Partial Differential Equations Notes I

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In these notes we will survey some basic results in the theory of partial differential equations (PDE), concentrating on elliptic equations, particularly the Laplace operator. We begin with a brief introduction to the terminology.

Some basic definitions: Let's fix a domain $D \subset \mathbb{R}^n$ in Euclidean space, and suppose that the boundary of D, written ∂D, is at least of class C^1. (Recall that, by definition, a domain is an open, connected set in $\mathbb{R}^n$.)

Definition 1. *For $k \geq 1$, let $u \in C^k(D)$. An order k partial differential operator defined in D, acting on the function u, has the form $F(x, u, Du, D^2u, \ldots, D^k u)$, where Du is the gradient of u, D^2u is the Hessian matrix of u, and so on, and F is a function of all these variables.*

We'll concentrate on second order differential operators, which have the form $F(x, u, Du, D^2u)$. It's worthwhile to consider some examples.

- The most important differential operator is the Laplace operator,
$$\Delta u = \mathrm{div}(\mathrm{grad}\, u) = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$
The Laplace operator is linear and elliptic, and in fact the model of all such operators.

- The mean curvature operator is
$$\mathcal{H}(u) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \frac{\Delta u(1 + |\nabla u|^2) - |\nabla u|^2}{(1 + |\nabla u|^2)^{3/2}}.$$
This operator is nonlinear (so that $\mathcal{H}(u + v) \neq \mathcal{H}(u) + \mathcal{H}(v)$) but it is elliptic.

- The Monge-Ampere operator is
$$\mathcal{M}(u) = \det(D^2 u).$$
This is also nonlinear, and it is only elliptic when linearized about a convex function.

It's useful to write a second-order differential operator as $F = F(x, u, p, r)$ where $x \in D$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$, and r is a symmetric $n \times n$ matrix. Here p is supposed to represent the gradient ∇u and r is supposed to represent the Hessian matrix D^2u. We can usually write a differential equation in the form
$$F(x, u, p, r) = \phi(x)$$
for some continuous function ϕ.

Definition 2. *A (classical) solution to the differential equation $F(x, u, p, r) = \phi$ is a function $u \in C^2(D)$ such that for all $x \in D$ we have $F(x, u(x), \nabla u(x), D^2u(x)) = \phi(x)$.*

Again, it's worthwhile to have some examples. We leave it to the reader to check that

- For any vector $v \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the linear function $l(x) = \langle v, x\rangle + b$ is a solution to $\Delta(l) = 0$. For $x \neq 0$ and $n = 2$, then $\Delta(\ln(|x|)) = 0$ on $\mathbb{R}^2\backslash\{0\}$, and if $n \geq 3$ then $\Delta(|x|^{2-n}) = 0$ on $\mathbb{R}^n\backslash\{0\}$.

- For any positive number R, the function $u(x) = \sqrt{R^2 - |x|^2}$ is a solution to
$$\mathcal{H}(u) = \frac{1}{R}.$$

- The function $u(x) = \frac{1}{2}|x|^2 = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)$ is a solution to

$$\mathcal{M}(u) = 1.$$

In this course we will consider three fundamental questions for the differential equation $F(x, u, p, r) = \phi(x)$:

1. Existence: Under what circumstances does a solution exist? This is not an idle question. For instance, one can't always solve $\mathcal{M}(u) = 1$ with prescribed boundary data on a non-convex domain.

2. Uniqueness: If a solution does exist, is it unique?

3. Estimates: How big can a solution be? How much can it oscillate? If $F(x, u, Du, D^2u) = \phi$, can one estimate either the supremum of u or the oscillation of u in terms of ϕ?

Also, these questions are not of a purely theoretical nature. You may have seen some PDEs in physics, mathematical models, and/or applications. In this situation, you may have taken some of your data and plugged it into a numerical PDE solver or some other computational tool, and taken the answer this tool spits out as gospel. However, if the PDE doesn't have a unique solution, how do you know that the computational tool found the solution you're looking for? If a small change in the right had side ϕ can produce a large change in a solution u to the PDE $F(x, u, p, r) = \phi$, how much faith can you put in the answer your computational tool gives you?

Linear and nonlinear operators: The simplest sort of second order differential operator is a linear operator.

Definition 3. *A second order, linear, differential operator has the form*

$$L(x, u, p, r) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u(x).$$

The function a_{ij}, b_k, c are called the coefficients of L and we adopt the convention that $a_{ij} = a_{ji}$.

Observe that $L : C^2(D) \to C^0(D)$ is a linear operator: if $u, v \in C^2(D)$ and $\alpha, \beta \in \mathbb{R}$ then

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v).$$

The nonlinearities in a nonlinear differential operator can occur in several different severities. A **semi-linear** differential operator has the form

$$L(x, u, p, r) + f(u),$$

where L is linear. For instance, here are some semi-linear operators listed below:

$$\Delta u + u^2, \qquad \Delta u + u(1 - u^2), \qquad \Delta u + e^u.$$

A **quasilinear** differential operator is at least linear in the second derivative terms, so it has the form $Q(x, u, p, r)$, where Q is linear in the r variables. For instance, the mean curvature operator

$$\mathcal{H}(u) = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right),$$

which we saw above, is quasilinear. A differential operator which is not even quasilinear is called **fully nonlinear**. The Monge-Ampere operator $\mathcal{M}(u) = \det(D^2u)$ is fully nonlinear.

You can always linearize a nonlinear differential operator about a particular function.

Definition 4. *Let $F(x, u, p, r)$ be a second order differential operator, and choose $v \in C^2(D)$. The linearization of F about v is the second-order, linear, differential operator*

$$L_v(u) = \frac{d}{dt}\bigg|_{t=0} F(x, v + tu, \nabla v + t\nabla u, D^2 v + tD^2 u).$$

This is a little bit like computing the derivative of a nonlinear function at a point. In the same way that the derivative gives the slope of the line of best fit for this nonlinear function, the linearization L_v of F at v gives the linear operator which, at least close to v, comes closest to the nonlinear operator F.

Exercise: If $L(u) = \sum a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_k b_k(x)\frac{\partial u}{\partial x_k} + c(x)u(x)$ is a linear, homogeneous, differential operator, prove that it is its own linearization.

Exercise: If $F(x, u, p, r)$ is a second order differential operator, show that the linearization of F about the function v has the form

$$L_v(u) = \sum_{ij} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_k b_k(x)\frac{\partial u}{\partial x_k} + c(x)u(x)$$

where

$$a_{ij} = \frac{\partial F}{\partial r_{ij}}\bigg|_v, \qquad b_k = \frac{\partial F}{\partial p_k}\bigg|_v, \; c = \frac{\partial F}{\partial u}\bigg|_v.$$

Exercise: Verify that the linearization of the mean curvature operator $\mathcal{H}$ about v is

$$L_v(u) = \frac{d}{dt}\bigg|_{t=0} \mathcal{H}(v + tu) = \frac{\Delta u(1 + |\nabla v|^2) - (1 + \Delta v)\langle \nabla u, \nabla v\rangle}{(1 + |\nabla v|^2)^{3/2}}.$$

Types of second order differential operators: Whether or not one can solve $F(x, u, p, r) = \phi$ depends a lot on the type of differential operator F. This is easiest to see with linear operators.

The story begins with second order differential operators on functions of two variables. It's useful to recall that a conic section in the variables (x, y) is given by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

This zero-set is an ellipse, a parabola, or a hyperbola, depending on the sign of the discriminant $\delta = B^2 - 4AC$. We have a hyperbola if $\delta = B^2 - 4AC > 0$, a parabola if $\delta = B^2 - 4AC = 0$, and an ellipse if $\delta = B^2 - 4AC < 0$.

Proposition 1. *Let*

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

be a second order, linear differential operator in two variables with constant coefficients and no first- or zero-order terms, and let $\delta = B^2 - 4AC$.

- *If $\delta > 0$ then there is a linear change of variables $(\eta, \xi) = T(x, y)$ such that in the new variables $L(u) = \frac{\partial^2 u}{\partial \eta \partial \xi}$, and so the solutions to $L(u) = 0$ are $u(\eta, \xi) = p(\eta) + q(\xi)$.*
- *If $\delta = 0$ then there is a linear change of variables $(\eta, \xi) = T(x, y)$ such that in the new variables $L(u) = \frac{\partial^2 u}{\partial \eta^2}$, and so the solutions to $L(u) = 0$ are $u(\eta, \xi) = p(\xi) + \eta q(\xi)$.*
- *If $\delta < 0$ there is a linear change of variables $(\eta, \xi) = T(x, y)$ such that in the new variables $\pm L(u) = \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2}$.*

We can characterize these constant coefficient operators as hyperbolic (if $\delta > 0$), parabolic (if $\delta = 0$) or elliptic (if $\delta < 0$).

Proof. Let

$$\mathcal{A} = \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}$$

be the matrix formed by the coefficients of L. If L is hyperbolic, then the eigenvalues of $\mathcal{A}$ have opposite signs, and so there exists an orthogonal change of variables

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

so that

$$\bar{\mathcal{A}} = T\mathcal{A}T^t = \begin{bmatrix} 0 & \frac{\bar{B}}{2} \\ \frac{\bar{B}}{2} & 0 \end{bmatrix}.$$

In fact, the columns of T are nothing more than the eigenvectors of $\mathcal{A}$. In these new variables L has the form

$$L(u) = \bar{B}\frac{\partial^2 u}{\partial\xi\partial\eta},$$

and we can rescale both η and ξ by $\frac{2}{\bar{B}}$ to give L the form we claim.

Next we write out the solutions to the equation $\frac{\partial^2 u}{\partial\xi}\partial\eta = 0$. We first fix η_0 and allow ξ to vary. Then $u(\eta_0, \xi)$ satisfies $\frac{\partial}{\partial\xi}(u(\eta_0, \xi)) = 0$, and so $u(\eta_0, \xi) = p(\eta_0)$. Now for each value of ξ we can integrate to get $u(\eta, \xi) = p(\eta) + q(\xi)$.

The proofs for the parabolic and elliptic cases are almost exactly the same, and are left to the reader. □

If $D \subset \mathbb{R}^2$ and we consider

$$L(u) = A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2},$$

a general second-order, linear differential operator with no first- or zero-order term. At each point $(x_0, y_0) \in D$, this operator will be either hyperbolic, parabolic, or elliptic, depending on the sign of the discriminant $\delta(x,y) = B^2(x,y) - 4A(x,y)C(x,y)$. If $\delta(x,y)$ maintains one sign throughout the region D, then we say L is hyperbolic if $\delta > 0$, parabolic if $\delta = 0$, and elliptic if $\delta < 0$. In general, we do have lower order terms, but the second order part of L plays a stronger part in determining the behavior of solutions to the equation $L(u) = 0$, so that we can again characterize the differential operators as hyperbolic, parabolic, or elliptic, depending on the sign of the discriminant δ.

The picture is a little more complicated in higher dimensions, and there are more types of differential operators which can occur. However, the types which are the best understood are still the hyperbolic, parabolic, and elliptic types. As before, we have a domain $D \subset \mathbb{R}^n$, and we write a general point in D as $x = (x_1, x_2, \ldots, x_n) \in D$. We write

$$L(u) = L_2(u) + L_1(u) + L_0(u)$$

where

$$L_2(u) = \sum_{ij} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad L_1(u) = \sum_k b_k(x)\frac{\partial u}{\partial x_k}, \qquad L_0(u) = c(x)u.$$

To simplify the analysis a little, we start with the coefficients a_{ij}, b_k and c all being constants. If $\xi = T(x)$ is a linear change of variables, then we have the change of variables

$$\frac{\partial u}{\partial \xi_k} = \sum_l T_{kl}\frac{\partial u}{del x_l}, \qquad \frac{\partial^2 u}{\partial\xi_i\partial\xi_j} = \sum_{m,p} T_{mi}T_{pj}\frac{\partial^2 u}{\partial x_m \partial x_p},$$

where T_{mp} is the entry in the mth row, pth column of the $n \times n$ matrix representing T. With respect to these new coordinates, we can write the operators L_1, L_2 as

$$L_2(u) = \sum_{ij} \bar{a}_{ij}(\xi) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}, \qquad L_1(u) = \sum_k \bar{b}_k(\xi) \frac{\partial u}{\partial \xi_k},$$

where the new coeffients are $[\bar{a}_{ij}] = T[a_{ij}]T^t$ and $\bar{b} = T(b)$.

The discussion above motivates the following definition.

Definition 5. *The linear differential operator $L(u) = \sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_k b_k(x) \frac{\partial u}{\partial x_k} + c(x)u$ is elliptic if for every $x \in D$ the matrix $[a_{ij}(x)]$ is positive definite. It is uniformly elliptic if there is a $\Lambda > 0$ such that for all $x \in D$ and for all $\xi \in \mathbb{R}^n \backslash \{0\}$ we have*

$$\frac{1}{\Lambda}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq |\xi|^2\Lambda.$$

If $F(x, u, p, r)$ is a second order differential operator and $v \in C^2(D)$ then it is elliptic at v if the linearized operator L_v is elliptic. If L_v is elliptic for every $v \in C^2$ we say F is elliptic, and if one can find $\Lambda > 0$ such that the above estimate holds for all linearizations L_v we say F is uniformly elliptic.

Proposition 2. *Let L be a second order, linear differential operator of the form*

$$L(u) = \sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where the coefficients $a_{ij} \in C^2(D)$. Moreover, suppose L is elliptic. Then, at every point $x_0 \in D$, there is a linear change of variables $\xi = T(x)$ such that near x_0 we have

$$L(u) = \sum_{i=1}^n \bar{a}_{ii} \frac{\partial^2 u}{\partial \xi_i^2} + L_1(u),$$

where $L_1 = \sum_{k=1}^n b_k(\xi) \frac{\partial u}{\partial \xi_k}$ with $b_k(\xi_0) = 0$ if $\xi_0 = T(x_0)$.

Effectively, this proposition says that an elliptic operator behaves locally like the Laplace operator. Thus, if we understand solutions to the equation $\Delta u = 0$ we can transfer most of that understanding to solutions of the equation $L(u) = 0$, whenever L is uniformly elliptic. We'll see many nice properties of the Laplace operator in the next set of notes.

Proof. Let A be the $n \times n$ matrix with entries $a_{ij}(x_0)$. The fact that L is elliptic means A is positive definite, and so there is a linear change of variables T (in fact, it's an orthogonal change of variables) so that

$$TAT^t = \bar{A} = \begin{bmatrix} \bar{a}_{11} & 0 & \cdots & \cdots \\ 0 & \bar{a}_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 0 & \bar{a}_{nn} \end{bmatrix}.$$

Now perform this change of variables, using the formula for how L transforms above, and write out a second-order Taylor expansion for the coefficients of L in the new variables. □

Exercise: Prove that our model operator $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ is uniformly elliptic on any domain.

Exercise: Consider the domain $D = B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$, and consider the mean curvature operator

$$\mathcal{H}(u) = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

on D. Show that $\mathcal{H}$ is elliptic about any function v, but it might not be uniformly elliptic. (Hint: $v = \sqrt{R^2 - |x|^2}$.) Can you find conditions on v such that $\mathcal{H}$ is uniformly elliptic at v?

Exercise: Show that the Monge-Ampere operator $\mathcal{M}(u) = \det(D^2 u)$ is elliptic about v if and only if v is convex. (It might be helpful to do this computation first for $n = 2$ or $n = 3$.)