \qquad u|_{\partial\Omega\cap\{|x-x_0|<\epsilon\}} = 0.$$

Then there exists $\delta > 0$ such that $u_1 < 0$ on $\Omega \cap \{|x - x_0| < \delta\}$.

Proof. We have $\frac{\partial u}{\partial N} \leq 0$, so (after possibly decreasing ϵ) we can assume $u_1 \leq 0$ on $S = \partial\Omega \cap \{|x - x_0| < \epsilon\}$. If the lemma were not true, there would exist a sequence of points $x^j \to x_0$ such that $u_1(x^j) \geq 0$. Let l_j be the segment in the x_1 direction joining x^j to $\partial\Omega$. Along this segment, u_1 changes sign, going from positive to negative, so there is a point $y^j \in l_j$ where $u_1(y^j) = 0$ and $u_{11}(y^j) \leq 0$. Taking the limit as $j \to \infty$ we have

$$u_1(x_0) = 0, \qquad u_{11}(x_0) \leq 0. \tag{10}$$

Suppose $f(0) \geq 0$. Then

$$\Delta u + b_1 u_1 + f(u) - f(0) \leq 0 \Rightarrow \Delta u + b_1 u_1 c_1 u = 0$$

for some function c_1. Then by Theorem 3 we have $\frac{\partial u}{\partial N}(x_0) < 0$, which implies $u_1(x_0) < 0$, which contradicts (10). On the other hand, if $f(u) < 0$ the $\Delta u(x_0) = -f(0) > 0$. Then $u_{ij}(x_0) = -f(0)N_i(x_0)N_j(x_0)$, so in particular $u_{11}(x_0) > 0$, which again contradicts (10). □

Lemma 11. *Suppose that for some λ in $[\lambda_1, \lambda_0)$ we have*

$$u_1 \leq 0, \qquad u(x) \leq u(x^\lambda), \qquad u(x) \not\equiv u(x^\lambda)$$

in the cap $\Sigma(\lambda)$. Then $u(x) < u(x^\lambda)$ and $u_1 < 0$ on $T_\lambda \cap \Omega$.

Proof. In $\Sigma'(\lambda)$, consider the function $v(x) = u(x^\lambda)$. In the reflected cap, we have

$$\Delta v - b_1(x^\lambda)v_1 + f(v) = 0, \qquad v_1 \geq 0.$$

Taking differences, in $\Sigma'(\lambda)$ we obtain

$$\Delta(v - u) + b_1(x)(v_1 - u_1) + f(v) - f(u) = (b_1(x^\lambda) + b_1(x))v_1 \geq 0,$$

where we have used that b_1 and v_1 are both non-negative. If $w = v - u$ in $\Sigma'(\lambda)$, then we have

$$\Delta w + b_1 w + cw \geq 0$$

for some function c. However, on T_λ we have $x^\lambda = x$ so $w = 0$, and then Theorem 1 implies $w = 0$ on $\Sigma'(\lambda)$. Thus $u(x) > u(x^\lambda)$ for $x \in \Sigma'(\lambda)$, and so (reflecting across T_λ) for $x \in \Sigma$ we have $u(x) < u(x^\lambda)$. Also, Theorem 3 implies and $w_1 = -2u_1 > 0$ on T_λ. □

Finally we prove Proposition 9.

Proof. Lemma 10 says we can assume (9) holds for $\lambda_0 - \delta < \lambda < \lambda_0$ for some small, positive δ. Let

$$\mu = \inf\{\tilde{\lambda} : \text{ (9) holds for } \tilde{\lambda} < \lambda < \lambda_0\} \geq \lambda_1.$$

We want to show that $\mu = \lambda_1$. By continuity, for $x \in \Sigma(\mu)$ we have

$$u(x) \leq u(x^\mu), \qquad u_1 < 0.$$

Now suppose $\mu > \lambda_1$ and take $x_0 \in \partial(\Sigma(\mu)) \backslash T_\mu$. Then $x_0^\mu \in \Omega$ so $0 = u(x_0) < u(x^\mu)$, so $u(x) \not\equiv u(x^\mu)$ and Lemma 11 implies $u(x) < u(x^\mu)$ in $\Sigma(\mu)$ and $u_1 < 0$ on $T_\mu \cap \Omega$. By continuity, there is some $\epsilon > 0$ such that $u_1 < 0$ in $\Omega \cap \{x_1 > \mu - \epsilon\}$.

By our construction of μ, there is a sequence $\lambda^j \to \mu^-$ and $x_j \in \Sigma(\lambda^j)$ such that $u(x_j) \geq u(x_j^{\lambda^j})$. A subsequence of $\{x_j\}$ converges to $\bar{x} \in \bar{\Sigma}(\mu)$ and $x_j^{\lambda^j} \to \bar{x}^\mu$. Because (9) holds in $\Sigma(\mu)$, we must have $\bar{x} \in \partial\Sigma(\mu)$. However, because $\mu > \lambda_1$, if $\bar{x} \in \partial\Sigma(\mu)\backslash T_\mu$ then $u(\bar{x}^\mu) > 0 = u(\bar{x})$, which we just showed can't happen. Thus $\bar{x} \in T_\mu \cap \Omega$ and so, for large j, the segement joining x_j to $x_j^{\lambda_j}$ is contained in Ω. On this segment, u increases moving in the positive x_1 direction, so each segment contains a point y_j such that $u_1(y_j) \geq 0$. Taking a limit we see $u_1(\bar{x}) \geq 0$ for some $\bar{x} \in T_\mu$, which contradicts $u_1 < 0$ in $\Omega \cap \{x_1 > \mu - \epsilon\}$. We conclude that $\mu = \lambda_1$.

In the case of equality, suppose we have $u_1(p) = 0$ for some $p \in T_{\lambda_1} \cap \Omega$. Then Lemma 11 tells us $u(x) = u(x^{\lambda_1})$ in $\Sigma(\lambda_1)$. Next, for any $x \in \partial(\Sigma(\lambda_1))\backslash T_{\lambda_1}$, we have $u(x) = 0 = u(x^{\lambda_1})$, and so $x^{\lambda_1} \in \partial\Omega$, which implies Ω is symmetric. Finally, suppose $b(x) > 0$ for some $x \in \Omega$. Then, comparing $u(x)$ to $u(x^{\lambda_1})$ and using (8) we have $b_1(x)u_1(x) = b_1(x^{\lambda_1})u_1(x^{\lambda_1})$. However, one side of this last equation is positive while the other side is negative, which is impossible. □

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