The Monotonicity Formula

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In this note we prove the monotonicity formula for minimal submanifolds of $\mathbb{R}^n$, and discuss some of its consequences.

Let $\Sigma^k \subset \mathbb{R}^n$ be a k-dimensional submanifold of Euclidean space, and let H be its mean curvature vector. Recall that Σ is minimal if $H \equiv 0$. Because H is the first variation of the k-dimensional volume of Σ, the requirement $H \equiv 0$ is equivalent to Σ being a critical point of the volume functional under compactly supported variations. In other words, Σ is minimal means precisely

$$\int_\Sigma \operatorname{div}_\Sigma(X) = 0 \tag{1}$$

for any vector field X (with values in $\mathbb{R}^n$) with compact support.

Theorem 1. *Let $\Sigma^k \subset \mathbb{R}^n$ be minimal. Fix $x_0 \in \mathbb{R}^n$ and $R > r > 0$, and denote the (ambient) ball centered at x_0 with radius ρ by B_ρ. Then*

$$R^{-k}\operatorname{Vol}(B_R \cap \Sigma) - r^{-k}\operatorname{Vol}(B_r \cap \Sigma) = \int_{(B_R \setminus B_r)\cap\Sigma} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}}. \tag{2}$$

Before we begin the proof, observe that mean curvature scales: for any $\lambda > 0$

$$H(\lambda\Sigma) = \frac{1}{\lambda}H(\Sigma).$$

In particular, any rescaling of a minimal submanifold is still minimal. Indeed, this is the key observation for understanding the monotonicity formula. One can understand monotonicity giving a variational characterization of the fact that minimality is preserved by rescaling. Knowing this, one should first try to prove monotonicity by applying the divergence theorem as in equation (1) to a radial vector field. In fact, this is the approach we will take. Intuitively, the vector field you want to consider is $r\partial_r$ restricted to a ball. This in not quite smooth enough to apply the variational formula, so we will need to regularize it.

The proof below is essentially the proof in Simon's book [Si], and easily adapts to the case where Σ is a minimal varifold or a minimal current. One can find another proof in Colding and Minicozzi's book [CM].

Proof. To start, we will show

$$R^{-k}\operatorname{Vol}(B_R \cap \Sigma) - r^{-k}\operatorname{Vol}(B_r \cap \Sigma) \geq 0,$$

and then recover the right hand side of the inequality above by examining our calculation more carefully.

Let $\rho = |x - x_0|$, and for a given compactly supported function η consider the vector field $X_\eta = \eta(\rho)(x - x_0)$. For $p \in \Sigma$, choose an adapted orthonormal frame

$$\{E_1, \ldots, E_n\}, \qquad \{E_1, \ldots, E_k\} \in T_p\Sigma, \qquad \{E_{k+1}, \ldots, E_n\} \in (T_p\Sigma)^\perp.$$

Then

$$\begin{aligned}
\operatorname{div}_\Sigma(X) &= \sum_{j=1}^k \langle \nabla_{E_j} \eta(x-x_0), E_j\rangle = \sum_{j=1}^k \eta(\rho)\langle E_j, E_j\rangle + \sum_{j=1}^k \langle (E(\eta))(x-x_0), E_j\rangle \\
&= k\eta(\rho) + \sum_{j=1}^k \frac{\eta'(\rho)}{\rho}\langle (x-x_0)_j(x-x_0), E_j\rangle = k\eta(\rho) + \frac{\eta'(\rho)}{\rho}|(x-x_0)^T|^2 \\
&= k\eta(\rho) + \rho\eta'(\rho)\left(\frac{|(x-x_0)^T|^2}{|x-x_0|^2}\right) = k\eta(\rho) + \rho\eta'(\rho)\left(\frac{|x-x_0|^2 - |(x-x_0)^\perp|^2}{|x-x_0|^2}\right) \\
&= k\eta(\rho) + \rho\eta'(\rho)\left(1 - \frac{|(x-x_0)^\perp|^2}{\rho^2}\right).
\end{aligned}$$

For $0 < \epsilon < 1$ choose $\phi(t)$ such that

$$\phi'(t) \le 0, \qquad \phi(t) = \begin{cases} 1 & t \le 1-\epsilon \\ 0 & t \ge 1. \end{cases}$$

Let $\eta(\rho) = \phi(\rho/r)$ and define

$$I(r) = \int_\Sigma \eta(\rho) = \int_\Sigma \phi\left(\frac{\rho}{r}\right). \tag{3}$$

By the dominated convergence theorem,

$$\lim_{\epsilon \to 0^+} I(r) = \operatorname{Vol}(B_r \cap \Sigma).$$

Also,

$$\begin{aligned}
\rho\eta'(\rho) &= \rho\frac{\partial}{\partial\rho}(\phi(\rho/r)) = \frac{\rho}{r}\phi'(\rho/r) \\
r\frac{\partial}{\partial r}(\phi(\rho/r)) &= r\left(-\frac{\rho}{r^2}\phi'(\rho/r)\right) = -\frac{\rho\phi'(\rho/r)}{r} = -\rho\eta'(\rho).
\end{aligned}$$

Now evaluate equation (1) with $X = \eta(\rho)(x-x_0)$. We have

$$\begin{aligned}
0 &= \int_\Sigma k\eta(\rho) + \rho\eta'(\rho)\left(1 - \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}\right) \\
&= \int_\Sigma k\phi(\rho/r) - r\frac{\partial}{\partial r}(\phi(\rho/r)) + r\frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)^\perp|^2}{\rho^2} \\
&\ge \int_\Sigma k\phi(\rho/r) - r\frac{\partial}{\partial r}(\phi(\rho/r)) = k\int_\Sigma \phi(\rho/r) - r\frac{\partial}{\partial r}\int_\Sigma \phi(\rho/r) \\
&= kI(r) - rI'(r).
\end{aligned}$$

Now differentiate

$$\frac{d}{dr}(r^{-k}I(r)) = -kr^{-k-1}I(r) + r^{-k}I'(r) = -r^{-k-1}(kI - rI') \ge 0.$$

We conclude that $r^{-k}I(r)$ is a monotone nondecreasing function of r. Let $\epsilon \to 0^+$, we then see $r^{-k}\operatorname{Vol}(B_r \cap \Sigma)$ is monotone nondecreasing. In other words,

$$R^{-k}\operatorname{Vol}(B_R \cap \Sigma) - r^{-k}\operatorname{Vol}(B_r \cap \Sigma) \ge 0$$

for $0 < r < R$.

To obtain equation (2) we analyze the term we discarded:

$$\int_\Sigma r\frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)\perp|^2}{|x-x_0|^2}.$$

Notice that

$$\frac{\partial}{\partial r}(\phi(\rho/r)) \neq 0 \Leftrightarrow (1-\epsilon)r \leq \rho \leq r,$$

or, equivalently,

$$\frac{\partial}{\partial r}(\phi(\rho/r)) \neq 0 \Leftrightarrow (1-\epsilon)^k\rho^{-k} \leq r^{-k} \leq \rho^{-k}.$$

Therefore

$$(1-\epsilon)^k\int_\Sigma \frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}} \leq r^{-k}\int_\Sigma \frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)^\perp|^2}{|x-x_0|^2} \leq \int_\Sigma \frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}}.$$

Letting $\epsilon \to 0^+$ and using the dominated convergence theorem, we obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\int_{B_r\cap\Sigma}\frac{|(x-x_0)^\perp|^2}{|(x-x_0|^{k+2}} &= \lim_{\epsilon\to0^+}\int_\Sigma r^{-k}\frac{\partial}{\partial r}(\phi(\rho/r))\frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}\\
&= \lim_{\epsilon\to0^+} r^{-k}\int_\Sigma r\frac{\partial}{\partial r}(\phi(\rho/r)) - k\phi(\rho/r) = \lim_{\epsilon\to0^+}\frac{d}{dr}(r^{-k}I(r))\\
&= \frac{d}{dr}(r^{-k}\operatorname{Vol}(B_r\cap\Sigma)).
\end{aligned}$$

Integrating this last equation between r and R, we get

$$\int_{(B_R\setminus B_r)\cap\Sigma}\frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}} = R^{-k}\operatorname{Vol}(B_R\cap\Sigma) - r^{-k}\operatorname{Vol}(B_r\cap\Sigma),$$

as claimed.

□

Along, the same lines, one can prove a mean value theorem for for functions $f \in C^2(\Sigma)$. Again, if $0 < r < R$ then

$$\begin{aligned}
R^{-k}\int_{B_R\cap\Sigma} f - r^{-k}\int_{B_r\cap\Sigma} f &= \int_{(B_R\setminus B_r)\cap\Sigma} f\frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}}\\
&\quad +\frac{1}{2}\int_r^R \rho^{-k-1}\left(\int_{B_\rho\cap\Sigma}(\rho^2-|x-x_0|^2)\Delta_\Sigma f\right)d\rho.
\end{aligned} \tag{4}$$

The proof of (4) is the same as the proof of (2), except that one must weight the volume form of Σ by the function f.

In fact, even the weak form

$$R^{-k}\operatorname{Vol}(B_R\cap\Sigma) - r^{-k}\operatorname{Vol}(B_r\cap\Sigma) \geq 0 \tag{5}$$

of monotonicity is useful. First we notice that (5) is an equality if and only if $(x-x_0)^\perp = 0$ on $(B_R\setminus B_r)\cap\Sigma$. This can only occur if the part of Σ in the annulus $B_R\setminus B_r$ is contained in the cone over x_0. If equality in (5) holds for all $r > 0$ then $B_R\cap\Sigma$ is a k-dimensional cone over x_0. If, additionally, Σ is smooth at x_0, then Σ can only be a k-dimensional plane in $\mathbb{R}^n$.

Next, we define the function

$$\Theta_{x_0}(r) = \frac{\operatorname{Vol}(B_r \cap \Sigma)}{r^k \operatorname{Vol}(B_1 \subset \mathbb{R}^k)}, \tag{6}$$

where $x_0 \in \Sigma$ and $r > 0$. By equation (5), for $0 < r < R$ we have

$$\Theta_{x_0}(R) - \Theta_{x_0}(r) = \frac{1}{\operatorname{Vol}(B_1 \subset \mathbb{R}^k)} \left(R^{-k} \operatorname{Vol}(B_R \cap \Sigma) - r^{-k} \operatorname{Vol}(B_r \cap \Sigma) \right) \geq 0,$$

and so $\Theta_{x_0}(r)$ is a nondecreasing function of r for each $x_0 \in \Sigma$. The limit as $r \to 0^+$ exists, and we use it to define the density of Σ at x_0 as

$$\Theta_{x_0} = \lim_{r \to 0^+} \Theta_{x_0}(r). \tag{7}$$

Observe that, if Σ is a smooth, embedded submanifold in a neighborhood of x, then $\Theta_x = 1$. Geometrically, Θ_x measures the number of smooth sheets of Σ in a small ball centered at x. In general, $\Theta_x \geq 1$, and strict inequality can occur, as the following example shows. Let Σ be the union of two k-dimensional planes in $\mathbb{R}^n$ which intersect transversally. For all x in this intersection, $\Theta_x = 2$. Indeed, this example shows that in general Θ is not continuous. Denote the two k-planes as Σ_1 and Σ_2, and take a sequence of points $x_j \in \Sigma_1 \backslash \Sigma_2$, with $x_j \to x \in \Sigma_1 \cap \Sigma_2$. Then $\Theta_{x_j} = 1$ while $\Theta_x = 2$, so we see the density function can suddenly jump up. The regularity statement below is thus the best one can hope to achieve.

Proposition 2. *The density function Θ_{x_0} is an upper semi-continuous function on Σ.*

Proof. We want to show

$$\Theta_x \geq \limsup_{j \to \infty} \Theta_{x_j}$$

if $x_j \to x$. Let $\delta > 0$ and choose $r > 0$ such that $\Theta_x(r) \leq \Theta_x + \delta$. If $0 < \epsilon < 1$ and x_j satisfies $|x - x_j| < \epsilon r$ then $B_{(1-\epsilon)r}(x_j) \subset B_r(x)$. In this case

$$\begin{aligned} \Theta_{x_j} &\leq \Theta_{x_j}((1-\epsilon)r) = \frac{\operatorname{Vol}(B_{(1-\epsilon)r}(x_j) \cap \Sigma)}{(1-\epsilon)^k r^k \operatorname{Vol}(B_1 \subset \mathbb{R}^k)} \\ &\leq \frac{\operatorname{Vol}(B_r(x) \cap \Sigma)}{(1-\epsilon)^k r^k \operatorname{Vol}(B_1 \subset \mathbb{R}^k)} = (1-\epsilon)^{-k} \Theta_x(r) \\ &\leq (1-\epsilon)^{-k} (\Theta_x + \delta). \end{aligned} \tag{8}$$

Now choose ϵ small enough so that

$$(1-\epsilon)^{-k} \Theta_x \leq \Theta_x + \delta, \qquad (1-\epsilon)^{-k} \leq 2.$$

Putting these two inequalities together with (8) implies $\Theta_{x_j} \leq \Theta_x + 3\delta$. However, δ was arbitrary, so

$$\Theta_x \geq \limsup_{j \to \infty} \Theta_{x_j},$$

completing the proof. □

This phenomenon persists for limit of minimal submanifolds. If Σ^j is a sequence of minimal submanifolds converging (in some sense) to Σ, then the corresponding density functions cannot suddenly jump down. Intuitively, this says you can't suddenly lose volume when taking a limit of minimal submanifolds. On the other hand, you can suddenly gain volume (consider the limit of rescaled catenoids, or rescaled helicoids, where the scaling parameter goes to zero).

References

[CM] T. Colding and W. Minicozzi. Minimal Surfaces. Courant Lecture Notes in Mathematics, 1999.

[Si] L. Simon. Lecture on Geometric Measure Theory. Proc. Centre Math. Anal. Australia Nat. Univ. No. 3, 1983.