A Connoisseur's Guide to Constant Mean Curvature Surfaces in $\mathbb{R}^3$

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1 Introduction

The study of mean curvature extends back to the latter part of the 18th century. Indeed, Lagrange first found the minimal surface equation in 1761 when he looked for a necessary condition to minimizing a certain integral. The notion of mean curvature was first formally defined by Meusnier in 1776. Throughout the 19th century important mathematicians such as Gauß and Weierstraß devoted much of their studies to these surfaces. And yet constant mean curvature surfaces remain somewhat mysterious to this day.

In these notes we present some of the basic theorems from the field of constant mean curvature surfaces, most of which date from the first fifty years (or so) of the 20th century. However, one should not read these notes with the impression that the field is inactive; in fact, this is one of the more active fields relating geometry and analysis today. Most of the theorems below have more modern (less than 20 years old) extensions all of which build from the ideas presented in these notes. In addition, we will not even touch upon other exciting developments in this field. One should think of these notes as a brief guided tour to some of our favorite theorems and techniques from the field of constant mean curvature.

We caution the reader that this is very far from a complete survey of constant mean curvature surfaces (even in $\mathbb{R}^3$). For a more complete treatment, refer to [O], [Ho], [CM1], [DHKO], [C], [Str], and [L]. In addition, one should also consult a general text in differential and Riemannian geometry, for instance [Sp], [Lee] or [G].

2 Notions of Mean Curvature

In this section we will present some basic notions of mean curvature. We will start from its variational formulation, continue with a geometric interpretation and finally present some examples of constant mean curvature surfaces and alternative characterizations of surfaces with constant (or 0) mean curvature.

2.1 Geometric Variational Problems

We will start with some basic variational problems in differential geometry which give rise to the concept of mean curvature.

2.1.1 The Plateau Problem

Suppose γ is a closed Jordan curve in Euclidean Space. The Plateau Problem is to find a regular immersed surface with least area having γ as its boundary. If one does not restrict the topology of the surface in question, it may happen that one can choose a sequence of surfaces (with the same boundary curve γ) with increasingly complicated topology such that their areas converge to 0.

For that reason, we fix the topological type and try to minimize among parametric surfaces given by maps of a fixed two-manifold with boundary $X : \Sigma \to \mathbb{R}^3$. The simplest case is to consider maps from the closed unit disc $\bar{D}$ in the plane. A mapping $X : \bar{D} \to \mathbb{R}^3$ is called piecewise C^1 if it is continuous, and if except along ∂D and along a finite number of regular C^1 arcs and points in the interior of D, X is of class C^1. A restricted version of the Plateau problem is to find a $\bar{X} : \bar{D} \to \mathbb{R}^3$ (where $\bar{X}\big|_{\partial \bar{D}}$ is a diffeomorphism onto γ) which minimizes the area of all such parameterized surfaces.

Then we define the area functional Area(X), when $X : \bar{D} \to \mathbb{R}^3$ with $X|_{\partial \bar{D}}$ parameterizing γ by the following (generally improper) integral:

$$\text{Area}(X) = \int_{\bar{D}} \sqrt{\det(g_{ij})}\, dx_1 dx_2$$

Here $(x_1, x_2) \in \bar{D}$ are coordinates in the disc and

$$X_i = \frac{\partial X}{\partial x_i}, \qquad g_{ij} = \langle X_i, X_j \rangle \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^3$. We can orient our surface by using the normal vector

$$\nu = \frac{X_1 \times X_2}{|X_1 \times X_2|}.$$

The map $\nu : X(\bar{D}) \to S^2$ is called the Gauß map. Using the notation $|V|^2 = \langle V, V \rangle$ to denote the square length of a vector, then the integrand can be interpreted as the area of the parallelogram spanned by X_1 and X_2. If θ is the angle between X_1 and X_2, then the squared area of a parallelogram is

$$\begin{aligned} |X_1|^2 |X_2|^2 \sin^2\theta &= |X_1|^2 |X_2|^2 (1 - \cos^2\theta) \\ &= |X_1|^2 |X_2|^2 \left(1 - \frac{\langle X_1, X_2 \rangle^2}{|X_x|^2 |X_y|^2}\right) = \det(g). \end{aligned}$$

One must be careful in choosing the boundary contour γ, as it is possible to find Jordan curves γ such that Area$(X) = \infty$ for all $X : \bar{D} \to \mathbb{R}^3$ with $X|_{\partial \bar{D}}$ parameterizing γ. See [L] for such an example. The point is that one must assume there exists $X : \bar{D} \to \mathbb{R}^3$ with $\partial X(\bar{D}) = \gamma$ and Area$(X(\bar{D}))$ finite in trying to solve the Plateau problem.

The most significant difficulty in solving the variational problem arises from the fact that the area is independent of parameterization. Thus there is a loss of compactness for minimizers. Douglas found a way to finesse this difficulty, which will be explained in section 5.

2.1.2 Constant Mean Curvature for Graphs

One way to eliminate the invariance under reparameterizations is to restrict attention to nonparametric surfaces, those that can be given as graphs of functions. Thus $X(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$ where $u \in C^1(\bar{D})$. So $X_1 = (1, 0, u_1)$, $X_2 = (0, 1, u_2)$,

$$g = \begin{bmatrix} 1 + u_1^2 & u_1 u_2 \\ u_1 u_2 & 1 + u_2^2 \end{bmatrix}$$

and $\det(g_{ij}) = 1 + |Du|^2$. Also, the Gauß map is given by $\nu = \frac{(-u_1, -u_2, 1)}{\sqrt{1+|Du|^2}}$. The area of $X(\bar{D})$ is given by

$$\text{Area}(X) = \int_{\bar{D}} \sqrt{1 + |Du|^2}\, dx_1 dx_2$$

Consider variations of u which have the same boundary values on ∂D, say $u = \psi$ on ∂D. Assume that $v \in C_0^2(D)$ is a function with compact support and consider the variation

$$X[t] = (x_1, x_2, u(x_1, x_2) + tv(x_1, x_2)).$$

Then critical points of Area among variations that fix the volume

$$\mathrm{Vol}(X) = \int_D u dx_1 dx_2 = c,$$

where c is a constant, also satisfy the Euler Lagrange equation. Therefore there is some constant λ, the Lagrange multiplier, such that after integrating by parts

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \Big\{ \mathrm{Area}(X[t]) + \lambda \,\mathrm{Vol}(X[t]) \Big\} \\ &= \int_B \left\{ \frac{\langle Du, Dv \rangle}{\sqrt{1+|Du|^2}} + \lambda v \right\} dx_1 dx_2 \\ &= \int_B \left\{ -\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) + \lambda \right\} v \, dx_1 dx_2. \end{aligned}$$

Since v was arbitrary, the constrained optimizers satisfy the constant mean curvature equation (CMC equation.)

$$\begin{aligned} \mathcal{M}(u) = \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) &= \lambda. \qquad \text{on } D, \\ f &= \psi \qquad \text{on } \partial D. \end{aligned} \tag{2}$$

Definition 1 *The mean curvature of the graph of a function $u : \Omega \to \mathbb{R}$ is given by*

$$\mathcal{M}(u) = \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = \frac{u_{11}(1+u_2^2) + u_{22}(1+u_1^2) - 2u_1 u_2 u_{12}}{(1+|Du|^2)^{3/2}}.$$

If $\lambda = 0$ or if there was no volume constraint, then the critical points for area satisfy the resulting elliptic divergence structure equation, which we can rearrange to read

$$u_{11}(1+u_2^2) + u_{22}(1+u_1^2) - 2u_1 u_2 u_{12} = 0. \tag{3}$$

This equation is called the minimal surface equation.

2.2 Geometric Interpretation of the Mean Curvature Equation

In this section we formulate a coordinate-independent version of the mean curvature of an immersed surface and interpret it geometrically.

Recall that if we have a parameterization $X : \Omega \to \mathbb{R}^3$ of a surface for some planar domain Ω then we can measure distances and area in the surface using the induced metric (see equation (1))

$$g_{ij} = \langle X_i, X_j \rangle.$$

For a pair of general tangent vectors of the form $V = a_1 X_1 + a_2 X_2$ and $W = b_1 X_1 + b_2 X_2$ we can compute the inner product of V and W (in the surface) by

$$\langle V, W \rangle_X = \sum_{i,j} a_i b_i g_{ij}.$$

If f is a real C^1 function in a neighborhood of $p \in \Sigma$, its directional derivative in the $V \in \Sigma_p$ direction, where $V = \gamma'(0)$ for some curve through $\gamma(0) = p$, is given by

$$Vf(p) = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma.$$

The map $T_pX \to \mathbb{R}$ given by $V \mapsto Vf(p)$ is linear over $C^1(\Omega)$: $((f+g)V)h = f(Vh) + g(Vh)$. [1] It is also linear on the real vector space of functions $V(c_1 f + c_2 g) = c_1 Vf + c_2 Vg$ and satisfies Leibnitz rule $V(fg) = fVg + gVf$ where $c_i \in \mathbb{R}$ and f, g are C^1 real functions defined in a neighborhood of p.

The gradient of $f \in C^1(\Omega)$ is the unique vector field in Σ_p defined by

$$\langle \mathrm{grad} f, V \rangle_\Sigma = Vf, \qquad \text{for all } V \in T_pX.$$

Thus if $f, g_{ij} \in C^k$ then $\mathrm{grad} f$ is a C^{k-1} vector field.[2] One checks that the gradient is linear over $\mathbb{R}$ and satisfies the Leibnitz rule $\mathrm{grad}(c_1 f + c_2 g) = c_1 \mathrm{grad} f + c_2 \mathrm{grad} g$ and $\mathrm{grad}(fg) = f\mathrm{grad} g + g\mathrm{grad} f$ where $c_i \in \mathbb{R}$ and f, g are C^1 real functions defined in a neighborhood of p.

We also need to differentiate vector fields $Z(p)$ along a surface which are $\mathbb{R}^3$ valued functions on M. Thus $Z \in C^1(\Omega, \mathbb{R}^3)$. If V, W are tangent to $X(\Omega)$ then the componentwise directional derivative VZ makes sense. Note that the directional derivative is a C^1 linear function of V and an $\mathbb{R}$ linear in Z for which the Leibnitz rule holds $(fV + gW)Z = f(VZ) + g(VZ)$, $V(c_1 Z + c_2 Z) = c_1 VZ + c_2 VZ$ and $V(fZ) = (Vf)Z + fVZ$ for all $f, g \in C^1(U)$ and C^1 tangent vector fields V, W. Associated to the linear map, such as $V \mapsto VZ$, where Z is any C^1 vector field along $X(\Omega)$, such as ν, the unit normal vector field to X, are quadratic forms

$$\begin{aligned} Q(Z)[V,W] &= \langle VZ, W \rangle; \\ A[V,W] &= -\langle V\nu, W \rangle; \\ \mathrm{Hess}(f)[V,W] &= \langle V\,\mathrm{grad} f, W \rangle. \end{aligned}$$

Definition 2 *The bilinear forms A and $\mathrm{Hess}(f)$ defined above are called the second fundamental form and the Hessian of f, respectively. Given a C^1 vector field Z along $X(\Omega)$, we can also define the divergence $\mathrm{div}\, Z$, mean curvature H and the Laplacian of a function Δf as*

$$\begin{aligned} \mathrm{div}\ Z &= \mathrm{tr}_g\, Q(Z); \\ H &= \mathrm{tr}_g\, A; \\ \Delta f &= \mathrm{tr}_g\, \mathrm{Hess}(f) = \mathrm{div}(\mathrm{grad} f); \end{aligned}$$

where tr_g denotes the trace with respect to g, i.e. if $\mathcal{Q}$ is a quadratic form the in local coordinates $\mathrm{tr}_g\, \mathcal{Q} = \sum_{ij} g^{ij} \mathcal{Q}(X_i, X_j)$ where g^{ij} is the inverse matrix to g_{ij}. None of these objects we have defined depend on the choice of coordinates.

It is a good exercise to check that if $X : \Omega \to \mathbb{R}^3$ is of the form $X(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$ then H turns out to be the same thing as $\mathcal{M}(u)$, as defined in definition 1. (Hint: take the directional derivative of the equation $\langle \nu, W \rangle = 0$ in the V direction and show $A[V,W] = \langle \nu, VW \rangle$.)

The divergence is linear, $\mathrm{div}(c_1 W + c_2 Z) = c_1\, \mathrm{div}\, W + c_2\, \mathrm{div}\, Z$ and Leibnitz rule holds $\mathrm{div}(fW) = \langle \mathrm{grad} f, W \rangle_X + f\, \mathrm{div}(W)$. If V and g_{ij} are C^k then $\mathrm{div}(V)$ is C^{k-1}.[3] The Laplacian is $\mathbb{R}$ linear and satisfies $\Delta(c_1 f + c_2 g) = c_1 \Delta f + c_2 \Delta g$ and $\Delta(fg) = f\Delta g + \langle \mathrm{grad} f, \mathrm{grad} g \rangle + g\Delta f$.

[1] So $Vf = v^i \frac{\partial f}{\partial x_i}(p)$. In this notation, $\partial_i f = \partial f / \partial x_j$.

[2] In local coordinates, the equation implies $g_{ij}(\mathrm{grad} f)^i v^j = v^j \partial_j f$ for all $v^j \partial_j$ so $\mathrm{grad} f = g^{ij}(\partial_j f)\partial_i$ where g^{ij} is the inverse matrix of g_{ij}.

[3] Thus, in local coordinates $\mathrm{div}(V) = g^{ij}\langle \partial_i V, \partial_j \rangle_\Sigma$. In case $Z = z^i \partial_i$, $\mathrm{div}(V) = g^{pq}\langle \partial_p(z^i X_i), X_q \rangle = \partial_i z^i + g^{pq} z^i \langle X_{pi}, X_q \rangle$. However, $\partial_i \log g = g^{pq} g_{pqi} = g^{pq}(\langle X_{pi}, X_q \rangle + \langle X_p, X_{qi} \rangle) = 2g^{pq}\langle X_{pi}, X_q \rangle$. It follows that $\mathrm{div}\, Z = \partial_i z^i + \frac{1}{2} g^{pq} z^i g_{pqi} = g^{-1/2}(g^{1/2}\partial_i z^i + \frac{1}{2} g^{-1/2} z^i g g^{pq} g_{pqi}) = g^{-1/2}\partial_i(z^i g^{1/2})$ where $g = \det(g_{ij})$.

It is sometimes called the Laplace-Beltrami operator.[4] One way to remember the formula for Δ is to remember that is it contrived so that integration by parts works. More precisely, for $\phi, \psi \in C^2(\Omega)$,

$$\int_\Omega \phi\Delta\psi + \langle \mathrm{grad}\phi, \mathrm{grad}\psi\rangle \, d\mathrm{Area} = \int_\Omega \mathrm{div}(\phi \, \mathrm{grad}\psi) \, d\mathrm{Area} = \oint_{\partial\Omega} \phi \frac{\partial\psi}{\partial \mathrm{n}} \, d\,\mathrm{Length}$$

where n is the outward normal to $X(\Omega)$ in M.

It turns out that $\mathrm{Hess}(f)$ and A are symmetric quadratic forms. To see that the Hessian is symmetric, consider

$$\langle V\mathrm{grad}f, W\rangle = V\langle \mathrm{grad}f, W\rangle - \langle \mathrm{grad}f, VW\rangle = V(Wf) - (VW)f$$

which is symmetric since mixed second derivatives commute.[5] The second fundamental form is symmetric for a similar reason.[6]

The principal curvatures κ_i are the eigenvalues of the second fundamental form relative to the metric. [7] Because A is symmetric they are real and are given by the Rayleigh quotients

$$\kappa_1 = \max_{V\in T_pX\setminus\{0\}} \frac{A[V,V]}{\langle V,V\rangle}, \qquad \kappa_2 = \min_{V\in T_pX\setminus\{0\}} \frac{A[V,V]}{\langle V,V\rangle}.$$

From the definition of the principle curvatures it is immediate that

$$H = \kappa_1 + \kappa_2.$$

It is natural to also define the Gauss curvature by

$$K = \kappa_1\kappa_2.$$

Gauß proved that the Gauß curvature is an intrinsic quantity, which means that it can be computed from the metric alone and does not depend on how the surface is immersed in Euclidean space.

It follows that the squared norm of the second fundamental form satisfies

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = (\kappa_1+\kappa_2)^2 - 2\kappa_1\kappa_2 = H^2 - 2K. \tag{4}$$

For example, the sphere of radius r about the origin has $\nu = r^{-1}X$. Thus, the second fundamental form satisfies

$$A[V,W] = -\langle V\nu, W\rangle = -\frac{1}{r}\langle VX, W\rangle = -\frac{1}{r}\langle V, W\rangle,$$

so $\kappa_1 = \kappa_2 = 1/r$ and the surface is umbillic (i.e. the principal curvatures are equal everywhere). Therefore $H = 2/r$ and $K = r^{-2}$.

Let $f = \langle X, Z\rangle$ and $h = \langle \nu, Z\rangle$ where $Z \in \mathbb{R}^3$ is any fixed vector. Then the gradient, [8] Laplacian [9] and Hessian,[10] of f are

$$\begin{aligned} \mathrm{grad}f &= Z - \langle \nu, Z\rangle\nu \\ \Delta f &= -Hh \\ \mathrm{Hess}(f)[V,W] &= -hA[V,W]; \\ \Delta h &= -\langle \mathrm{grad}H, Z\rangle - |A|^2 h. \end{aligned}$$

[4] In local coordinates, $\Delta f = g^{pq}\langle \partial_i \mathrm{grad}f, X_j\rangle = g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j f)$.

[5] In local coordinates, $V(Wf) - (VW)f = v^i w^j \partial^2 f/\partial x_i \partial x_j$.

[6] Since $\langle \nu, X_i\rangle = 0$, by differentiating we have $A[V,W] = -v^i w^j \langle \nu_i, X_j\rangle = v^i w^j \langle \nu, X_{ij}\rangle$ but $X_{ij} = X_{ji}$.

[7] If $\{\mathbf{e}_i\}$ is a ds^2 orthonormal basis for T_pX then κ_i are the eigenvalues of the matrix relative to this basis $A_{ij} = A(\mathbf{e}_i, \mathbf{e}_j)$. Also $H = A_{11} + A_{22}$ and $K = \det(A_{ij})$ for this matrix. In the skew coordinates however, $K = \det(\langle \nu, X_{ij}\rangle)/g$.

[8] If we write Z in the basis $Z = z^i X_i + z^3\nu$ then $\langle X_k, Z\rangle = z^i g_{ik}$ so $Z - \langle \nu, Z\rangle\nu = g^{ik}\langle X_k, Z\rangle X_i = \mathrm{grad}f$.

[9] Since $\langle \nu, X_i\rangle = 0$, $\Delta f = g^{pq}\langle \partial_p \mathrm{grad}f, X_q\rangle = g^{pq}\langle \partial_p(Z - \langle \nu, Z\rangle\nu), X_q\rangle = -g^{pq}\langle \nu, Z\rangle\langle \nu_p, X_q\rangle = Hh$.

[10] $\mathrm{Hess}(f)[V,W] = \langle V\mathrm{grad}f, W\rangle = \langle V(Z - \langle \nu, Z\rangle\nu), W\rangle = -\langle \nu, Z\rangle\langle V\nu, W\rangle = hA[V,W]$.

Finally, we give geometric interpretations of the second fundamental form and the Gauß and mean curvatures. If we set $Z = \nu(p)$ then we get $\mathrm{grad} f = 0$ and $\mathrm{Hess}(f) = v^i w^j \langle X_{ij}, Z\rangle$ at p. If $Z = X_k(p)$ then $\mathrm{grad} f = X_k$ and $\mathrm{Hess}(f)(V,V) = 0$ at p. [11] Thus the surface may be approximated near $p = X(p_1, p_2)$ by

$$\begin{aligned} X(p_1 + x_1, p_2 + x_2) &= X(p) + \sum_i x_i X_i(p) + \frac{1}{2}\sum_{i,j} X_{ij}(p) x_i x_j + \mathbf{o}(x_1^2 + x_2^2) \\ &= X(p) + \sum_i x_i X_i(p) + \frac{1}{2}\sum_{i,j} A[x_i X_i, x_j X_j]\nu(p) + \mathbf{o}(x_1^2 + x_2^2) \end{aligned}$$

as $(x_1, x_2) \to 0$. Thus the second fundamental form gives the second order part of how a surface leaves its tangent plane. If the surface is minimal, $\kappa_1 + \kappa_2 = H \equiv 0$. Then the eigenvalues have opposite sign and the surface viewed as a graph over its tangent plane is indefinite. If the principal curvatures don't vanish at p then the minimal surface lies on both sides of the p-tangent plane T_pX near p.

The meaning of the principal curvatures is now clear. If we consider a vector $V \in T_pX \backslash \{0\}$ and the plane $\mathcal{W}$ containing p spanned by V and $\nu(p)$, then the intersection $\mathcal{W} \cap X(\Omega)$ is a curve $X \circ \gamma(t) \in X(\Omega)$ so that $X \circ \gamma(0) = p$ and $(X \circ \gamma)'(0) = V$. Then the curvature of the curve in the plane $\mathcal{W}$, which is called the normal curvature, is given by [12]

$$\kappa(V) = \frac{\langle (X \circ \gamma)''(0), \nu(p)\rangle}{\langle V, V\rangle} = \frac{A[V,V]}{\langle V, V\rangle}.$$

Hence the principal curvatures, which are the extrema of the Rayleigh Quotient, are nothing more than the extrema of the normal curvatures $\kappa(V)$ as V is swept around the tangent plane.

The geometric interpretation of $K(p)$ is given by the following. Let $p = X(q)$ for some point $q \in \Omega$ and let $X_i = X(B_{r_i}(q))$ where $r_i \to 0$. Also let $\tilde{X}_i$ be the image of X_i under the Gauß map ν. Then

$$K(p) = \pm \lim_{i\to\infty} \frac{\mathrm{Area}(\tilde{X}_i)}{\mathrm{Area}(X_i)},$$

where the plus sign is chosen if ν preserves orientation at p and the minus sign if it reverses the orientation. [13]

2.3 Delaunay surfaces

As an example, let us find all surfaces of revolution with constant mean curvature. The general surface of revolution, if it is not an annulus in a plane, takes the form

$$X(t, \theta) = (h(t)\cos\theta, h(t)\sin\theta, t)$$

for some function $h(t) > 0$. Differentiating yields

$$X_1 = (\dot{h}\cos\theta, \dot{h}\sin\theta, 1), \qquad X_2 = (-h\sin\theta, h\cos\theta, 0),$$

which are orthogonal. Writing $w = (1 + \dot{h}^2)^{-1/2}$,

$$E_1 = w(\dot{h}\cos\theta, \dot{h}\sin\theta, 1), \qquad E_2 = (-\sin\theta, \cos\theta, 0), \qquad E_3 = w(\cos\theta, \sin\theta, -\dot{h}),$$

[11] $\mathrm{Hess}(f)[V,W] = v^p w^p \langle \partial_p(g^{ij}\langle X_i, Z\rangle X_j), X_q\rangle$
$= v^p w^q (g^{ij}\langle X_{ip}, Z\rangle g_{jq} + g^{ij}\langle X_i, Z\rangle\langle X_{jp}, X_q\rangle - g^{is}\langle X_{sq}, X_p\rangle\langle X_i, Z\rangle - g^{is}\langle X_s, X_{qp}\rangle\langle X_i, Z\rangle)$.

[12] $(X \circ \gamma)''(0) = X_{ij}(p) v^i v^j + X_i(p)(v^i)'(0)$.

[13] This is just the change of variables formula under the map ν. First, the area form on $X(\Omega)$ is $g^{1/2}\, dx_1\, dx_2$. The area form on S^2 is $\tilde{g}^{1/2}\, dx_1\, dx_2$ where $\tilde{g} = \det(\tilde{g}_{ij})$ where the metric on $S^2 \subset \mathbb{R}^3$ is induced as $\tilde{g}_{ij} = \langle \nu_i, \nu_j\rangle$. Writing $A_{ij} = A[X_i, X_j]$ we have $\nu_i = -A_{ij} g^{jk} X_k$ so $\tilde{g}^{1/2} = g^{-1/2}\det(A_{ij}) = K g^{1/2}$ whence $d\widetilde{\mathrm{Area}} = K\, d\mathrm{Area}$.

which gives (using the definition of mean curvature in definition 2)

$$-H = -w^3\ddot{h} + \frac{w}{h}.$$

Setting $c = -H$ constant, we can integrate. If $c = 0$ then it is a minimal surface of revolution. Hence

$$h\ddot{h} = 1 + \dot{h}^2.$$

Since this ODE is independent of t we regard $\zeta = \dot{h}$ as a function of h to get

$$\ddot{h} = \frac{d\zeta}{dh}\frac{dh}{dt} = \zeta\frac{d\zeta}{dh} = \frac{1+\zeta^2}{h}.$$

Separating variables and integrating implies,

$$\ln\left(\sqrt{1+\zeta^2}\right) = \ln c_1 + \ln h$$

for some constant $c_1 > 0$, or

$$\frac{dh}{dt} = \zeta = \sqrt{c_1^2 h^2 - 1}.$$

Another integration yields

$$h = \frac{1}{c_1}\cosh\left(c_1 t + c_2\right)$$

for a second constant c_2. Hence this part of the surface is a catenoid, a catenary of revolution.

If $c \neq 0$ the equation becomes

$$0 = ch\dot{h} - \frac{\dot{h}}{\sqrt{1+\dot{h}^2}} + \frac{h\dot{h}\ddot{h}}{\left(1+\dot{h}^2\right)^{3/2}} = \frac{d}{dt}\left(\frac{c}{2}h^2 - \frac{h}{\sqrt{1+\dot{h}^2}}\right),$$

and has a first integral for some constant c_2:

$$\frac{c}{2}h^2 - \frac{h}{\sqrt{1+\dot{h}^2}} = c_2.$$

These surfaces are known as Delaunay surfaces because he discovered that the function h may be given as the roulette of an ellipse. That means if an ellipse whose major axis is $1/|c_1|$ and minor axis is $\sqrt{2|c_2|}$ is rolled without sliding along the t axis in the plane, then the focus of the ellipse traces the curve $(t, h(t))$. (See [Eells].)

2.4 Complex Analysis and Isothermal Coordinates.

(Σ, ds^2) can be thought of as a Riemannian manifold, that is for each local chart there is a symmetric, positive definite $ds^2 = g_{ij}\,dx_1\,dx_2$. It turns out, that by a (local) diffeomorphism, it is possible to choose (x_1, x_2) to be isothermal coordinates, i.e. so that the metric takes the form $ds^2 = e^{2\phi}((dx_1)^2 + (dx_2)^2)$. This enables one to give Σ a local complex structure.

Theorem 1 *Suppose Σ is a surface with boundary, homeomorphic to the unit disc $\bar{D}$ in the plane via the chart $X : \bar{D} \to \Sigma$. Suppose the coefficients g_{ij} defined in this chart are bounded measurable functions with $\det(g_{ij}) \geq c > 0$ in D. Then Σ admits a conformal representation $\tau \in H^{1,2} \cap C^\alpha(\bar{B}, \bar{D})$, where $\bar{B}$ is the unit disc and τ satisfies almost everywhere the conformality relations*

$$|\tau_1|^2 = |\tau_2|^2, \qquad \langle\tau_1, \tau_2\rangle = 0.$$

Here (x_1, x_2) denote the coordinates in $\bar{B}$ and the inner product is given by the metric of Σ so in terms of g_{ij} on $\bar{D}$. Moreover τ can be normalized by the three point condition, namely three prescribed points on the boundary of D can be made to correspond, respectively to three points on the boundary of $\bar{B}$. Furthermore, τ is as regular as M, i.e. if Σ is of class $C^{k,\alpha}$ ($k \in \mathbb{N}, 0 < \alpha < 1$) or C^∞ then $\tau \in C^{k,\alpha}(\bar{B})$ or $C^\infty(\bar{B})$, resp. In particular, if $k \geq 1$ then the conformality relations are satisfied everywhere and τ is a diffeomorphism.

For a proof of this, see Jost [J]. The local version, known as the Korn-Lichtenstein theorem, was proved by Lavrenitiev and Morrey for this generality. Morrey and Jost extended it a global result on multiply connected domains. Their proof is variational.

Recall from section 2.2 that the Laplacian of a function on a surface is given by $\Delta f = \mathrm{div}(\mathrm{grad} f)$. One can check that in isothermal coordinates

$$\Delta f = e^{-2\phi}(f_{11} + f_{22}).$$

Also, in these coordinates the Gauß curvature has the expression

$$K = e^{-2\phi}(\phi_{11} + \phi_{22}).$$

In addition, one can check that

$$\Delta X = H\nu,$$

which is easiest to compute using isothermal coordinates. Therefore, a surface in $\mathbb{R}^3$ is minimal if and only if its coordinate functions are harmonic.

If x_1 and x_2 are isothermal coordinates, one can define a complex coordinate by $z = x_1 + ix_2$. Then the metric $ds^2 = e^{2\phi}|dz|^2$ and the Laplacian $4\Delta f = \partial_z \partial_{\bar{z}} f$, where $df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}$ where $dz = dx_1 + i dx_2$, $d\bar{z} = dx_1 - i dx_2$ and so $2\partial_z = \partial_1 - i\partial_2$ and $2\partial_{\bar{z}} = \partial_1 + i\partial_2$.

3 Alexandrov's Theorem

In this section we present Alexandrov's theorem, which we state now.

Theorem 2 *Let Σ be a compact, connected surface without boundary and $X : \Sigma \to \mathbb{R}^3$ a CMC embedding. The $X(\Sigma)$ is a round sphere.*

In this section we will identify $X(\Sigma)$ with Σ for notational convenience.

The steps in Alexandrov's proof of his theorem are:

1. Show that for all unit vectors $\vec{u}$ there is a plane $\pi \perp \vec{u}$ such that π is a plane of reflection symmetry for Σ.

2. Note that the center of mass

$$\bar{x} = \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma x d\,\mathrm{Area}$$

   is in each plane π (because π is a plane of symmetry). Without loss of generality, set $\bar{x} = 0$.

3. Thus each plane π through 0 is a plane of reflection symmetry for Σ.

4. Since any rotation about 0 is a composition of an even number of reflections, Σ is invariant under all rotations about 0. Therefore Σ is a round sphere.

The first step in Alexandrov's proof is the key one, and the way in which he completed it has come to be known as the method of moving planes. The idea behind this method is to move

a plane perpendicular to $\vec{u}$ in its normal direction and compare Σ to its reflection through the plane using the maximum principle. For any $t \in \mathbb{R}$ let

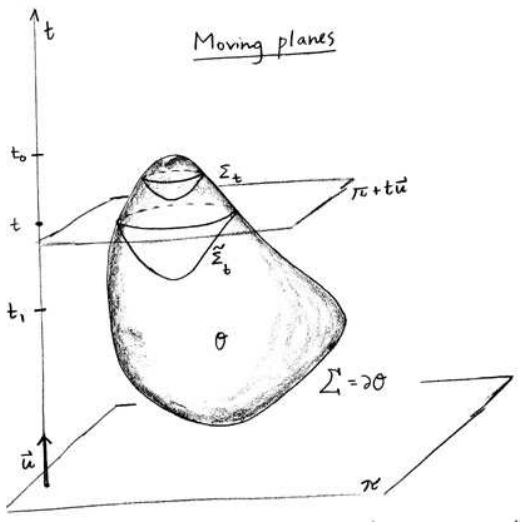
$$\pi_t = \{\langle x, \vec{u}\rangle = t\} \subset \mathbb{R}^3, \qquad \pi = \pi_0, \qquad \Pi_t = \{\langle x, \vec{u}\rangle \geq t\} \subset \mathbb{R}^3.$$

Also, given $t \in \mathbb{R}$ and any subset $G \subset \mathbb{R}^3$ let

$$G_t = G \cap \Pi_t \qquad \tilde{G}_t = \{p + (t-r)\vec{u} \mid p \in \pi, p + (t+r)\vec{u} \in G_t\}.$$

Notice G_t is the part of G above (or in) π_t and $\tilde{G}_t$ is its reflection through π_t.

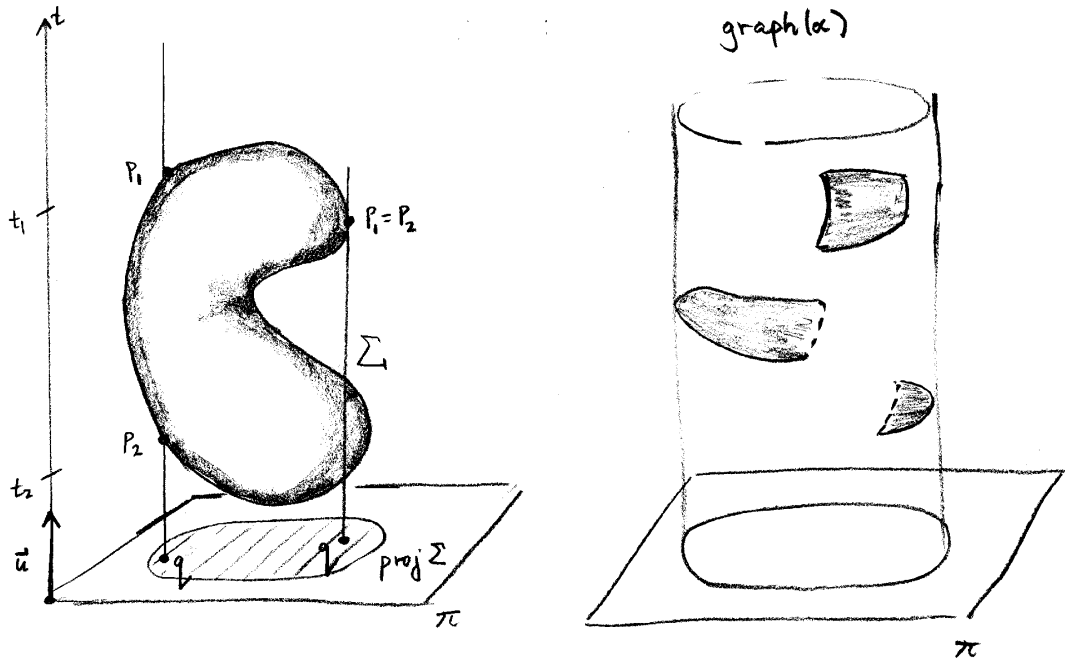
Because Σ is compact and embedded we may write $\Sigma = \partial\mathcal{O}$ for a bounded region $\mathcal{O}$ in $\mathbb{R}^3$. For t sufficiently large both Σ_t and $\tilde{\Sigma}_t$ will be empty. Start with such a value of t and decrease it until the first $t = t_0$ at which $\pi_t \cap \Sigma$ is nonempty. As t initially decreases from t_0, $\tilde{\Sigma}_t$ will be a subset of $\bar{\mathcal{O}}$, and both $\tilde{\Sigma}_t$ and Σ_t will be graphs in the $\vec{u}$ direction. As t continues to decrease there must occur a critical time (labeled $t = t_1$ in the "Moving planes" figure, although t_1 has a different meaning later and the critical height is denoted by t_3) when $\tilde{\Sigma}_t$ has first "tangential" contact with Σ, i.e. at $t = t_1$ we still have that $\tilde{\Sigma}_t$ lies in $\bar{\mathcal{O}}$ and additionally there exist contact points P on $\tilde{\Sigma}_t \cap \Sigma_t$ at which the tangent planes are equal, $T_P\Sigma_t = T_P\tilde{\Sigma}_t$.



To make the concept of first touching precise, we will define the Alexandrov function for points $q \in \text{proj}_\pi \Sigma$. Let t_1 be the supremum of t such that $q + t\vec{u} \in \bar{\mathcal{O}}$. Then $P_1 = q + t_1\vec{u}$ is the point of highest contact of the line $\{q + t\vec{u}\}$ with Σ. If this first contact is transverse, let t_2 be the supremum of $t < t_1$ such that $q + t\vec{u} \notin \mathcal{O}$. Then $P_2 = q + t_2\vec{u}$ is the point where the line $\{q + t\vec{u}\}$ first leaves $\mathcal{O}$ as t decreases from t_1. In case the contact between line and surface at P_1 is not transverse, let $P_2 = P_1$. Then (as in [KKS]) we define

$$\alpha(q) = \frac{t_1 + t_2}{2}$$

for points $q \in \text{proj}_\pi \Sigma$. Note that $\alpha(q)$ is the t-value for which P_1 reflects to P_2. It is not hard to show that α is upper semicontinuous, by considering the cases of transverse and non-transverse intersection separately. Thus α attains its maximum on the compact set $\text{proj}_\pi \Sigma$. In the figures below we show a particular surface Σ with a chosen direction $\vec{u}$ and the graph of its associated Alexandrov function α.



The maximal value t_3 of α is the value of t where $\tilde{\Sigma}_t$ has first tangential contact with Σ. Notice that the first touching points may be interior or boundary points of $\tilde{\Sigma}_t$. In the boundary point case the tangent plane $T_P\Sigma = T_P\tilde{\Sigma}_{t_3}$ by construction. In the interior point case the tangent planes must coincide because the contact between the two surfaces is one-sided.

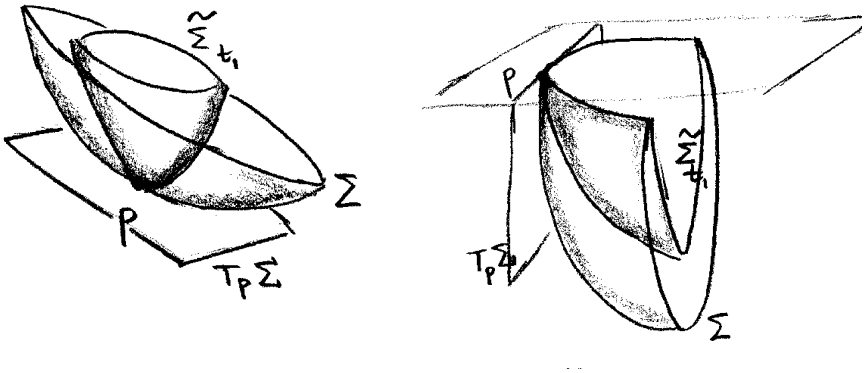
Near such a first touching point P we can write $\tilde{\Sigma}_{t_3}$ as the graph of u and Σ as the graph of v over their common tangent plane, in the direction of the inner normal to $\mathcal{O}$. Notice that by construction we have

$$u \geq v$$

in the common domain of definition Ω. Also if P has coordinates p in the plane $T_P\Sigma$, then

$$u(p) = v(p), \qquad Du(p) = Dv(p).$$

Below we have drawn the two possibilities.



Moreover, u and v satisfy the same nonlinear PDE, equation (2). Then we apply the strong maximum principle (SMP) below to conclude that

$$\{P \in \Sigma \mid \text{the reflection of } P \text{ through } \pi_{t_3} \text{ is also in } \Sigma\}$$

is an open set. Because fixed point sets of reflections are also closed, we deduce that all of Σ is preserved by reflection through π_{t_3}.

Here is a version of the strong maximum principle which we used above, along with the auxiliary weak maximum principle which we will need to prove it.

Theorem 3 (Strong Maximum Principle) *Let Ω be a smooth (connected) domain, and suppose $u, v : \Omega \to \mathbb{R}$ satisfy*

1. *$\Phi(x, u, Du, D^2u) = \Phi(x, v, Dv, D^2v)$, where Φ is a uniformly elliptic operator and*
2. *$u \geq v$ in $\bar{\Omega}$.*

If there exists $p \in \bar{\Omega}$ such that

$$u(p) = v(p), \qquad Du(p) = Dv(p)$$

then $u \equiv v$.

Theorem 4 (Weak Maximum Principle) *Suppose the linear operator L defined by*

$$L(u) = \sum_{ij} a_{ij} u_{ij} + \sum_k b_k u_k$$

is uniformly elliptic. Let $\bar{\Omega}$ be compact, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and suppose

- *$u < v$ implies $L(u) < L(v)$ in Ω (really, $L(u) \leq L(v)$ suffices) and*
- *$u \geq v$ on $\partial\Omega$.*

Then $u \geq v$ in $\bar{\Omega}$.

Proof of Theorem 4: Let $w = u - v$ and suppose w has a negative minimum at $p \in \Omega$. Then since p is an interior point,

$$Dw(p) = 0, \qquad D^2w(e,e) := \left.\frac{d^2}{dt^2}\right|_{t=0} w(p + t\vec{e}) \geq 0 \quad \forall \vec{e}.$$

But the second item is $\sum_{ij} w_{ij} e_i e_j$, and so at p (letting Q be the orthogonal matrix with diagonalizes $[A]$, with columns $\vec{e_i}$)

$$\sum_{ij} a_{ij} w_{ij} = \mathrm{tr}[A][D^2w] = \mathrm{tr}[Q^t A Q Q^t D^2 w Q]$$

$$= \mathrm{tr} \begin{bmatrix} \lambda_1 & & 0 \\ & \cdots & \\ 0 & & \lambda_n \end{bmatrix} [Q^t D^2 w Q] = \sum \lambda_i D^2 w(\vec{e_i}, \vec{e_i}) \geq 0.$$

Therefore $L(w)(p) \geq 0$, which implies $L(u) \geq L(v)$. This contradicts our hypotheses and proves the weak maximum principle. $\square$.

Before proving the strong maximum principle we will present a quick but illustrative digression in which we apply the weak maximum principle to understand some geometry for constant mean curvature graphs. The first place a student usually sees the weak maximum principle is in the construction of barriers for solutions to semilinear equations, for example in studying the boundary behavior of Perron solutions to (the usual) Laplace's equation. So it shouldn't be surprising that the weak maximum principle has useful applications in other geometric situations.

Let Σ be a graph over a plane π with constant mean curvature H such that $\partial\Sigma \subset \pi$ and Σ lies above π (i.e. Σ is the graph of a non-negative function having zero boundary values). Orient

Σ with the downward normal ν to Σ (this choice makes H positive). For example, Σ could be an upper hemisphere with equator in π and radius equal to $2/H$.

Recall the intrinsic Laplacian defined in section 2.4. Then

$$\Delta\vec{x} = H\vec{\nu}, \qquad \Delta\vec{\nu} = -|A|^2\vec{\nu}.$$

Let $u \equiv 0$ and $v = Hx_3 + 2\nu_3$. Then by hypothesis and since $(\kappa_1 + \kappa_2)^2 \leq 2(\kappa_1^2 + \kappa_2^2)$,

$$u \geq v \text{ on } \partial\Sigma, \qquad \Delta v = (H^2 - 2|A|^2)\nu_3 \geq 0 = \Delta u \text{ on } \Sigma.$$

Hence by the weak maximum principle $u \geq v$. We can rewrite this to read

$$Hx_3 + 2\nu_3 \leq 0, \text{ or } x_3 \leq \frac{-2\nu_3}{H} \leq \frac{2}{H},$$

which is an a priori height estimate. In fact, it is the interesting fact that the north pole of the radius $2/H$ upper hemisphere maximizes the height among all compact constant mean curvature graphs with boundary in π. We can rearrange the inequality $u \geq v$ to read

$$-\nu_3 \geq \frac{H}{2}x_3,$$

with is a gradient estimate. In fact, it says the steepness of the upper hemisphere at a given height is an upper bound for the steepness of any of our comparison mean curvature H graphs, at corresponding heights. Height and gradient estimates such as these have many interesting applications beyond their intrinsic beauty.

We return to the task of proving the the strong maximum principle using the weak maximum principle and a barrier function. For simplicity of exposition we will restrict to a uniformly elliptic differential operator of the form $\Phi = \Phi(Du, D^2u) = \Phi(p, r)$. We assume $u \geq v$ on $\bar{\Omega}$, $u(p) = v(p)$ and $Du(p) = Dv(p)$ for some $p \in \bar{\Omega}$. We have three cases to consider:

- $u = v$ on Ω, in which case we are done,
- $u > v$ on the interior of Ω so $p \in \partial\Omega$ or
- $u > v$ on a strict subdomain of Ω, so there are interior points p with $u(p) = v(p)$ and (since $u \geq v$ in Ω) $Du(p) = Dv(p)$.

We wish to eliminate the last two or our three possibilities. In case two, we use the smoothness of Ω to pick a closed ball $\bar{B_R} \subset \bar{\Omega}$ which contacts $\bar{\Omega}$ only at p. Then $u > v$ on the open ball, so pick any $0 < r < R$ and define the annulus $\mathcal{A} = B_R \backslash \bar{B}_r \subset \Omega$, where B_r is concentric to B_R. We will denote the boundary curves of $\mathcal{A}$ as $\Gamma_0 = \partial B_r$ and $\Gamma_1 = \partial B_R$. Then we have $u \geq v$ on $\bar{\mathcal{A}}$, $u = v$ at $p \in \Gamma_1$, and $u - v \geq \epsilon$ on Γ_0 for some $\epsilon > 0$.

In case three we pick an annulus with identical properties by judicious choice of p: Pick $q \in \Omega$ so that the distance R from q to $\{u = v\}$ is less than the distance from q to $\partial\Omega$. Let p be a nearest point to q in the set $\{u = v\}$, Let B_R be the ball centered at q. Then proceed as in case two to define the annulus $\mathcal{A}$.

Now let $w = u - v$ and

$$\chi(t) = \Phi(tDu + (1-t)Dv, tD^2u + (1-t)D^2v)$$

then

$$0 = \Phi(Du, D^2u) - \Phi(Dv, D^2v) = \chi(1) - \chi(0) = \chi'(c) = \sum_{ij} a_{ij}w_{ij} + \sum_k b_k w_k := L(w).$$

Now L is a uniformly elliptic linear operator(i.e. $\lambda|\xi|^2 \leq \sum_{ij} a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ for some $0 < \lambda < \Lambda$) (this is an hypothesis on Φ), and we have

$$w|_{\bar{\mathcal{A}}} \geq 0, \qquad w|_{\Gamma_0} \geq \epsilon, \qquad w(p) = 0.$$

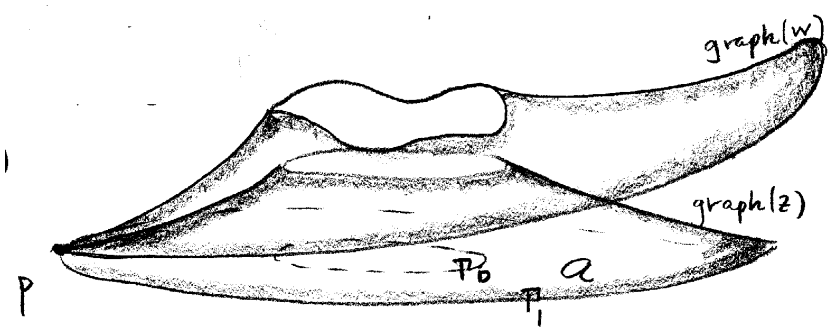
To apply the weak maximum principle, we only need to find a barrier function z such that

$$z|_{\Gamma_0} = \epsilon, \qquad z|_{\Gamma_1} = 0, \qquad L(z) > 0.$$

We can then deduce by the weak maximum principle, applied to the annulus, that $w \geq z$. If in addition our barrier z satisfies $\frac{\partial z}{\partial \rho}(p) < 0$ for radial direction ρ on the annulus $\mathcal{A}$, then also

$$\frac{\partial w}{\partial \rho}(p) \leq \frac{\partial z}{\partial \rho}(p) < 0.$$

Note that $Dw(p) \neq 0$ contradicts the working hypothesis in the latter two cases we were considering, namely that $Du(p) = Dv(p)$. Thus only the first case was possible, and we have proved the strong maximum principle. We also proved the Hopf boundary point lemma, the fact if $u > v$ in Ω then at boundary points where $u = v$, we cannot have $Du = Dv$. Below we have drawn the graphs of w and the barrier function z, although we have yet to prove the existence of z.



For the barrier function, we define the center of $\mathcal{A}$ to be the origin, and take a radial function $z(x) = f(|x|^2/2)$. Then

$$z_i = f'x_i, \qquad z_{ij} = f''x_ix_j + f'\delta_{ij},$$

and so

$$Lz = f''\sum_{ij} a_{ij}x_ix_j + f'\sum_i a_{ii} + f'\sum_k b_k x_k \geq f''(\lambda r^2) - (n\Lambda + C)|f'| > 0$$

provided $f'' >> |f'|$. For example,

$$f(s) = \frac{\epsilon(e^{-Ms} - e^{-MR^2/2})}{e^{-Mr^2/2} - e^{-MR^2/2}}$$

works for large M. Notice that $z(x) = f(|x|^2/2)$ satisfies all the barrier conditions, as well as $\frac{\partial z}{\partial \rho}(p) < 0$. □.

We will briefly mention some other applications of the method of moving planes. In the compact case, one can apply the same result to show that closed hypersurfaces in $\mathbb{R}^n$ with any constant "elliptic" curvature must be a sphere (e.g. Gauss curvature or the intermediate curvatures for higher dimensional hypersurfaces). Gidas, Ni and Nirenberg [GNN] showed that positive solutions u to certain elliptic partial differential equations in balls, with zero boundary

data, must be rotationally symmetric. Here one uses vertical planes only and considers the compact region $\bar{\mathcal{O}}$ bounded between the graph of u and the ball. One deduces that the graph of u is rotationally symmetric.

In the noncompact setting, there have been several applications of Alexandrov's technique. Schoen [S] proved that if Σ is a properly embedded minimal surface with finite total curvature ($-\int_\Sigma K d\text{Area} < \infty$) and two ends then Σ is a catenoid. There are several results for properly embedded CMC (but not minimal) surfaces. Meeks [Me] showed that there cannot be any one-ended example. Korevaar, Kusner and Solomon [KKS] showed that if Σ has two ends and finite genus then it must be one of the Delaunay examples. They also showed that if Σ has finite topology then each end of Σ must be asymptotic to some Delaunay surface. Caffarelli, Gidas and Spruck [CGS] showed that any positive solutions to the PDE

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$$

on the whole of $\mathbb{R}^n$ must be radially symmetric with respect to some point in $\mathbb{R}^n$. They also showed that positive weak solutions to this equation with a point singularity at 0 are radially symmetric. In this case one must replace reflection through planes with reflection through spheres (i.e. the Kelvin transform). This particular semilinear equation has geometric meaning having to do with the intrinsic notion of constant scalar curvature. In fact the positive solutions on all of $\mathbb{R}^n$ correspond to the spherical metric, and the positive weak solutions with point singularities at the origin (and infinity) are analogs to Delaunay surfaces. Solutions to semilinear elliptic P.D.E.'s of this form behave in ways which are very similar to the geometry of analogous surfaces of prescribed mean curvature, despite the different elliptic operators which govern the two situations. A persistent mystery is why the analogy is as strong as it is.

4 Bernstein's Theorem

Bernstein's theorem states that any minimal graph over the entire plane must be a flat plane. In this section we present two different proof of Bernstein's theorem. The first proof we present is originally due to Leon Simon [Si], and has a more variational flavor; it relies on an interior estimate for $\int_\Sigma |A|^2 d\text{Area}_\Sigma$. The second proof is more classical; after some manipulations it follows from Liouville's theorem.

4.1 Simon's proof

Along the way to proving Bernstein's theorem, we will derive some interesting bounds for minimal graphs. Throughout this section, we will let $u : \Omega \to \mathbb{R}$ be a solution to the minimal surface equation and we will let $\Sigma_u \subset \mathbb{R}^3$ be the graph of u.

The two important ingredients in this proof are

- an a priori bound on the area growth of a minimal graph and
- a relationship between $|A|$ and K for a minimal surface.

Before we proceed, we will first recall some of the facts mentioned previously. First recall that Σ_u is minimal if and only if

$$u_{11}(1+u_2^2) + u_{22}(1+u_1^2) - 2u_1u_2u_{12} = 0.$$

Also, the Gauß curvature is given by

$$K = \frac{u_{11}u_{22} - u_{12}^2}{(1+|Du|^2)^2}.$$

Thus we have (using equation (4) and the fact that $H = 0$)

$$|A|^2 d\,\text{Area} = -2K d\,\text{Area} = \frac{2u_{12}^2 - 2u_{11}u_{22}}{(1+|Du|^2)^{\frac{3}{2}}} dx_1 \wedge dx_2.$$

However, one can check that if

$$\beta = \arctan(u_1) d\left(\frac{u_2}{\sqrt{1+u_1^2+u_2^2}}\right) = \arctan(u_1) \frac{(u_{12} + u_1^2 u_{12} - u_1 u_2 u_{11})dx_1 + (u_{22} + u_1^2 u_{11} - u_1 u_2 u_{12})dx_2}{(1+u_1^2+u_2^2)^{3/2}}$$

then [14]

$$d\beta = \frac{u_{11}u_{22} - u_{12}^2}{(1+u_1^2+u_2^2)^{3/2}} dx_1 \wedge dx_2.$$

Also (using the minimal surface equation),

$$|\beta|^2 = g^{ij}\beta_i\beta_j = -|\arctan(u_1)|^2 \frac{1+u_1^2}{1+u_1^2+u_2^2} K \leq \frac{\pi^2}{8}|A|^2 < 4|A|^2,$$

or $|\beta| \leq 2|A|$.

Below we will denote extrinsic balls in $\mathbb{R}^3$ of radius R as $\mathbb{B}_R$, so as not to confuse them with intrinsic balls on the surface.

Lemma 5 *For all $\eta \in C^1(\Omega)$ with compact support*

$$\int_{\Sigma_u} \eta^2 |A|^2 \leq 64 \int_{\Sigma_u} |d\eta|^2.$$

Proof: First observe that

$$\eta^2|A|^2 d\,\text{Area} = -2\eta^2 K d\,\text{Area} = -2\eta^2 d\beta = -2d(\eta^2 d\beta) + 4\eta d\eta \wedge \beta.$$

The integral of the first term in the last expression will be zero by Stokes' theorem. Choose an $R > 0$ large enough so that $\operatorname{supp}\eta \subset \mathbb{B}_R$. Then

$$\begin{aligned}\int_{\Sigma_u} \eta^2|A|^2 d\,\text{Area} &= \int_{\Sigma_u \cap \mathbb{B}_R} \eta^2|A|^2 d\,\text{Area} = \int_{\Sigma_u \cap \mathbb{B}_R} -2\eta^2 d\beta = \int_{\Sigma_u \cap \mathbb{B}_R} -2d(\eta^2\beta) + 4\eta d\eta \wedge \beta \\ &\leq 4\int_{\Sigma_u \cap \mathbb{B}_R} |\eta||\beta||d\eta| \leq 8\int_{\Sigma_u \cap \mathbb{B}_R} |\eta||A||d\eta| \leq 8\left(\int_{\Sigma_u \cap \mathbb{B}_R} \eta^2|A|^2\right)^{1/2}\left(\int_{\Sigma_u \cap \mathbb{B}_R} |d\eta|^2\right)^{1/2}.\end{aligned}$$

The desired inequality follows by squaring both sides of the inequality

$$\left(\int_{\Sigma_u} \eta^2|A|^2 d\,\text{Area}\right)^{1/2} \leq 8\left(\int_{\Sigma_u} |d\eta|^2 d\,\text{Area}\right)^{1/2}.$$

□

Remark 1 *The same proof holds when η is piecewise C^1, or even just Lipschitz.*

Next we will need the important fact that the area of a minimal graph grows at most quadratically. To make this precise, note $\Sigma_u \cap \mathbb{B}_R$ divides $\partial\mathbb{B}_R$ into several components, which we can group into two categories: the part above the graph and the part below it. The union of the surfaces in one of these groups (either above or below the graph) must have area at most $2\pi R^2$ (the area of a hemisphere of radius R). Using the fact that $\Sigma_u \cap \mathbb{B}_R$ is area minimizing (see the appendix) we have

$$\text{Area}(\Sigma_u \cap \mathbb{B}_R) \leq 2\pi R^2.$$

[14] How did we find β? The short answer is we used moving frames; a nice introduction to the method of moving frames is in [Sp], Vol. II, chapter 7.

Corollary 6 *If Ω contains a disc of radius R (for $R > 2$) centered at the origin, then there exists a constant c_1 independent of R such that*

$$\int_{\mathbb{B}_{\sqrt{R}} \cap \Sigma_u} |A|^2 \leq \frac{c_1}{\log(\sqrt{R})}.$$

Proof: We define the cutoff function $\eta : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$\eta(x) = \begin{cases} 1 & |x| \leq \sqrt{R} \\ \frac{\log(R/|x|)}{\log \sqrt{R}} & \sqrt{R} \leq |x| \leq R \\ 0 & |x| \geq R. \end{cases}$$

Notice that

$$|D\eta|_{\Sigma_u}(x) \leq |D\eta|_{\mathbb{R}^3} = \begin{cases} 0 & |x| > R, |x| > \sqrt{R} \\ \frac{1}{|x| \log(\sqrt{R})} & \sqrt{R} < |x| < R. \end{cases}$$

Then by the above lemma,

$$\begin{aligned} \int_{\mathbb{B}_{\sqrt{R}} \cap \Sigma_u} |A|^2 &\leq \int_{\Sigma_u} \eta^2 |A|^2 \leq 64 \int_{\Sigma_u} |d\eta|^2 \leq \frac{64}{(\log(\sqrt{R}))^2} \int_{\Sigma_u \cap (\mathbb{B}_R \setminus \mathbb{B}_{\sqrt{R}})} |x|^{-2} \\ &= \frac{64}{(\log(\sqrt{R}))^2} \left[R^{-2} \operatorname{Area}(\Sigma_u \cap (\mathbb{B}_R \setminus \mathbb{B}_{\sqrt{R}})) + \int_{\Sigma_u \cap (\mathbb{B}_R \setminus \mathbb{B}_{\sqrt{R}})} (|x|^{-2} - R^{-2}) \right] \\ &\leq \frac{64}{(\log(\sqrt{R}))^2} \left[2\pi + 2 \int_{\sqrt{R}}^{R} \rho^{-3} \operatorname{Area}(\Sigma_u \cap \mathbb{B}_\rho) d\rho \right] \\ &\leq \frac{64}{(\log(\sqrt{R}))^2} \left[2\pi + 4\pi \int_{\sqrt{R}}^{R} \rho^{-1} d\rho \right] = \frac{64}{(\log(\sqrt{R}))^2} \left[2\pi + 4\pi \log R - 4\pi \log(\sqrt{R}) \right] \\ &= \frac{64}{(\log(\sqrt{R}))^2} \left[2\pi + 4\pi \log(\sqrt{R}) \right] \leq \frac{64(4\pi + 1)}{\log(\sqrt{R})}. \end{aligned}$$

Here we have used the identity $2 \int_{|x|}^{R} \rho^{-3} d\rho = |x|^{-2} - R^{-2}$ and Fubini's theorem in the fifth inequality. □

The type of cutoff function we defined above is often referred to as a "logarithmic cutoff." It can be quite useful in obtaining bounds for surfaces with (at most) quadratic area growth.

Theorem 7 (Bernstein's Theorem) *If $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation then $u(x, y) = ax + by + c$ for some constants a, b, c.*

Proof: By the above corollary, for $R > 2$ we have

$$\int_{\mathbb{B}_{\sqrt{R}} \cap \Sigma_u} |A|^2 \leq \frac{c_1}{\log(\sqrt{R})}$$

for all $R > 1$ and some constant $c_1 > 0$. Letting $R \to \infty$, we see that $|A|^2 \equiv 0$, and so $D^2 u \equiv 0$. The theorem follows. □

4.2 Another proof

For this proof of Bernstein's theorem we will need to know the following fact.

Theorem 8 (Jörgen's Theorem) *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function satisfying*

$$\det D^2 f = 1$$

then f is a quadratic polynomial.

We will not prove Jörgen's theorem. However, the idea behind the proof is the following. Define the function

$$\Phi = \frac{f_{22} - f_{11} + 2if_{12}}{2 + f_{11} + f_{22}} : \mathbb{C} \to \mathbb{C}.$$

One can show that

- Φ is analytic (using the equation f satisfies),
- Φ is bounded and
- one can solve for f_{11}, f_{22} and f_{12} in terms of Φ.

Then, by Liouville's theorem Φ must be constant, which in turn implies $D^2 f$ must be constant as well. One can find a complete proof in [Sp], vol. 4, chapter 7.

Now we are ready to prove Bernstein's theorem (again). Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a global solution to the minimal surface equation. Because

$$\partial_{x_1}\left(\frac{u_1}{\sqrt{1+u_1^2+u_2^2}}\right) + \partial_{x_2}\left(\frac{u_2}{\sqrt{1+u_1^2+u_2^2}}\right),$$

we can find functions $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\alpha_1 = \frac{1+u_1^2}{\sqrt{1+|Du|^2}} \qquad \alpha_2 = \frac{u_1 u_2}{\sqrt{1+|Du|^2}} \qquad \beta_1 = \frac{u_1 u_2}{\sqrt{1+|Du|^2}} \qquad \beta_2 = \frac{1+u_2^2}{\sqrt{1+|Du|^2}}.$$

Here is the only place we are using the fact that u solves the minimal surface equation in this proof. Because $\alpha_2 = \beta_1$ we can find a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\Phi_1 = \alpha \qquad \Phi_2 = \beta.$$

But then

$$\det D^2\Phi = \Phi_{11}\Phi_{22} - \Phi_{12}^2 = \alpha_1\beta_2 - \alpha_2^2 = \frac{(1+u_1^2)(1+u_2^2) - u_1^2u_2^2}{1+|Du|^2} = 1,$$

and thus Φ is a quadratic polynomial. Therefore

$$\frac{1+u_1^2}{\sqrt{1+|Du|^2}} \qquad \frac{1+u_2^2}{\sqrt{1+|Du|^2}} \qquad \frac{u_1u_2}{\sqrt{1+|Du|^2}}$$

are all constants, from which it follows that u_1 and u_2 are constant.

4.3 Some generalizations

In this subsection we present some generalizations of Bernstein's theorem and the interior curvature estimates of the previous sections. For most of this discussion we will consider an embedded surface Σ in a 3-manifold M, where M possibly has some curvature restrictions. However, this discussion will all be valid for minimal surfaces in $\mathbb{R}^3$, and the reader may wish to only read this section thinking of $\mathbb{R}^3$ as the ambient space.

First we need to present the idea of stability for a minimal surface. By definition, an embedded surface $\Sigma \subset M$ is minimal if and only if

$$H_\Sigma = 0.$$

We mentioned the local formulation of mean curvature of a surface in $\mathbb{R}^3$ back in section 2.2; we will not explicitly write down the formulation of mean curvature for a more general ambient

space, but it is similar. Recall that we arrived at the notion of mean curvature by seeking critical points of the area functional. More precisely, let $\Sigma \subset M$ be an embedded surface and let Σ_t be a one-parameter family of surfaces which is a normal variation of Σ. In other words, we can write

$$\Sigma_t = \exp_{tu\nu_\Sigma}(\Sigma)$$

where $u : \Sigma \to \mathbb{R}$ is a compactly supported function and exp is the exponential map of M. In local coordinates, this means we can write Σ_t as the normal graph of tu over Σ. Then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma u H_\Sigma d\,\mathrm{Area}_\Sigma\,.$$

In this sense, H_Σ is the first derivative of the area functional and minimal surfaces are the critical points of the area functional.

To find area minimizers we must examine the second variation of the area functional in M. One can think of the second variation as follows: again, take Σ to be an embedded surface in M and $u : \Sigma \to \mathbb{R}$ to be a compactly supported function. Let $\Sigma_t = \exp_{tu\nu_\Sigma}(\Sigma)$ be the normal variation of Σ associated to u. Then the second variation of the area functional, evaluated on u is

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(\Sigma_t) = \left.\frac{d}{dt}\right|_{t=0} \int_\Sigma u H_{\Sigma_t} d\,\mathrm{Area}_\Sigma\,.$$

Under the above conditions (because u is compactly supported in Σ) one can differentiate underneath the integral sign and just compute

$$\left.\frac{d}{dt}\right|_{t=0} H_{\Sigma_t}\,.$$

Definition 3 *The stability operator L_Σ of an embedded surface is given by*

$$L_\Sigma(u) = \left.\frac{d}{dt}\right|_{t=0} H_{\Sigma_t} = \Delta_\Sigma u + |A|^2 u + \mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) u.$$

A minimal surface Σ is called stable if L_Σ is a positive definite operator.

The stability operator is also often called the Jacobi operator, and functions such that $L_\Sigma u = 0$ are often called Jacobi fields.

(Digression: the term $\mathrm{Ric}(\nu_\Sigma, \nu_\Sigma)$ in the above formula is the Ricci curvature of M evaluated on the unit normal to Σ. One should think of Ricci curvature of a Riemannian manifold as measuring how the volume growth of a thin wedge about a given solid angle differs from that of Euclidean space. The scalar curvature R is the average of the Ricci curvature over all directions; it determines how the volume growth of small balls differs from that of Euclidean space. For instance, $\mathbb{R}^n$ with its usual metric has Ricci curvature 0 everywhere, S^n with its usual metric has Ricci curvature n everywhere and n-dimensional hyperbolic space with its usual metric has Ricci curvature $-n$ everywhere. Some very nice introductions to curvature are [Sp] and [Lee].)

One should think of stable minimal surfaces as surfaces which minimize area among all other nearby surfaces. The stability condition is equivalent to asking that the second variation of the area functional (evaluated at the surface in question) is positive definite, which is precisely the condition we need for a surface to minimize area among all nearby competitors. Indeed, *most minimal surfaces are not area minimizing*, because their stability operators have zero (or negative numbers) is their spectra.

Recall that the bottom of the (L^2) spectrum of an operator of the form $\Delta - q$ on a domain Ω is given by

$$\lambda_1(\Omega) = \inf\left\{ \int_\Omega (|Du|^2 + qu^2) d\,\mathrm{Area}_\Omega \;\mid\; \mathrm{supp}(u) \subset \Omega, \int_\Omega u^2 d\,\mathrm{Area}_\Omega = 1 \right\}.$$

In 1980 Fischer-Colbrie and Schoen [FCS] proved the following theorem.

Theorem 9 *When considering the operator $\Delta - q$, the following conditions are equivalent:*

- $\lambda_1(\Omega) \geq 0$ *for every bounded domain* $\Omega \subset \Sigma$
- $\lambda_1(\Omega) > 0$ *for every bounded domain* $\Omega \subset \Sigma$
- *There is a positive function* u *satisfying* $\Delta_\Sigma u - qu = 0$

Note that any graph in $\mathbb{R}^3$ satisfies the last property for the operator $L_\Sigma = \Delta_\Sigma + |A|^2$. Indeed, the Jacobi field one obtains from vertically translating a graph is a positive Jacobi field.

Using this theorem, Fischer-Colbrie and Schoen then proved the following theorem, which has Bernstein's theorem as a corollary.

Theorem 10 *Let M be a complete oriented 3-manifold with non-negative scalar curvature and let $\Sigma \subset M$ be an oriented, complete, stable, minimal surface.*

- *If Σ is compact, then Σ is conformally equivalent to the standard S^2 or a totally geodesic flat torus. If the scalar curvature of M is strictly positive then Σ must be a sphere.*
- *If Σ is not compact then Σ is conformally equivalent to $\mathbb{C}$ or the cylinder. If Σ is a cylinder with finite total curvature then it is flat and totally geodesic. If the scalar curvature of M is strictly positive then Σ cannot be a cylinder with finite total curvature.*

Another generalization is the curvature estimate of Choi and Schoen [CS]. In 1985 they proved the following theorem.

Theorem 11 (Choi and Schoen) *Let M be an n-dimensional Riemannian manifold with positive Ricci curvature. There exist $\epsilon = \epsilon(M) > 0$ and $\rho = \rho(M) > 0$ such that if $r_0 < \rho$ and $\Sigma \subset M$ is a compact minimal surface with $\partial\Sigma \subset \partial B_{r_0}(x)$, $0 < \delta < 1$ and*

$$\int_{B_{r_0}(x)\cap\Sigma} |A|^2 < \delta\epsilon$$

then for $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x)$

$$\sigma^2|A|^2(y) \leq \delta.$$

One should think of this theorem as saying that if the Gauß curvature of a minimal surface is small on average then it has to be small uniformly. In particular, the Gauß curvature of a minimal surface cannot have small L^2-norm on a disc and yet collect to be very large at some points on that disc.

5 Douglas' Solution to the Plateau Problem

Appendices

A Minimal Graphs are Area Minimizing

In this section we present the result that minimal graphs are actually area minimizing in quite a strong way. The easiest proof of this fact involves the idea of calibrations, which we will present in some generality.

Definition 4 *Let M be an oriented n-dimensional Riemannian manifold. A calibration on M is a closed k-form ξ such that*

$$|\xi_p(e_1, \ldots, e_k)| \leq 1$$

for all orthonormal sets of tangent vectors $e_1, \ldots, e_k$ in T_pM, for all points $p \in M$. In addition, a submanifold $\Sigma \subset M$ is calibrated by ξ if

$$\xi|_{T\Sigma} = d\,\mathrm{Vol}_\Sigma\,.$$

Notice that an equivalent way to say that ξ is a calibration is to say that $|\xi_p(V)| \leq d\,\mathrm{Vol}_V$ for any oriented k-plane V in T_pM. Also notice that if ξ is nonvanishing then calibrated submanifolds are automatically oriented.

The first example of a calibration is the form $\xi = dx_1 \wedge \cdots \wedge dx_k$ on $\mathbb{R}^n$. Then calibrated submanifolds are horizontal k-planes (with the correct orientation).

The key fact about compact calibrated submanifolds (with or without boundary) is that they are area minimizing in their homology class. Indeed, let Σ be a compact calibrated submanifold and $\tilde{\Sigma}$ be another compact submanifold in the same homology class (so in particular $\partial\Sigma = \partial\tilde{\Sigma}$). Then

$$\mathrm{Vol}(\Sigma) = \int_\Sigma d\,\mathrm{Vol}_\Sigma = \int_\Sigma \xi = \int_{\tilde{\Sigma}} \xi \leq \int_{\tilde{\Sigma}} d\,\mathrm{Vol}_{\tilde{\Sigma}} = \mathrm{Vol}(\tilde{\Sigma}).$$

The second equality uses the fact that Σ is calibrated by ξ, the third uses the fact that Σ and $\tilde{\Sigma}$ are homologous and that ξ is closed and the fourth uses the fact that ξ is a calibration.

Let $u : \Omega \to \mathbb{R}$ be a solution to the minimal surface equation on some domain $\Omega \subset \mathbb{R}^2$. To show that Σ_u, the graph of u, is area minimizing among all homologous surface with the same boundary one only need to check that

$$\omega_u = (1 + |Du|^2)^{-\frac{1}{2}}(dx_1 \wedge dx_2 - u_1 dx_2 \wedge dx_3 - u_2 dx_3 \wedge dx_1)$$

is a calibration with Σ_u as a calibrated submanifold. Checking that Σ_u is calibrated by ω_u is much easier once one observes that

$$\omega_u(U, V) = \det(U, V, \nu_{\Sigma_u})$$

for any pair of vectors U, V in $\mathbb{R}^3$.

B Hopf's Theorem

Are all immersed closed C^3 surfaces of constant mean curvature in Euclidean space round spheres? All the evidence was positive for a long time. In 1900, Liebman showed that it is true for convex surfaces. The minimizing solution of the isoperimetric variational problem was known to be the round sphere to Steiner in 1836, but he didn't prove compactness of the minimizing sequence. Perron pointed out the flaw and H. Schwartz gave the first rigorous proof in 1884. For not necessarily minimizing closed CMC surfaces, some called this H. Hopf's conjecture since Hopf pondered the question and showed that it is true assuming that the surface is topologically a sphere in 1950. We describe Hopf's argument following ([Ho]), which involves a bit of complex analysis. A. D. Alexandrov proved that any embedded CMC surface has to be the round sphere in 1958. His reflection technique is an important method to prove many uniqueness results in PDE and geometry. A good source for various proofs of this and many related results is Huck, et.al. [HRSV].

H. Hopf's question was resolved in the negative by H. Wente in 1986. He found a CMC immersed torus by solving an elliptic problem and proving that the resulting solution forms a closed surface.

Let $X : \Sigma \to \mathbb{R}^3$ be a CMC immersion of a closed surface Σ. It turns out that for H constant,

$$\Phi = e^{2\phi}(A_{11} - A_{22} - 2iA_{12})$$

is a holomorphic function on $\bar{B}$. We will not prove this fact; see [Ho] for a proof. Actually the holomorphic quadratic differential Φdz^2 is a globally defined object on Σ called the Hopf differential.

The Hopf differential vanishes if and only if $A_{11} = A_{22}$ and $A_{12} = 0$, in other words when the surface is umbilic.

Theorem 12 (H. Hopf's Theorem) *Let $X : S^2 \to \mathbb{R}^3$ be a C^2 immersion with constant mean curvature. The $X(S^2)$ is the round sphere.*

Proof: The existence of isothermal coordinates allows us to view the parameter sphere as a Riemann surface. The Hopf differential is a quadratic holomorphic differential on the sphere, hence identically zero, by Lemma 13. Hence $X(S^2)$ is totally umbilic, and thus a sphere ([G], p.218.) □

We show that quadratic holomorphic differentials vanish on the sphere. A different argument may be constructed by considering the line field of imaginary directions which would exist if the Hopf differential didn't vanish. The existence of such a line field would lead to a topological contradiction [Ho], [HRSV].

Lemma 13 [Ho] *On a compact Riemann surface S of genus zero there are no holomorphic quadratic differentials $\Xi = \Phi dw^2$ other than the trivial one $\Xi \equiv 0$.*

Proof: By the uniformization theory of Riemann surfaces, there is only one conformal type of a compact Riemann surface of genus zero. Thus, after a conformal reparameterization we may assume S is covered by two isothermal coordinate charts given by two copies of $\mathbb{C}$, say $z \in \mathbb{C}$ and $w \in \mathbb{C}$. The transition function for $z \in \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ is given by $w = 1/z$. The quadratic differential is therefore given as $\Xi = \Phi dw^2 = \Psi dz^2$ where $\Phi(w)$ and $\Psi(z)$ are entire functions. The transformation rule is

$$\Phi(w) = \Phi(w(z)) \left(\frac{dz}{dw}\right)^2 = \frac{\Psi(z)}{w^4} = \Psi(z)z^4.$$

Now Since $\Psi(z)$ is regular near $z = 0$ so $\Phi(\infty) = 0$ and $\Phi(w)$ is bounded. Therefore, by Liouville's Theorem, $\Xi \equiv 0$. □

C Some recent developments

In this section we will mention some of the more important recent developments. This list reflects the biases of the authors and is not meant to be anywhere near complete. We apologize in advance to those whose recent work we have not mentioned below.

First, Colding and Minicozzi have greatly generalized the Choi and Schoen result. The main part of their work (so far) has been an understanding of how a sequence of minimal surfaces in $\mathbb{R}^3$ can degenerate. To consider some toy models for how this degeneration can occur, think of rescaling a catenoid or a helicoid. Roughly stated, their work says that if $|A|$ is large at a point on the surface then it must look something like a helicoid. See [CM2] and [CM3] for more details.

Using the Colding-Minicozzi curvature estimates, Meeks and Rosenberg have recently proved the following theorem (see [MR]).

Theorem 14 *A simply connected, complete, noncompact minimal without boundary in $\mathbb{R}^3$ is either a flat plane or a helicoid.*

They have some extensions of this theorem, which is still work in progress. Meeks, Pérez and Ros have also developed some uniqueness theorems regarding Riemann's minimal surface, see [P].

It is a classical result that any embedded minimal surface is $\Sigma \subset \mathbb{R}^3$ has a cousin minimal surface $\tilde{\Sigma} \subset \mathbb{R}^3$ (not necessarily embedded) which is isometric to Σ. Geometrically, one can obtain $\tilde{\Sigma}$ from Σ by rotating the vectors of each tangent plane of Σ by $\pi/2$ in that tangent plane. Karcher [Kar] explored the relationship between mean curvature 1 surfaces in $\mathbb{R}^3$ and minimal surfaces in S^3. Roughly speaking, the correspondence is the same as the classical one for minimal surfaces (i.e. one can obtain one surface from the other by rotating the tangent vectors in their tangent planes), but one needs to first make sense of that in S^3. Using these methods, Große-Brauckmann in [GB1] created new examples of mean curvature 1 surfaces in $\mathbb{R}^3$. Later Große-Brauckmann, Kusner and Sullivan [GKS] used this technique to classify all genus zero, three ended CMC surfaces in $\mathbb{R}^3$. Their result states that the moduli space of such surfaces (identifying congruent surfaces) is the space of distinct triples of points on S^2, modulo rotations. Also, Cosín and Ros [CR] used the same techniques to classify properly immersed genus zero minimal surfaces of finite total curvature with a plane of reflection symmetry in $\mathbb{R}^3$. These surfaces turn out to be classified by planar polygons which are the boundary of an immersed disc. Große-Brauckmann wrote a nice survey of these developments in [GB2].

Related to the classification results for CMC surfaces, the search for explicit examples continues. In a series of papers, Mazzeo, Pacard and Pollack construct many new CMC surfaces by gluing together known examples. In some sense, these kinds of constructions go back to Kapouleas [Kap], but the techniques of Mazzeo, Pacard and Pollack are new. In [MP] Mazzeo and Pacard construct CMC surfaces of genus zero and with arbitrarily many ends by attaching Delaunay ends onto a central core. In [MPP1] Mazzeo, Pacard and Pollack show that under certain conditions one can perform the connected sum construction in the CMC category. In other words, given two compact embedded CMC surfaces with boundary Σ_1 and Σ_2 first arrange them so that they have first order contact at a point, with their tangent planes oriented oppositely. Then one can find a one-parameter family of embedded CMC surfaces Σ_ϵ such that away from the contact point Σ_ϵ converges to $\Sigma_1 \cap \Sigma_2$ uniformly on compact sets. In [MPP2] Mazzeo, Pacard and Pollack show that one can attach a Delaunay end onto any point of a CMC surface. Using this construction of adding an end to a CMC surface, they explore the topology of the moduli space of CMC surfaces with fixed topological type. In particular, they show that (for $k > 3$ ends) these moduli spaces are not simply connected and that (in the genus 0, $k > 3$ ends case) every conformal type of a finitely punctured sphere realized as a CMC surface. In related work, Kusner (to appear) has characterized the ways which a sequence of constant mean curvature surfaces in $\mathbb{R}^3$ can degenerate.

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