The Maximum Principle and Applications

In these notes we prove some versions of the maximum principle and some applications, particularly using the moving planes argument of Alexandrov [A] (see also [GNN]). The standard references for the maximum principle are [GT] and [PW].

Definitions: A second order linear differential operator L has the form

$$L(u) = a_{ij}(x)u_{ij} + b_k(x)u_k + c(x)u, \tag{1}$$

where subscripts denote partial derivatives and we sum over repeated indices. The operator L is elliptic at a point x if the coefficient matrix $[a_{ij}(x)]$ is positive definite, and L is uniformly elliptic on a domain $\Omega \subset \mathbb{R}^n$ if there is $\Lambda > 0$ such that

$$\frac{1}{\Lambda}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{2}$$

for all $x \in \Omega$.

Let $F = F(x, u, Du, D^2u)$ be a (nonlinear) differential operator which is C^1 in all its arguments, and let w be a C^2 function. The linearization of F about w is the linear differential operator defined by

$$L_w(f) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(w + \epsilon f) = a_{ij}(x)f_{ij} + b_k(x)f_k + c(x)f, \tag{3}$$

and we say F is uniformly elliptic if there is a number $\Lambda > 0$, which is independent of x and w, such that (2) holds, where a_{ij} are the coefficients of the linearization L_w.

It is worthwhile to consider some examples. The mean curvature operator

$$H(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

is elliptic, but not uniformly elliptic. You lose control of Λ is $|\nabla u| \to \infty$, which is precisely what happens when $u(x) = \sqrt{R^2 - |x|^2}$ and $|x| \to R^-$. On the other hand, if $|\nabla u|$ is uniformly bounded then the nonlinear operator $H(u)$ is uniformly elliptic about u. The Monge-Ampere operator

$$M(u) = \det D^2u$$

is elliptic about w if and only if w is convex, that is, if and only if D^2w is positive definite.

Basic Maximum Principles: We start with the weak maximum principle.

Theorem 1. *Let L be a uniformly ellptic, linear operator of the form* (1) *with $c \leq 0$, and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq 0, \qquad u|_{\partial\Omega} \leq 0.$$

Then, unless $u \equiv 0$, for all $x \in \Omega$ we have $u(x) < 0$.

Proof. We suppose the theorem is not true, which means u has a non-negative maximum at some $p \in \Omega$, and derive a contradiction. This is easy if $L(u) > 0$, because

$$u(p) \geq 0, \qquad \nabla u(p) = 0, \qquad \left. D^2u\right|_p (e, e) \leq 0, \tag{4}$$

where e is any unit vector. Now let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $a_{ij}(p)$, which are all positive, and let e_i be the eigenvector associated to λ_i. Then

$$L(u)(p) = a_{ij}(p)u_{ij}(p) + b_k(p)u_k(p) + c(p)u(p) = \sum_{i=1}^{n} \lambda_i \left. D^2u\right|_p (e_i, e_i) + c(p)u(p) \leq 0, \tag{5}$$

which contradicts $L(u) > 0$.

For the general case, we build a barrier function as follows. Recall that $[a_{ij}(p)]$ is positive definite, so (after a rotation) we can assume $a_{11}(p) > 0$. We define

$$w(x) = u(x) + \epsilon z(x) = u(x) + \epsilon(e^{\alpha(x_1 - p_1)} - 1),$$

where α and ϵ are constants we choose later. Observe that

$$L(z)(p) = e^{\alpha(x_1 - p_1)}(\alpha^2 a_{11}(p) + \alpha b_1(p)) + c(p)(e^{\alpha(x_1 - p_1)} - 1),$$

and we can choose $\alpha > 0$ sufficiently large to that $L(z)(p) > 0$. By continuity we also have $L(z) > 0$ is a small neighborhood of p. Because p is a local maximum for u, we can find a nearby $q \in \Omega$ such that $u(q) < u(p)$, and now choose a positive

$$0 < \epsilon < \frac{u(p) - u(q)}{z(q)}.$$

Then

$$w(q) = u(q) + \epsilon z(q) < u(p), \qquad w(p) = u(p),$$

and so w has a positive interior maximum and satisfies $L(w) > 0$, which contradicts (5) as applied to w. $\square$

Taking differences, we immediately obtain the following comparison theorem:

Corollary 2. *Let L be a uniformly ellptic, linear operator of the form* (1) *with $c \leq 0$, and let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq L(v), \qquad u|_{\partial\Omega} \leq v|_{\partial\Omega}.$$

Then, unless $u \equiv v$, for all interior points $x \in \Omega$ we have $u(x) < v(x)$.

Proof. Apply Theorem 1 to $w = u - v$. $\square$

We have a condition on the normal derivative of u at $\partial\Omega$ as well.

Theorem 3. *Let L be uniformly elliptic, and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$L(u) \geq 0, \qquad u \leq 0, \qquad u|_{\partial\Omega} = 0.$$

Then, unless $u \equiv 0$, for each $p \in \partial\Omega$ we have

$$\frac{\partial u}{\partial N} > 0,$$

where N is the unit outward normal vector for $\partial\Omega$.

Observe that we do not place a condition on the sign of c here.

Proof. We first prove this result for $c \equiv 0$. Fix $p \in \partial\Omega$, and choose $r_1 > 0$ small enough so that the ball $B_{r_1}(\tilde{x})$ tangent to $\partial\Omega$ at p lies completely inside Ω. Let B_1 be this ball and let $B_2 = B_{r_1/2}(p)$. Now, for some constants α and ϵ we define

$$w = u + \epsilon z = u + \epsilon(e^{-\alpha|x-\tilde{x}|^2} - e^{-\alpha r_1^2}).$$

Observe that

$$z|_{B_1} > 0, \qquad z|_{\partial B_1} = 0, \qquad z < 0 \quad \text{otherwise}.$$

By Theorem 1, we may assume $u < 0$ inside Ω, so in particular $u < 0$ on $\bar{B}_1 \backslash \{p\}$. Now pick $\epsilon > 0$ small enough so that $w = u + \epsilon z \leq 0$ on $(\partial B_2) \cap B_1$, and (as before) pick $\alpha > 0$ large enough so that $L(w) > 0$. Then, applying Theorem 1 to w on $B_1 \cap B_2$, we see w attains its maximum at p, so

$$0 \leq \frac{\partial w}{\partial N}(p) = \frac{\partial u}{\partial N}(p) + \epsilon \frac{\partial z}{\partial N}(p).$$

A quick computation shows

$$\frac{\partial z}{\partial N}(p) = -2\alpha e^{-\alpha r_1^2} \sum_{i=1}^{n} N_i x_i < 0,$$

which implies $\frac{\partial u}{\partial N}(p) > 0$.

Now we use the result above and a barrier to prove the theorem in the general case. Let $v = e^{-\beta x_1} u$, where $\beta > 0$ is a constant we choose later, and as before we can take $a_{11}(p) > 0$. Then

$$0 \leq L(u) = e^{\beta x_1} L'(v) + vL(e^{\beta x_1}),$$

where L' is a uniformly elliptic, linear operator with no zero order term. Rearranging the above inequality we get

$$0 \leq L'(v) + v(a_{11}\beta^2 + b_1\beta + c) = L'(v) + c'v.$$

Choose $\beta > 0$ large enough so that $L'(v) > 0$, at least near p. By what we have just proved,

$$\frac{\partial v}{\partial N}(p) > 0 \Rightarrow \frac{\partial u}{\partial N}(p) = e^{\beta p_1} \frac{\partial v}{\partial N}(p) > 0.$$

□

Finally, we prove a somewhat simplified version of the strong maximum principle, which will suffice for our purposes.

Theorem 4. *Let $F = F(x, Du, D^2u)$ be a uniformly elliptic nonlinear differential operator which is C^1 in all its arguments, and let $u, v, \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$u \geq v, \qquad F(x, Du, D^2u) = F(x, Dv, D^2v).$$

If there is a point $p \in \bar{\Omega}$ such that

$$u(p) = v(p), \qquad Du(p) = Dv(p)$$

then $u \equiv v$.

It is possible to prove the strong maximum principle for operators which also depend on u, provided F is monotone in the function values of u, see Section 17.1 of [GT]. The proof is more complicated, and we omit it here.

Proof. As before, we assume $u(p) = v(p)$, $Du(p) = Dv(p)$, and that there are points where $u > v$, and derive a contradiction. There are two possible cases: either $u(q) > v(q)$ for all interior points $q \in \Omega$, or there are some interior points p with $u(p) = v(p)$. In the first case, let B_R be a ball of radius R such that $p \in \partial B_R$ and $\bar{B}_R \backslash \{p\} \subset \Omega$, and then let $A = B_R \backslash B_{\tilde{R}}$ where $\tilde{R} < R$ and $B_{\tilde{R}}$ has the same center as B_R. Also let $\Gamma_0 = \partial B_{\tilde{R}}$ and $\Gamma_1 = \partial B_R$. In the second case we can also choose an annulus A with the same form, provided we make a smart choice of p. We choose $p \in \partial\{u(x) = v(x)\}$ and observe that $\{u(x) \neq v(x)\}$ is a nonempty open set in Ω, so there is a q near p such that $u(q) > v(q)$. Now let q be the center of our annulus, $R = \text{dist}(p, q)$, such that $R < \text{dist}(q, \partial\Omega)$, and construct A as above. In either case, we now have an annulus A where $u \geq v$ on A, $u(p) = v(p)$ and $Du(p) = Dv(p)$ for some $p \in \Gamma_1$ and $u - v \geq \epsilon$ on Γ_0 for some $\epsilon > 0$.

For $0 \leq t \leq 1$ define

$$\chi(t) = F(x, tDu + (1-t)Dv, tD^2u + (1-t)D^2v).$$

Then by the mean value theorem

$$0 = F(x, Du, D^2u) - F(x, Dv, D^2v) = \chi(1) - \chi(0) = \chi'(t_0) = L(w)$$

for some $t_0 \in (0,1)$, where $w = u - v$ and L is the linearization of F, linearized about $t_0u + (1-t_0)v$. By the hypothesis on F, the linear operator L is a uniformly elliptic operator of the form (1) with $c = 0$. In addition to $L(w) = 0$, we also have

$$w|_A \geq 0, \qquad w|_{\Gamma_0} \geq \epsilon, \qquad w(p) = 0.$$

We complete the proof by finding a barrier z with

$$L(z) > 0, \qquad z|_{\Gamma_0} = \epsilon, \qquad z|_{\Gamma_1} = 0, \qquad \frac{\partial z}{\partial r} < 0. \tag{6}$$

Indeed, once we construct z we use Corollary 2 on A to get $w \geq z$ and so

$$\frac{\partial w}{\partial r}(p) \leq \frac{\partial z}{\partial r} < 0,$$

which contradicts $Du(p) = Dv(p)$. It is straightforward to check that, for M large enough, the function

$$z(x) = f(|x|^2/2), \qquad f(s) = \frac{\epsilon(e^{-Ms} - e^{-MR^2/2})}{e^{-M\tilde{R}^2/2} - e^{-MR^2/2}}$$

satisfies all the conditions in (6).

The following corollary is a special case of the strong maximum principle.

Corollary 5. *Let $F = F(x, Du, D^2u)$ be a uniformly elliptic, nonlinear differential operator which is homogeneous, i.e. $F(x, 0, \ldots, 0) = 0$ for all x. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

$$u \leq 0, \qquad F(x, Du, D^2u) = 0$$

then either $u \equiv 0$ or $u < 0$ on the interior of Ω.

□

Moving planes and constant mean curvature surfaces: We include here Alexandrov's proof [A] that the only compact, embedded, constant mean curvature surface in $\mathbb{R}^3$ without boundary is the round sphere (see also [Ho]). Let $\Sigma \subset \mathbb{R}^3$ be a compact, embedded, constant mean curvature surface without boundary, and let Ω be the 3-dimensional region it bounds. The strategy is to use moving planes to show that Σ has a plane of symmetry perpendicular to any unit vector $\gamma \in S^2$. Once we do this, we still need to show that all these symmetry planes pass through a common point, which is easy. If x_0 is the center of mass of Σ (*i.e.* the average of all the position vectors of Σ), then each symmetry plane must contain x_0. After translation, we can assume $x_0 = 0$, and so Σ is invariant under all reflections through planes passing through the origin, which implies Σ is invariant under all rotations fixing 0. Thus Σ must be a round sphere.

Now fix some direction $\gamma \in S^2$. For $\lambda \in \mathbb{R}$ we let

$$T_\lambda = \{\langle x, \gamma \rangle = \lambda\}, \qquad \Sigma(\lambda) = \{x \in \Sigma : \langle x, \gamma \rangle > \lambda\},$$

and we let $\Sigma'(\lambda)$ be the reflection of $\Sigma(\lambda)$ through T_λ. Also define

$$\lambda_0 = \sup\{\lambda : \Sigma(\lambda) \neq \varnothing\}, \qquad \lambda_1 = \inf\{\lambda : \Sigma'(\tilde{\lambda}) \subset \Omega, \lambda < \tilde{\lambda} < \lambda_0\}.$$

Then $\Sigma'(\lambda_1)$ must contact Σ to first order at some point $p \in \Sigma\backslash\Sigma(\lambda_1)$. Near this point p, we can write Σ as the graph of u and $\Sigma'(\lambda_1)$ as the graph of v such that

$$u \geq v, \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 1 = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right).$$

Here u and v are functions defined on a small neighborhood of p in the the common tangent plane to Σ and $\Sigma'(\lambda_1)$. Moreover, because Σ and $\Sigma'(\lambda_1)$ contact to first order we have

$$u(p) = v(p), \qquad \nabla u(p) = \nabla v(p).$$

The strong maximum principle tells us $u \equiv v$ in a small neighborhood of p. However, soltutions to the equation $H(u) = 1$ are analytic, and so $\Sigma'(\lambda_1) = \Sigma\backslash\Sigma(\lambda_1)$, which means T_{λ_1} is a plane of symmetry for Σ. This completes the proof of Alexandrov's theorem.

The proof above yields the following more general theorem.

Theorem 6. *Let $\Sigma \subset \mathbb{R}^n$ be a compact, embedded hypersurface without boundary, and let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be its principle curvatures. If*

$$F(\kappa_1, \ldots, \kappa_n) = a,$$

where $a \in \mathbb{R}$ and F is a homogeneous, C^1 function, then Σ is a round sphere.

Moving planes and nonlinear equation on bounded domains: We present here some classical results of Gidas, Ni, and Nirenberg about positive solutions to nonlinear partial differential equations. Our model equation is

$$\Delta u + f(u) = 0, \qquad u > 0, \tag{7}$$

where f is a C^1 function. We consider this equation in either a bounded domain Ω with smooth boundary, or in the whole space of $\mathbb{R}^n$.

Theorem 7. *Let $\Omega = B_R = \{|x| < R\}$ and let $u \in C^2(\bar{\Omega})$ be a positive solution to* (7) *on Ω with the boundary condition $u|_{\partial\Omega} = 0$. Then $u(x) = u(r)$ and for $0 < r < R$ we have $\frac{\partial u}{\partial r} < 0$.*

Theorem 8. *Let $\Omega = B_R\backslash B_{\tilde{R}}$ and let $u \in C^2(\bar{\Omega})$ be a positive solution to* (7) *with the boundary condition $u|_{|x|=R} = 0$. Then*

$$\frac{R+\tilde{R}}{2} \leq |x| < R \quad \Rightarrow \quad \frac{\partial u}{\partial r} < 0.$$

Notice that in this last theorem we do not place a boundary condition on the inner sphere $|x| = \tilde{R}$.

We use Alexandrov's technique of moving planes. For a fixed direction $\gamma \in S^{n-1}$ and $\lambda \in \mathbb{R}$ we take

$$T_\lambda = \{\langle x, \gamma\rangle = \lambda\}, \qquad \Sigma(\lambda) = \{x \in \Omega : \langle x, \gamma\rangle > \lambda\},$$

and we let $\Sigma'(\lambda)$ be the reflection of $\Sigma(\lambda)$ across T_λ. We also define

$$\lambda_0 = \sup\{\lambda : \Sigma(\lambda) \neq \varnothing\}, \qquad \lambda_2 = \inf\{\tilde{\lambda} : \Sigma'(\lambda) \subset \Omega, \tilde{\lambda} < \lambda < \lambda_0\},$$

and we let λ_1 be the time of first contact of $\partial(\Sigma'(\lambda))$ with $\partial\Omega$. This first contact occurs either at a point $p \in \partial\Omega$ where $\partial(\Sigma'(\lambda_1))$ is tangent to $\partial\Omega$, or at a point $p \in T_{\lambda_1} \cap \partial\Omega$ where $T_{\lambda_1} \perp \partial\Omega$. We call $\Sigma(\lambda_1)$ the maximal cap, and $\Sigma(\lambda_2)$ the optimal cap. Observe that $\lambda_2 \leq \lambda_1 < \lambda_0$, and it is possible to have $\lambda_2 < \lambda_1$. For a point $x \in \Sigma(\lambda)$, we denote the reflection of x across T_λ by x^λ.

For the following technical results we take $\gamma = e_1$, so that

$$\Sigma(\lambda) = \{x \in \Omega : x_1 > \lambda\}, \qquad (x_1, x')^\lambda = (2\lambda - x_1, x'), \qquad \Sigma'(\lambda) = \{x : x^\lambda \in \Sigma(\lambda)\}.$$

We take $u \in C^2(\bar{\Omega})$ with $u > 0$ in Ω and

$$\Delta u + b_1 u_1 + f(u) = 0, \qquad u|_{\partial\Omega \cap \{x_1 > \lambda\}} = 0. \tag{8}$$

Proposition 9. *Let u satisfy* (8)*and assume $b_1 \geq 0$ on $\Sigma(\lambda_1) \cup \Sigma'(\lambda_1)$. Then for any $\lambda \in (\lambda_1, \lambda_0)$ we have, for $x \in \Sigma(\lambda)$,*

$$u_1(x) < 0, \qquad u(x) < u(x^\lambda), \tag{9}$$

and therefore $u_1 < 0$ in $\Sigma(\lambda_1)$. Moreover, if $u_1(p) = 0$ for some $p \in \Omega \cap T_{\lambda_1}$ then $u(x) = u(x^{\lambda_1})$, and

$$\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup (T_{\lambda_1} \cap \Omega),$$

and $b_1 \equiv 0$.

We first use this technical proposition to prove Theorems 7 and 8. In the case of a ball, the proposition tells us $u_1 < 0$ in $\{x : |x| < R, x_1 > 0\}$, and, by continuity, $u_1(x) \leq 0$ for $x_1 = 0$. However, we can also apply the same argument with $\gamma = -e_1$, and so $u_1(x) = 0$ for $x_1 = 0$. The equality case of Proposition 9 tells us $u(x_1, x') = u(-x_1, x')$. Now rotate to obtain this symmetry for any choice of unit vector e.

In the case of the annulus, Proposition 9 shows $u_1 < 0$ in the maximal cap $\{x : |x| < R, x_1 > (R + \tilde{R})/2\}$. Moreover, if $u_1(x) = 0$ for some $x_1 = (R + \tilde{R})/2$ then Ω has to be symmetric about the plane $x_1 = (R + \tilde{R})/2$, which is it not. Thus $u_1 < 0$ on $\{x : |x| < R, x_1 \geq (R + \tilde{R})/2\}$. Again, we can apply this argument for any unit vector e to obtain Theorem 8.

To prove Proposition 9 we need the following lemmas.

Lemma 10. *Let $x_0 \in \partial\Omega$ with $N_1(x_0) > 0$, and let $u \in C^2(\Omega \cap \{|x - x_0| < \epsilon\})$ for some $\epsilon > 0$ with $u > 0$ in Ω and*

$$\Delta u + b_1(x))u_1 + f(u) = 0, \qquad u|_{\partial\Omega \cap \{|x - x_0| < \epsilon\}} = 0.$$

Then there exists $\delta > 0$ such that $u_1 < 0$ on $\Omega \cap \{|x - x_0| < \delta\}$.

Proof. We have $\frac{\partial u}{\partial N} \leq 0$, so (after possibly decreasing ϵ) we can assume $u_1 \leq 0$ on $S = \partial\Omega \cap \{|x - x_0| < \epsilon\}$. If the lemma were not true, there would exist a sequence of points $x^j \to x_0$ such that $u_1(x^j) \geq 0$. Let l_j be the segment in the x_1 direction joining x^j to $\partial\Omega$. Along this segment, u_1 changes sign, going from positive to negative, so there is a point $y^j \in l_j$ where $u_1(y^j) = 0$ and $u_{11}(y^j) \leq 0$. Taking the limit as $j \to \infty$ we have

$$u_1(x_0) = 0, \qquad u_{11}(x_0) \leq 0. \tag{10}$$

Suppose $f(0) \geq 0$. Then

$$\Delta u + b_1 u_1 + f(u) - f(0) \leq 0 \Rightarrow \Delta u + b_1 u_1 c_1 u = 0$$

for some function c_1. Then by Theorem 3 we have $\frac{\partial u}{\partial N}(x_0) < 0$, which implies $u_1(x_0) < 0$, which contradicts (10). On the other hand, if $f(u) < 0$ the $\Delta u(x_0) = -f(0) > 0$. Then $u_{ij}(x_0) = -f(0) N_i(x_0) N_j(x_0)$, so in particular $u_{11}(x_0) > 0$, which again contradicts (10). □

Lemma 11. *Suppose that for some λ in $[\lambda_1, \lambda_0)$ we have*

$$u_1 \leq 0, \qquad u(x) \leq u(x^\lambda), \qquad u(x) \not\equiv u(x^\lambda)$$

in the cap $\Sigma(\lambda)$. Then $u(x) < u(x^\lambda)$ and $u_1 < 0$ on $T_\lambda \cap \Omega$.

Proof. In $\Sigma'(\lambda)$, consider the function $v(x) = u(x^\lambda)$. In the reflected cap, we have

$$\Delta v - b_1(x^\lambda) v_1 + f(v) = 0, \qquad v_1 \geq 0.$$

Taking differences, in $\Sigma'(\lambda)$ we obtain

$$\Delta(v - u) + b_1(x)(v_1 - u_1) + f(v) - f(u) = (b_1(x^\lambda) + b_1(x)) v_1 \geq 0,$$

where we have used that b_1 and v_1 are both non-negative. If $w = v - u$ in $\Sigma'(\lambda)$, then we have

$$\Delta w + b_1 w + cw \geq 0$$

for some function c. However, on T_λ we have $x^\lambda = x$ so $w = 0$, and then Theorem 1 implies $w = 0$ on $\Sigma'(\lambda)$. Thus $u(x) > u(x^\lambda)$ for $x \in \Sigma'(\lambda)$, and so (reflecting across T_λ) for $x \in \Sigma$ we have $u(x) < u(x^\lambda)$. Also, Theorem 3 implies and $w_1 = -2u_1 > 0$ on T_λ. □

Finally we prove Proposition 9.

Proof. Lemma 10 says we can assume (9) holds for $\lambda_0 - \delta < \lambda < \lambda_0$ for some small, positive δ. Let

$$\mu = \inf\{\tilde{\lambda} : \text{ (9) holds for } \tilde{\lambda} < \lambda < \lambda_0\} \geq \lambda_1.$$

We want to show that $\mu = \lambda_1$. By continuity, for $x \in \Sigma(\mu)$ we have

$$u(x) \leq u(x^\mu), \qquad u_1 < 0.$$

Now suppose $\mu > \lambda_1$ and take $x_0 \in \partial(\Sigma(\mu))\backslash T_\mu$. Then $x_0^\mu \in \Omega$ so $0 = u(x_0) < u(x^\mu)$, so $u(x) \not\equiv u(x^\mu)$ and Lemma 11 implies $u(x) < u(x^\mu)$ in $\Sigma(\mu)$ and $u_1 < 0$ on $T_\mu \cap \Omega$. By continuity, there is some $\epsilon > 0$ such that $u_1 < 0$ in $\Omega \cap \{x_1 > \mu - \epsilon\}$.

By our construction of μ, there is a sequence $\lambda^j \to \mu^-$ and $x_j \in \Sigma(\lambda^j)$ such that $u(x_j) \geq u(x_j^{\lambda^j})$. A subsequence of $\{x_j\}$ converges to $\bar{x} \in \bar{\Sigma}(\mu)$ and $x_j^{\lambda^j} \to \bar{x}^\mu$. Because (9) holds in $\Sigma(\mu)$, we must have $\bar{x} \in \partial\Sigma(\mu)$. However, because $\mu > \lambda_1$, if $\bar{x} \in \partial\Sigma(\mu)\backslash T_\mu$ then $u(\bar{x}^\mu) > 0 = u(\bar{x})$, which we just showed can't happen. Thus $\bar{x} \in T_\mu \cap \Omega$ and so, for large j, the segement joining x_j to $x_j^{\lambda_j}$ is contained in Ω. On this segment, u increases moving in the positive x_1 direction, so each segment contains a point y_j such that $u_1(y_j) \geq 0$. Taking a limit we see $u_1(\bar{x}) \geq 0$ for some $\bar{x} \in T_\mu$, which contradicts $u_1 < 0$ in $\Omega \cap \{x_1 > \mu - \epsilon\}$. We conclude that $\mu = \lambda_1$.

In the case of equality, suppose we have $u_1(p) = 0$ for some $p \in T_{\lambda_1} \cap \Omega$. Then Lemma 11 tells us $u(x) = u(x^{\lambda_1})$ in $\Sigma(\lambda_1)$. Next, for any $x \in \partial(\Sigma(\lambda_1))\backslash T_{\lambda_1}$, we have $u(x) = 0 = u(x^{\lambda_1})$, and so $x^{\lambda_1} \in \partial\Omega$, which implies Ω is symmetric. Finally, suppose $b(x) > 0$ for some $x \in \Omega$. Then, comparing $u(x)$ to $u(x^{\lambda_1})$ and using (8) we have $b_1(x)u_1(x) = b_1(x^{\lambda_1})u_1(x^{\lambda_1})$. However, one side of this last equation is positive while the other side is negative, which is impossible. □

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