Pohozaev-type Identities

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In this note we collect various Pohozaev-type identities, starting with the classical inequality of [Poh].

Theorem 1. *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be an open, star-shaped (with respect to the origin) domain. Then the boundary value problem*

$$\Delta u + u^p = 0, \qquad u|_{\partial\Omega} = 0 \tag{1}$$

has a positive solution only if

$$p < \frac{n+2}{n-2}.$$

Proof. First observe that

$$\operatorname{div}(u\nabla u) = |\nabla u|^2 + u\Delta u = |\nabla u|^2 - u^{p+1}.$$

Now let

$$X = r\partial_r = \sum_{j=1}^n x_j \partial_j \Rightarrow \operatorname{div}(X) = n, \quad X(u) = \langle X, \nabla u\rangle = \sum_{j=1}^n x_j \frac{\partial u}{\partial x_j}.$$

Then

$$\begin{aligned}
\operatorname{div}\left[X(u)\nabla u - \left(\frac{1}{2}|\nabla u|^2 - \frac{u^{p+1}}{p+1}\right)X\right] &= X(u)\Delta u + \langle \nabla \sum_j x_j \partial_j u, \nabla u\rangle \\
&\quad - \left(\frac{n}{2}|\nabla u|^2 - \frac{nu^{p+1}}{p+1}\right) - \langle X, \nabla\left(\frac{1}{2}|\nabla u|^2 - \frac{u^{p+1}}{p+1}\right)\rangle \\
&= X(u)\Delta u + \langle \sum_{j,k} x_j \partial_j \partial_k u \partial_k + \sum_j \partial_j u \partial_j, \sum_j \partial_j u \partial_j\rangle \\
&\quad \left(\frac{n}{2}|\nabla u|^2 - \frac{nu^{p+1}}{p+1}\right) - \langle \sum_j x_j \partial_j, \sum_{j,k} \partial_j u \partial_j \partial_k u \partial_j - u^p \sum_j \partial_j u \partial_j\rangle \\
&= X(u)(\Delta u + u^p) + |\nabla u|^2 - \frac{n}{2}|\nabla u|^2 + \frac{nu^{p+1}}{p+1} \\
&\quad + \sum_{j,k} x_j \partial_k \partial_j \partial_k u - \sum_{j,k} x_k \partial_j u \partial_j \partial_k u \\
&= \left(\frac{2-n}{2}\right)|\nabla u|^2 - \frac{nu^{p+1}}{p+1}.
\end{aligned}$$

However,

$$\left(\frac{2-n}{2}\right)|\nabla u|^2 - \frac{nu^{p+1}}{p+1} = \left[\frac{2-n}{2} - \frac{n}{p+1}\right]u^{p+1} + \frac{2-n}{2}\operatorname{div}(u\nabla u).$$

Now integrate over Ω to get

$$\begin{aligned}\int_\Omega \left(\frac{2-n}{2} - \frac{n}{p+1}\right) u^{p+1} dV &= \int_\Omega \operatorname{div}\left[X(u)\nabla u - \left(\frac{1}{2}|\nabla u|^2 - \frac{u^{p+1}}{p+1}\right) X\right] dV \\ &= \int_{\partial\Omega} X(u)\langle \nabla u, N\rangle - \frac{1}{2}|\nabla u|^2 \langle X, N\rangle dA \\ &= \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle X, N\rangle dA.\end{aligned}$$

Here N is the unit normal to $\partial\Omega$, and we have integrated by parts, using the boundary condition $u|_{\partial\Omega} = 0$. Finally, the fact that Ω is star-shaped is exactly $\langle X, N\rangle > 0$, and so

$$\frac{2-n}{2} - \frac{n}{p+1} > 0 \Leftrightarrow p < \frac{n+2}{n-2}.$$

□

Equation (1) is the most elementary example of the critical exponent phenomena for semi-linear, elliptic PDE. In this case, a variational argument shows that the PDE has solutions for $p < \frac{n+2}{n-2}$, but we've just seen it has no solutions for $p \geq \frac{n+2}{n-2}$. This particular critical exponent of $p = \frac{n+2}{n-2}$ is strongly related to the critical exponent of the Sobolev embedding theorem. Indeed, the Sobolev embedding theorem states

$$W^{1,2}(\Omega) \hookrightarrow L^p(\Omega) \tag{2}$$

for $p \leq \frac{2n}{n-2} = \frac{n+2}{n-2} + 1$, and if the inequality is strict then the embedding is compact. In the case of equation (1), the nonexistence of solutions is related to the loss of compactness of the embedding (2).

One can understand theorem 1 more geometrically by observing that the vector field $X = \sum_{j=1}^n x_j \partial_j$ is a conformal Killing field. That is, the flow of X is a one-parameter family of conformal transformations of the ambient space $\mathbb{R}^n$. In the case of $X = \sum_{j=1}^n x_j \partial_j$, the conformal transformations are dilations. In general, if we write a conformal Killing field as $X = \sum_{j=1}^n X_j \partial_j$, then

$$\partial_j X_k + \partial_k X_j = \frac{2}{n}\operatorname{div}(X)\delta_{j,k}, \tag{3}$$

where $\delta_{j,k}$ is the Kronecker delta function. Using the same computation as in the proof of Theorem 1, we have (see section 9.1 of [PR]), provided $\Delta u + u^p = 0$,

$$\begin{aligned}\operatorname{div}\left(X(u)\nabla u - \left(\frac{1}{2}|\nabla u|^2 - \frac{u^{p+1}}{p+1}\right) X \right. &+ \left. \frac{n-2}{4n}\operatorname{div}(X)\nabla u^2 - u^2 \nabla \frac{2}{n}\operatorname{div}(X)\right) \\ &= \frac{1}{n}\operatorname{div}(X)\left(\frac{n}{p+1} - \frac{n-2}{2}\right) u^{p+1}.\end{aligned} \tag{4}$$

We can control the sign of the quantity on the right hand side of this equation, while the left hand side is the divergence of a vector field. This time, integrating by parts (and using $u|_{\partial\Omega} = 0$) yields

$$\int_\Omega \frac{1}{n}\operatorname{div}(X)\left(\frac{n}{p+1} - \frac{n-2}{2}\right) u^{p+1} dV = \int_{\partial\Omega} X(u)\langle\nabla u, N\rangle - \frac{|\nabla u|^2}{2}\langle X, N\rangle dA. \tag{5}$$

One can make sense of these sort of conservation laws for scalar curvature on an arbitrary Riemannian manifold with boundary. Let (Ω, g) be an n-dimensional, compact, Riemannian manifold with boundary, and let N be the unit outward normal vector along $\partial\Omega$. Denote its Ricci curvature by Ric and its scalar curvature by R. Also write the Lie derivative of any tensor A

with respect to the vector field X as $\mathcal{L}_x(A)$. As in the Euclidean case, a conformal Killing field is a vector field whose flow is a one-parameter family of conformal transformations. In other words, if we write $X = \sum_{j=1}^n X_j e_j$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame,

$$\nabla_{e_k} X_j + \nabla_{e_j} X_k = \frac{2}{n} \operatorname{div}(X) \delta_{j,k}, \tag{6}$$

where ∇_{e_j} is covariant differentiation in the e_j direction.

Schoen [Sc] proved the following general Pohozaev-type relation for scalar curvature.

Theorem 2. *Let (Ω, g) be a compact Riemannian manifold with boundary, and let X be a conformal Killing field on Ω. Then*

$$\int_\Omega \mathcal{L}_X(R) dV = \frac{2n}{n-2} \int_{\partial\Omega} \left(\operatorname{Ric}(X, N) - \frac{R}{n} g(X, N) \right) dA.$$

Proof. In our local frame,

$$\mathcal{L}_X(R) = \nabla R \cdot X = \sum_{j=1}^n X_j \nabla_{e_j} R.$$

Now integrate by parts:

$$\int_\Omega \mathcal{L}_X(R) dV = -\int_\Omega \operatorname{div}(X) R dV + \int_{\partial\Omega} R \langle X, N \rangle dA. \tag{7}$$

On the other hand,

$$\begin{aligned}
\int_\Omega \mathcal{L}_X(R) dV &= 2 \int_\Omega \sum_{i,j} X_i \nabla_{e_j} \operatorname{Ric}_{ij} dV = \int_\Omega \sum_{i,j} [X_i \nabla_{e_j} \operatorname{Ric}_{ij} + X_j \nabla_{e_i} \operatorname{Ric}_{ij}] dV \\
&= -\int_\Omega \sum_{i,j} [(\nabla_{e_j} X_i) \operatorname{Ric}_{ij} + (\nabla_{e_i} X_j) \operatorname{Ric}_{ij}] dV + 2 \int_{\partial\Omega} \operatorname{Ric}(X, N) dA \\
&= -\frac{2}{n} \int_\Omega \operatorname{div}(X) R + 2 \int_{\partial\Omega} \operatorname{Ric}(X, N) dA.
\end{aligned}$$

We have used the second Bianchi identity and integrated by parts. Adding this last equation to to equation (7), we have

$$\begin{aligned}
\left(1 - \frac{2}{n}\right) \int_\Omega \mathcal{L}_X(R) dV &= -\frac{2}{n} \int_\Omega \operatorname{div}(X) R dV + 2 \int_{\partial\Omega} \operatorname{Ric}(X, N) dA \\
&\quad + \frac{2}{n} \int_\Omega \operatorname{div}(X) R dV - \frac{2}{n} \int_{\partial\Omega} R \langle X, N \rangle dA \\
&= 2 \int_{\partial\Omega} \left(\operatorname{Ric}(X, N) - \frac{R}{n} \langle X, N \rangle \right) dA.
\end{aligned}$$

The theorem follows. ∎

The tensor $\operatorname{Ric} - \frac{R}{n} g$ is the trace-free part of the Ricci tensor.

In the special case that $g = u^{\frac{4}{n-2}} \sum_j dx_j^2$ is conformally flat the integral identity in Theorem 2 reduces to equation (5) for the conformal factor u. Previously, Kazdan and Warner [KW] had something similar to Theorem 2 in the special case of a sphere (with a metric conformal to the usual, round metric) and the dilation vector field one gets by pulling back $r\partial_r$ using stereographic projection.

One can apply Schoen's Pohozaev-type identity to study singular Yamabe metrics a finitely punctured sphere. In this case, we ask for a complete, constant scalar curvature, metric on

$S^n \backslash \Lambda$, which is conformal to the round metric. The completeness of the metric $g = u^{\frac{4}{n-2}}$ forces the conformal factor u to blow up as one approaches the singular set Λ. The dimension of the singular set Λ determines the sign of the scalar curvature. If $\dim(\Lambda) > \frac{n-2}{2}$, then scalar curvature must be negative [AM], while if $\dim(\Lambda) < \frac{n-2}{2}$ the scalar curvature is positive [SY].

In the case that the singular set $\Lambda = \{p_1, \ldots, p_k\}$ is a finite collection of points, the conformal factor u is asymptotically radial near each singular point p_j [CGS]. At this point, we pause to understand the radially symmetric asymptotes.

One can classify the radially symmetric constant scalar curvature metrics on $S^n \backslash \{\pm p\}$ by separating variables and solving the ODE for the conformal factor. If we reparameterize $S^n \backslash \{\pm p\} = S^{n-1} \times \mathbb{R}$, and let $u = u(t)$ be the conformal factor, then the equation requiring $R = n(n-1)$ becomes

$$\frac{d^2u}{dt^2} = \frac{(n-2)^2}{4} u - \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}.$$

In fact, this ODE has a conserved energy,

$$H\left(u, \frac{du}{dt}\right) = \left(\frac{du}{dt}\right)^2 - \frac{(n-2)^2}{4} u^2 + \frac{(n-2)^2}{4} u^{\frac{2n}{n-2}}. \tag{8}$$

The curve $(u(t), u'(t))$ associated to any radially symmetric, constant scalar curvature metric must thus lie on a level set of the energy H. In particular, there are periodic solutions $u_\epsilon(t)$, which are uniquely determined by their minimum value ϵ. The solution $u_\epsilon(t)$ lies on the level set

$$H = \frac{(n-2)^2}{4} (\epsilon^{\frac{2n}{n-2}} - \epsilon^2),$$

which is a closed curve in the phase plane. The constant solution

$$\left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$$

corresponds to a single point in the phase plane, and geometrically give a cylinder. As $\epsilon \to 0^+$, the solution converges to $u = \cosh^{\frac{2-n}{2}} t$, which gives the (incomplete) round metric on $S^n \backslash \{\pm p\}$. We will call the radially symmetric solution u_ϵ the Delaunay solution with necksize ϵ.

We can state the asymptotics theorem of [CGS] as follows. Let $g = u^{\frac{4}{n-2}} g_0$ be a constant positive scalar curvature metric which is complete in a closed, punctured ball $\bar{B} \backslash \{0\}$, and g_0 is the round metric (or the flat metric). Then for some ϵ with

$$0 < \epsilon \leq \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$$

we have

$$|u(x) - u_\epsilon(-\log(|x|))| = O(|x|).$$

Korevaar and Schoen proved a stronger asymptotic statement, see [KMPS].

We now use the Pohozaev identity of Theorem 2 to recover the energy $H(\epsilon)$ (and hence ϵ) as an integral invariant, following the proof given in [Pol]. Let $\Lambda = \{p_1, \ldots, p_k\} \subset S^n$ be a finite set of points, and let $(S^n \backslash \Lambda, g = u^{\frac{4}{n-2}} g_0)$ be a complete, constant positive scalar curvature metric Riemannian manifold, where g_0 is the usual round metric. Choose $\delta > 0$ small enough so that the balls $B_\delta(p_j)$ (in the round metric) are disjoint, and apply the integral identity of Theorem 2 to $\Omega = S^n \backslash \cup_{j=1}^k B_\delta(p_j)$. Because g has constant scalar curvature equal to $n(n-1)$, we have

$$0 = \sum_{j=1}^{k} \int_{\partial B_\delta(p_j)} (\mathrm{Ric}(X, N) - (n-1)\langle X, N \rangle)\, dA. \tag{9}$$

Notice that the boundary integrals are homology invariants, and so (in particular) they don't depend on δ, so long as δ is sufficiently small. We pick one singular point p_1, and move it to the north pole using a conformal motion. Now apply the Pohozaev identity with $X = r\partial_r$, the conformal dilation towards p_1. Reparameterize a neighborhood of p_1 using coordinates $S^{n-1} \times [0, \infty)$, where $-\log(|x|) = t \in [0, \infty)$, and write

$$u(t, \theta) = u_\epsilon(t) + O(e^{-t}), \qquad X = \partial_t, \qquad N = u^{-\frac{2}{n-2}} \partial_t.$$

Also, a computation gives

$$\mathrm{Ric}(X, N) - (n-1)\langle X, N\rangle = -\frac{2(n-1)}{n(n-2)} u_\epsilon'' u_\epsilon^{-\frac{n}{n-2}} + \frac{2(n-1)}{(n-2)^2} (u_\epsilon')^2 u_\epsilon^{\frac{2-2n}{n-2}} + O(e^{-t}).$$

Without loss of generality we can assume $\partial B_\delta(p_1) = \{t = T\}$ is a neck of u_ϵ, so that

$$u_\epsilon(T) = \epsilon, \qquad u_\epsilon'(T) = 0, \qquad u_\epsilon''(T) = \frac{(n-2)^2}{4}\epsilon - \frac{n(n-2)}{4}\epsilon^{\frac{n+2}{n-2}}.$$

Evaluating the integral, we recover the following theorem of [Pol]:

Theorem 3.

$$\int_{\partial B_\delta(p_1)} \mathrm{Ric}(X, N) - (n-1)\langle X, N\rangle dA = (n-1)(n-2)\,\mathrm{Vol}(S^{n-1}) \left(\epsilon^{\frac{2n}{n-2}} - \epsilon^2\right).$$

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