Extremal Domains for variational problems.

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Abstract

In this project we show the connection between three variational problems; eigenvalues of the Laplacian operator, density of the hyperbolic metric and expected lifetime of Brownian motion in a domain D. We show that both the first Dirichlet eigenvaule and the hyperbolic metric are monotone decreasing with expanding domain. We then show that finding the universal lower bound, with respect to the inradius of a domain, of the hyperbolic metric and finding the universal upper bound, with respect to the inradius of the domain, of the expected lifetime of a Brownian motion, both of which represent a Schlicht Bloch Landau problem. We also show that the expected lifetime of Brownian motion in a domain D and the torsion function from elasticity are one and the same. The first Dirichlet eigenvalue and the torsional rigidity have a similar variational characterization given by Rayeigh's theorem showing the connection between the first and the third variational problem. Armed with this connection we conjecture that the extremal domain of these three problems must be the same.

Contents

1 Introduction.

In this project we look at three variational problems: minimizing eigenvalues of the Laplacian operator, density of the minimizing hyperbolic metric, and maximizing the expected lifetime of Brownian motion in a domain D. We do this with an aim to show the connection between these variational problems and gain further insight into the extremal domains for each of these problems.

The first variational problem arises from the relation between the first Dirichlet eigenvalues and the geometry of underlying domains. We therefore make use Hayman's theorem [9] which states that for λ_D the first Dirichlet eigenvalue for the Laplacian in a simply connected domain D, there is a universal constant a such that $\lambda_D \geq \frac{a}{R_D^2}$ for R_D the radius of the largest disk contained in D. Studies have been done to find the best constant a and to identify the extremal domain in the inequality above. Our approach is to look at first Dirichlet eigenvalues for several basic domains so as to characterize the first Dirichlet eigenvalue from computations and relevant theorems.

The second variational problem goes on to look at properties of the density of the hyperbolic metric in simply connected domains. The density of the hyperbolic metric, $\sigma(z; D)$ represents a function of a conformal map of the unit disk onto a simply connected domain, D. Similar to variational problem 1, it is known from function theory that there is a universal constant c such that $\sigma_D = \inf_{z\in D} \sigma(z; D) \geq \frac{c}{R_D}$ and problems revolve around finding the best value of c known as the schlicht Bloch-Landau constant and the extremal domain.

The third variational problem looks at the expected lifetime of Brownian motion. For B_t the Brownian motion in domain D we denote $\tau_d(z) = \inf\{t > 0 : B_t \notin D\}$, where z is the initial starting point, as the first exit time of B_t from D. Thus $E(\tau_D(z))$ is the expectation of τ_D under the measure of the Brownian motion starting at the point z in D. It is known that, whenever D is a planar simply connected domain, then $\sup_{z\in D} E(\tau_D(z)) \leq bR_D^2$. Problems from the above inequality revolve around finding the best value of the constant b and the extremal domain for the inequalty. In our case we go on to look at the properties of the expeceted first exit time of Brownian motion in simply connected domains in relation to the the first two variational problems.

Looking at these problems simultaneously also allows for one to improve the value of the constant a in the first variational problem as was done by Bañuelos and Carroll [9] to show that $a > 0.6197$.

2 Eigenvalues of the Laplacian Operator.

This chapter is motivated by the following theorem from Hayman

Hayman's Theorem. *Let D be a simply connected domain in the complex plane. Let R_D be the inradius of the D, that is, the radius of the largest disk contained in D, and let λ_D be the first Dirichlet eigenvalue for the Laplacian in D. There is a universal constant a such that*

$$\lambda_D \geq \frac{a}{R_D^2}.$$

We compute first Dirichlet eigenvalues for various domains with a view to finding the extremal domain for the above inequality. We then look at general eigenvalue properties through several theorems thereafter.

Dirichlet eigenvalues are fundamental modes of vibrations of an idealized drum with a given shape. The fundamental mode is the pure tone of the lowest pitch or frequency and multiples of that frequency are called harmonic overtones. These help in deducing features of the shape of the drum. In this case the drum is thought of as an elastic membrane Ω which is represented as a planar domain whose boundary is fixed. Dirichlet boundary conditions specify that the boundary of the membrane is fixed.

2.1 Examples.

2.1.1 Taut string analogy.

We consider the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad \text{for } 0 < x < L, t > 0 \qquad (2.1.1)\\ u(x,0) &= \phi(x), u_t(x,0) = \psi(x), 0 \leq x \leq L \\ u(0,t) &= u(L,t) = 0, \text{for } t \geq 0. \end{aligned}$$

The above describes a taut string stretched between points at $x = 0$ and $x = L$ which are held fixed. At $t = 0$, the string takes an initial shape given by $\phi(x)$ and initial velocity profile $\psi(x)$.
Using separation of variables to obtain the general solution,

$$u(x,t) = X(x)T(t) \tag{2.1.2}$$

substituting (2.1.2) into (2.1.1) we get

$$X(x)T'' = c^2 X'' T(t)$$

which we can write as,

$$\frac{T''}{c^2 T(t)} = \frac{X''}{X(x)}$$

The left hand side of the equation above is a function of t only while the right hand side is a function of x only. The only way to maintain equality for all $0 < x < L$ and all $t > 0$ is if each of the sides is equal to a constant. We introduce a separation constant, λ such that:

$$\frac{T''}{c^2T(t)} = \frac{X''}{X(x)} = -\lambda$$

We now have two ordinary differential equations $\begin{cases} T'' &= -c^2\lambda T(t) \\ X'' &= -\lambda X(x) \end{cases}$

Looking at the boundary information first:

$$u(0,t) = 0 = X(0)T(t)$$

(Trivially $T(t) = 0$ is a solution but not an insightful one)

$$u(x,t) = 0 \qquad \text{for } t > 0, 0 < x < L$$
$$\text{also } u(L,t) = 0 = X(L)T(0) \text{ is also solved by } X(L) = 0.$$

Now $X'' = -\lambda X(x)$ subject to $X(0) = X(L) = 0$ thus solvable for certain values of λ, the eigenvalues.
Determination of the sign of λ:
assume λ is negative say $\lambda = -k^2$ then $X'' = k^2X(x)$ with general solution

$$\begin{aligned} X &= Ae^{kx} + Be^{-kx} && \text{(A,B) arbitrary constants} \\ X(0) &= X(L) = 0 \\ \text{we find: } 0 &= A + B \\ \text{and } 0 &= Ae^kL + Be^-kL \\ \text{so } 0 &= Ae^{kL} - Ae^{-kL} \\ \text{i.e. } e^{kL} &= e^{-kL} && A \neq 0. \end{aligned}$$

This gives a contradiction unless $k = 0$ which is not useful. So we have shown that λ is positive. Hence $\lambda = k^2$, with general solution $X = A\cos kX + B\sin kX$
Now boundary conditions yield

$$\begin{aligned} 0 &= A && \text{and} && 0 = B\sin kL \\ \text{hence from boundary conditions at } x = L : kL &= n\pi && n = 1, 2, 3, \ldots \\ k &= \frac{n\pi}{L} \\ \therefore X &= B\sin\frac{n\pi X}{L} \end{aligned}$$

Now $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ and $X_n(x) = B_n \sin\dfrac{n\pi X}{L}$.

Equation for T is $T'' = -c^2k^2T$ so the general solution is given by

$$T = C\cos kct + D\sin kct \qquad \text{i.e.} \qquad T_n = C_n\cos\frac{n\pi ct}{L} + D_n\sin\frac{n\pi ct}{L}$$

We can write (2.1.2) as

$$U_n(x,t) = X_n(x)T_n(t)$$

$U_n(x,t) = \{C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\}\sin\frac{n\pi X}{L}$ with B_n absorbed into C_n and D_n.

To satisfy initial conditions we form a linear combination of all possible solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right\}\sin\frac{n\pi X}{L}$$
$$u_t(x,t) = \sum_{n=1}^{\infty}\left\{\frac{-n\pi c}{L}a_n\sin\frac{n\pi ct}{L} + \frac{n\pi c}{L}b_n\cos\frac{n\pi ct}{L}\right\}\sin\frac{n\pi X}{L}$$
$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty}a_n\sin\frac{n\pi X}{L}.$$

Definition. *A set $\{e_i | i \in \mathbb{N}\}$ is a basis for a Hilbert space $\mathcal{H}$ if every x can be expressed uniquely in the form*

$$x = \sum_{i=1}^{\infty}x_i e_i$$

for some x_i in the field of scalars. If in addition $\{e_i | i \in \mathbb{N}\}$ is an orthonormal set, then we refer to it as an orthonormal basis, or a complete orthonormal sequence.

We know that $\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos nx, \sqrt{\frac{2}{\pi}}\sin nx\}_{n=1}^{\infty}$ forms an orthonormal basis for the real $L^2[0,\pi]$.

Using the Fourier sine series expansion of the above

$$a_n = \frac{2}{L}\int_0^L \phi(\zeta)\sin\frac{n\pi\zeta}{L}d\zeta$$

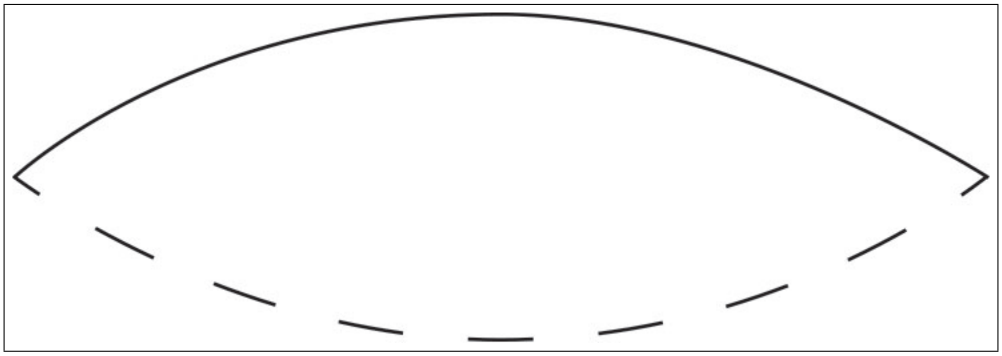
$$\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty}b_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\frac{L}{n\pi c}\int_0^L \psi(\zeta)\sin\frac{n\pi\zeta}{L}d\zeta$$

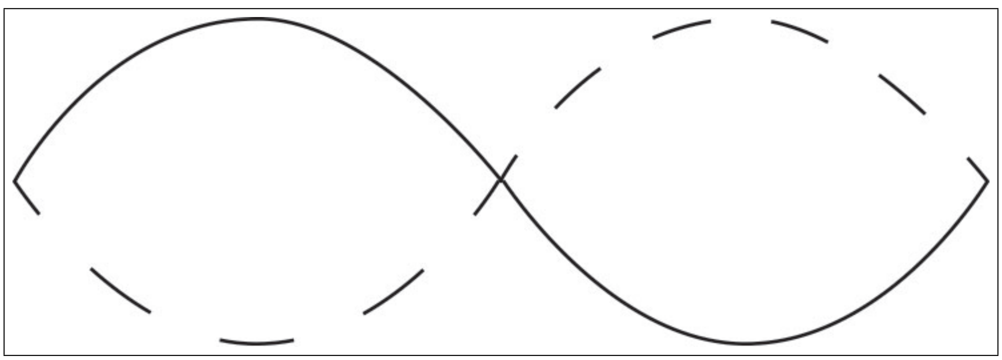
$$u(x,t) = \sum_{n=1}^{\infty}\left\{\left(\frac{2}{L}\int_0^L \phi(\zeta)\sin\frac{n\pi\zeta}{L}d\zeta\right)\cos\frac{n\pi ct}{L} + \left(\frac{2}{n\pi c}\int_0^L \psi(\zeta)\sin\frac{n\pi\zeta}{L}d\zeta\right)\sin\frac{n\pi ct}{L}\right\}\sin\frac{n\pi x}{L}.$$

Each succesive term involves the basic solutions. $\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$ and $\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$

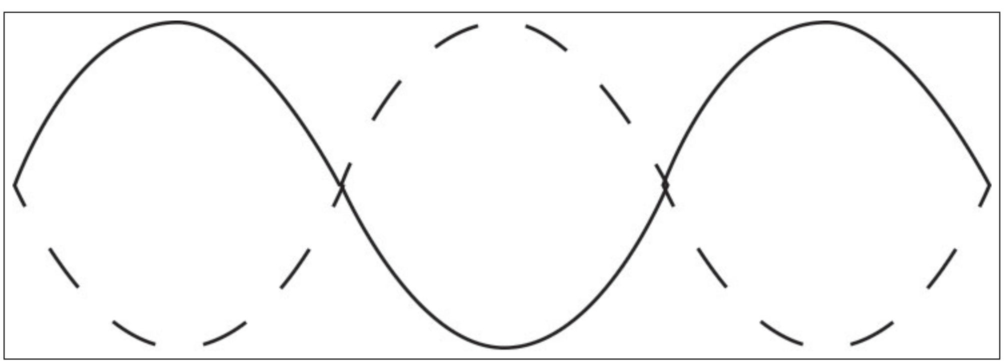
$n = 1$ fundamental mode: $\sin \frac{\pi ct}{L} \sin \frac{\pi x}{L}$ frequency: $w_1 = \frac{\pi c}{L}$



$n = 2$ first harmonic: $\sin \frac{2\pi ct}{L} \sin \frac{2\pi x}{L}$ frequency: $w_2 = \frac{2\pi c}{L}$



$n = 3$ second harmonic: $\sin \frac{3\pi ct}{L} \sin \frac{3\pi x}{L}$ frequency: $w_3 = \frac{3\pi c}{L}$



2.1.2 Rectangle analogy.

Consider now a taut elastic membrane which is confined to the region $0 \leq x \leq a$ and $0 \leq y \leq b$ and which has stationary boundaries at $x = 0, x = a, y = 0$ and $y = b$. Suppose also that there are initial conditions of shape $\phi(x, y)$ and zero velocity.That is, we have the problem

$$\begin{aligned} u_{tt} &= c^2(u_{xx}+u_{yy}) : 0 \le x \le a, 0 \le y \le b, t > 0 \\ u(0,y,t) &= u(a,y,t) = 0 : 0 \le y \le b, t > 0 \\ u(x,0,t) &= u(x,b,t) = 0 : 0 \le x \le a, t > 0 \\ u(x,y,0) &= \phi(x,y) : 0 \le x \le a, 0 \le y \le b \\ u_t(x,y,0) &= 0 : 0 \le x \le a, 0 \le y \le b. \end{aligned}$$

Again, using the usual sepation techniques we obtain eigenvalues of the form

$$\lambda_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}} \qquad \text{for } n = 1,2,\ldots m = 1,2,\ldots$$

2.1.3 Infinite strip analogy.

$$\begin{aligned} u_{tt} &= c^2(u_{xx}+u_{yy}) : -\infty \le x \le \infty, 0 \le y \le b, t > 0 \\ u(x,0,t) &= u(x,b,t) = 0 : -\infty \le x \le \infty, t > 0 \\ u(x,y,0) &= \phi(x,y) : -\infty \le x \le \infty, 0 \le y \le b \\ u_t(x,y,0) &= 0 : -\infty \le x \le \infty, 0 \le y \le b. \end{aligned}$$

The above problem is taken physically (and mathematically) as a limit of very long rectangles. After taking into consideration the above mentioned conditions, the first dirichlet eigenvalue of the infinite strip comes out to,

$$\lambda_1(D) = \frac{\pi^2}{b^2}.$$

In fact, it was proved by J. Hersch [12]that $\lambda_D \ge \frac{\pi^2}{4R_D^2}$ for convex D, with equality if and only if D is an infinite strip.

2.1.4 Disk analogy.

(We now look at the wave equation in two spatial dimensions)
$u_{tt} = c^2(u_{xx}+u_{yy})$.
To change to spherical coordinates we consider the following
$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\frac{y}{x}$

In general,

$$\Delta u = u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}\Delta_\theta u.$$

In our case, $(n = 2)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

So the wave equation changes to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right\}.$$

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) \qquad (0 < r < a, t > 0).$$

With boundary held fixed at $r = a$

$$u(a, \theta, t) = 0 \qquad 0 \leq \theta \leq 2\pi, t > 0$$

and some initial displacement

$$u(r, \theta, 0) = \phi(r, \theta) \qquad 0 \leq r \leq a, 0 \leq \theta \leq 2\pi$$

but zero initial velocity, say

$$u_t(r, \theta, 0) = 0 \qquad 0 \leq r \leq a, 0 \leq \theta \leq 2\pi$$

with usual assumption about separability

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t)$$

the partial differential equation becomes

$$R\Theta T'' = c^2 \left\{ R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T \right\}$$

dividing by $R\Theta T c^2$:

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

The left hand side is a functon of t, the right hand side is a function of r and θ

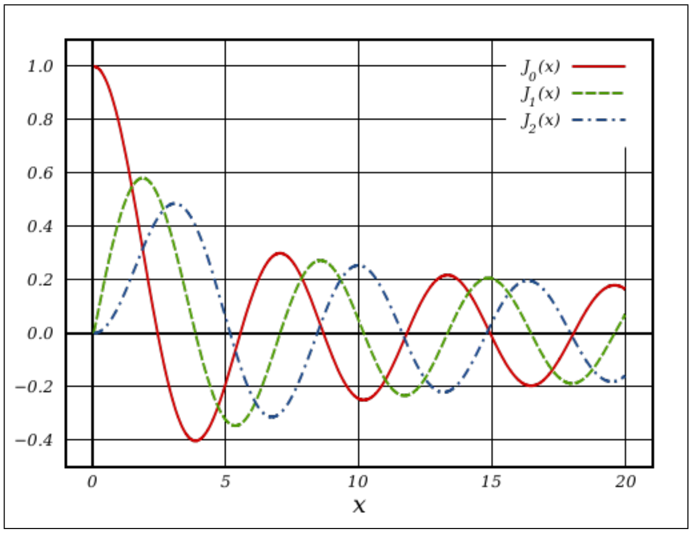
Now $$T'' = -c^2k^2T \qquad (2.1.3)$$

and $$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = -r^2k^2 \qquad (2.1.4)$$

with boundary conditions on $R(r)$: $R(a) = 0$, $R'(0) = 0$
For the T equation: $T = a\cos kct + b\sin kct$
further separation of (2.1.4)

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 = -\frac{\Theta''}{\Theta} = \nu^2$$

thus $$\Theta'' = -\nu^2\theta$$
$$\Theta = c\cos\nu\theta + d\sin\nu\theta \qquad \theta = \theta + 2\pi$$



Several orders of Bessel functions of the first kind

so the equation in R reads as follows

$$r^2R'' + rR' + r^2k^2R - \nu^2R = 0$$

$$\text{or} \qquad r^2R'' + rR' + (r^2k^2 - \nu^2)R = 0.$$

Above is the Bessel equation and the solution is given by

$$R = f\mathcal{J}_\nu(kr) + g\mathcal{Y}_\nu(kr)$$

with $\mathcal{J}_\nu$=bessel function of first kind, order ν
and $\mathcal{Y}_\nu$=bessel function of second kind, order ν.
Bessel functions of the second kind all go to $-\infty$ as $r \to 0$. Such functions cannot be solutions as the centre of the membrane must have finite displacement.

$$R(r) = f\mathcal{J}_n(kr)$$

using boundary conditions $u(a, \theta, t) = 0, 0 \leq \theta \leq 2\pi, t \geq 0$

$$R(a)\Theta(\theta)T(t) = 0$$

we have $R(a) = 0$ thus $\mathcal{J}_n(kr) = 0$ solves

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

to get, using series expansion, the series

$$y(x) = \mathcal{J}_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

Bessel functions are aperiodic thus, the spacing of zero is not constant. Subscripting the constant k

$$\mathcal{J}_n(k_{nm}a) = 0 \qquad \text{m=zero of } \mathcal{J}_n \text{ at } r = a.$$

Now,

$$u_{nm}(r, \theta, t) = (a_{nm} \cos k_{nm}ct + b_{nm} \sin k_{nm}ct)\mathcal{J}_n(k_{nm}r)(c_n \cos n\theta + d_n sinn\theta)$$

using initial condtions

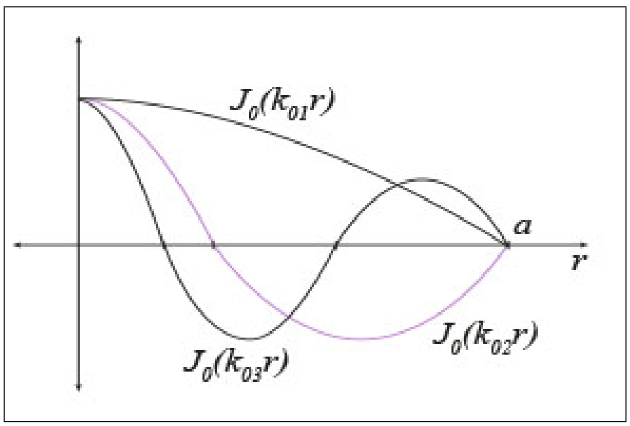
$$u_t(r, \theta, 0) = R(r)\Theta(\theta)T'(0) = 0.$$

So $T'(0) = 0$ which results in the choice of $b_{nm} = 0$

$$u_{nm}(r, \theta, t) = (\cos k_{nm}ct)\mathcal{J}_n(k_{nm}r)(c_n \cos n\theta + d_n sinn\theta)$$

forming double Fourier series (and also absorbing a_{nm} into c_{nm} and d_{nm})

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\cos k_{nm}ct)\mathcal{J}_n(k_{nm}r)(c_n \cos n\theta + d_n sinn\theta).$$



As for the eigenvalues, we recall that λ was our separation constant.$\lambda = c^2k^2$. Rewritten $k = \frac{\sqrt{\lambda}}{c}$. We therefore seek λ so that

$$\mathcal{J}_0\left(\frac{\sqrt{\lambda}}{c}r\right) = 0$$

Values of λ satisfying the above equation are the eigenvalues of this problem[4]. From our knowledge of Bessel functions, there is an infinte sequence of positive numbers $j_1, \ldots, j_n, \ldots$ which tend to ∞ as n increases, and satisfying $\mathcal{J}_0(j_n) = 0$. These are the positive zeros of $\mathcal{J}_0$. We now choose the numbers λ to satisfy

$$\frac{\sqrt{\lambda}}{c}r = j_n$$

with n any positive integer. The eigenvalues are the numbers

$$\lambda_n = \left(\frac{j_n c}{r}\right)^2.$$

To this end, knowing that first zero of the Bessel function of the first kind $j_0 \approx 2.4048$ and comparing the above mentioned 4 domains, it is clear that the infinte strip is the best candidate for the extremal domain for the variational problem proposed by Hayman[9]. Moreover, it is extremal for all convex domains as was proved by Hersch[12].

2.2 Eigenvalue Properties.

All the theorems, proofs and more in this section are given in greater detail in Chavel's book[3].

Spectral Theorem. *Let T be a compact self-adjoint operator on a Hilbert space $(\mathcal{H}, <\centerdot,\centerdot>)$. Then there exists a finite or infinite sequence $(\mu_k)_{k=1}^{\infty}$ of real eigenvalues and a finite or infinite orthonormal sequence $(x_k)_{k=1}^{\infty}$ of corresponding eigenvectors such that for each $x \in \mathcal{H}$*

$$T(x) = \sum_k \mu_k < x, x_k > x_k.$$

If the sequence $(x_k)_{k=1}^{\infty}$ is infinite, then it converges to zero.

Relating the above to the Laplacian operator, $\Delta = \mathbf{1} - T$ where T is a compact operator. Applying the spectral theorem to the Laplacian operator we see that $\mu_k = \frac{1}{\lambda_k}$.

For the Dirichlet eigenvalue problem ($\Delta\phi + \lambda\phi = 0$, satisfying boundary conditions $\phi = 0$ on ∂D), the set of eigenvalues consists of a sequence

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \uparrow +\infty,$$

and each associated eigenspace is finite dimensional. Eigenspaces belonging to distinct eigenvalues are orthogonal in $L^2(D)$ and $L^2(D)$ is the direct sum of all the eigenspaces. Furthermore, each eigenfunction is C^∞ (smooth) on $(\overline{D})$.

As soon as we know that the eigenfunction $\phi \in C^2(D) \cap C^1(\overline{D})$, then its eigenvalue λ must be nonnegative. One uses $\phi|_{\partial D} = 0$, sets $f = h = \phi$ and applies the following Green formula

$$\int_D \{h\Delta f + < \nabla h, \nabla f >\} dV = 0$$

to obtain

$$\begin{aligned} 0 &= \int_D \{\phi\Delta\phi + < \nabla\phi, \nabla\phi >\} dV \\ &= \int_D \{-\lambda\phi^2 + |\nabla\phi|^2\} dV \\ \implies \lambda &= \frac{\int_D |\nabla\phi|^2 dV}{\int_D \phi^2 dV} \geq 0. \end{aligned}$$

We also note that the orthogonality of eigenspaces is a direct consequence of the Green formulas

$$\int_D \{h\Delta f - f\Delta h\} dV = \int_{\partial D} \{h\frac{\partial f}{\partial n} - f\frac{\partial h}{\partial n}\} dA = 0$$

because $h = f = 0$ on ∂D.
($\frac{\partial f}{\partial n}$ is the directional derivative of f in the direction of the outward pointing normal $\overrightarrow{n}$).

Indeed, let ϕ, ψ be eigenfunctions of the respective eigenvalues λ, τ. Then

$$0 = \int_D \{\phi\Delta\psi - \psi\Delta\phi\} dV = (\lambda - \tau) \int_D \phi\psi dV$$

then orthogonality of eigenfunctions follows.
We refer to the dimension of such eigenspaces as the *multiplicity of the eigenvalue.* Henceforth, it is useful to write the eigenvalue sequence

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \uparrow +\infty,$$

with each eigenvalue repeated according to its multiplicity.

Rayleigh's Theorem. *We are given a normal domain with fixed eigenvalue problem having the function space $\mathcal{G}(D)$, a Hilbert space, and eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \ldots,$$

where each eigenvalue is repeated the number of times equal to its multiplicity. Then for any $f \in \mathcal{G}(D), f \neq 0$, we have

$$\lambda_1 \leq \frac{\int_D |\nabla f|^2 dV}{\|f\|^2}$$

with equality if and only if f is an eigenfunction of λ_1. If $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal basis of $L^2(D)$ such that ϕ_j is an eigenfunction of λ_j for each $j = 1, 2, \ldots$ then for $f \in \mathcal{G}(D), f \neq 0$, satisfying

$$(f, \phi_1) = \cdots = (f, \phi_{k-1}) = 0. \tag{*}$$

we have the inequality

$$\lambda_k \leq \frac{\int_D |\nabla f|^2 dV}{\|f\|^2}$$

with equality if and only if f is an eigenfunction of λ_k.

Proof. The argument is based on the consideration that if ϕ is an eigenfunction, and $f \in \mathcal{G}(D)$, then

$$-\lambda \int_D \phi f dV = \int_D < \Delta\phi, f > dV = -\int_D < \nabla\phi, \nabla f > dV$$

is valid.
For any given $f \in \mathcal{G}(D)$ set

$$\alpha_j = \int_D < f, \phi_j > dV.$$

For $k > 1$, (*) is equivalent to saying $\alpha_1 = \cdots = \alpha_{k-1} = 0$. So for all $k = 1, 2, \ldots$ and $r = k, k+1, \ldots$ we have

$$\begin{aligned}
0 &\leq \int_D < \nabla(f - \sum_{j=k}^{r} \alpha_j \phi_j), \nabla(f - \sum_{j=k}^{r} \alpha_j \phi_j) > dV \\
&= \int_D < \nabla f, \nabla f > dV - 2\sum_{j=k}^{r} \alpha_j \int_D < f, \phi_j > dV + \sum_{j,l=k}^{r} \alpha_j \alpha_l \int_D < \phi_j, \phi_l > dV \\
&= \int_D < \nabla f, \nabla f > dV + 2\sum_{j=k}^{r} \alpha_j \int_D < f, \Delta\phi_j > dV - \sum_{j,l=k}^{r} \alpha_j \alpha_l \int_D < \phi_j, \Delta\phi_l > dV \\
&= \int_D < \nabla f, \nabla f > dV - \sum_{j=k}^{r} \lambda_j \alpha_j^2.
\end{aligned}$$

We conclude that

$$\sum_{j=k}^{\infty} \lambda_j \alpha_j^2 \leq \int_D |\nabla f|^2 dV < +\infty$$

and

$$\int_D < \nabla f, \nabla f > dV \geq \sum_{j=k}^{\infty} \lambda_j \alpha_j^2 \geq \lambda_k \sum_{j=k}^{\infty} \alpha_j^2 = \lambda_k \|f\|^2$$

and the inequality for λ_1 follows easily.(case of $k = 1$) □

From Rayleigh's theorem we now have a variational characterization of the first eigenvalue:

$$\lambda_1(D) = \inf\left\{ \frac{\int_D |\nabla f|^2 dV}{\int_D f^2 dV} | f \in C_0^2(D) \right\}.$$

In light of this it is also important to note the domain monoticity of the first Dirichlet eigenvalue:
so if $D_1 \subset D_2$ then $\lambda_1(D_1) \geq \lambda_1(D_2)$. The larger the domain, the smaller the eigenvalue.

Nodal sets.
The nodal set,N, is the set of points in D such that the eigenfunctions of $-\Delta\phi = \lambda\phi$ in D are zero.

$$N = \{x \in D : \phi(x) = 0\}.$$

Nodal set allow us to visualize the sets where $\phi(x) > 0$ or $\phi(x) < 0$. They mark a division (natural one) of the domain into regions. First Dirichlet eigenvalue ensures the absence of nodes hence the first eigenfunction $\phi_1(x)$ has a sign in D thus $\phi_n(x)$ must change its sign in D.

Courant's Nodal Domain Theorem. *Let $\lambda_1 \leq \lambda_2 \leq \ldots \uparrow$ be our list of eigenvalues and $\{\phi_1, \phi_2, \ldots\}$ a complete orthonormal basis of $L^2(D)$ with each ϕ_j an eigenfunction of $\lambda_j, j = 1, 2, \ldots$. Then the number of nodal domains of ϕ_k is less than or equal to k, for every $k = 1.2. \ldots$.*

Proof. We prove by contradiction. Suppose there are at least $k+1$ nodal domains. Let $N_1, \ldots, N_k, N_{k+1}, \ldots$ be nodal domains of ϕ_k. For each $j = 1, \ldots, k$ define

$$\psi_j = \begin{cases} \phi_k|N_j & \text{on} N_j \\ 0 & \bar{D} - N_j \end{cases}$$

One then obtains, as above, the existence of a nontrivial function

$$f = \sum_{k=1}^{k} \alpha_j \phi_j$$

satisfying

$$(f, \phi_1) = \cdots = (f, \phi_{k-1}) = 0.$$

One verifies that $\phi_j \in \mathcal{G}(D)$ for each $j = 1, \ldots, k$. Then Rayleigh's theorem, the max-min method and the divergence theorem imply

$$\lambda_k \leq \frac{\int_D |\nabla f|^2 dV}{\|f\|^2} \leq \lambda_k.$$

So f is therefore, an eigenfunction of λ_k vanishing identically on N_{k+1}. But then the maximum principle states that for a bounded domain D with $\partial D \in C^1(D)$ and $f \in C^2(D) \cap C^0(\bar{D})$, $\sup_{x\in D} f(x) = \sup_{x \in \partial D} f(x)$ implying that f vanishes identically on D which is a contradiction. □

For completeness we make note of *Weyl's asymptotic theorem* without proof. According to the Weyl formula,

$$\lim_{\lambda \to +\infty} \left[\frac{N(\Omega, \lambda) - C_{d,W}\mu_d(\Omega)\lambda^d}{\lambda^d} \right] = 0, \quad \lambda \to +\infty,$$

where

- if we let Δ_D be the Dirichlet Laplacian on Ω then $N(\Omega, \lambda)$ is the number its eigenvalues lying below λ^2.

- $\mu_d(\Omega)$ is the d-dimensional volume of Ω.
- ω_d is the volume of the unit ball in $\mathbb{R}^d$ and $C_{d,W} := (2\pi)^{-d}\omega_d$ is the standard Weyl constant.

The following theorem is a classical eigenvalue inequality due separately to Faber[7] and Krahn[8].

Faber-Krahn's Theorem. *Let $D \subset \mathbb{R}^n$ be a bounded domain and let B be the ball centered at the origin with $Vol(D) = Vol(B)$. Then $\lambda_1(D) \geq \lambda_1(B)$, with equality if and only if $D = B$ almost everywhere.*
Here again, $\lambda_1(D)$ is the first eigenvalue of the Laplacian, with Dirichlet boundary conditions.

In other words, the first Dirichlet eigenvalue is no less than the corresponding Dirichlet eigenvalue of a Euclidean ball having the same volume. Furthermore, the inequality is sharp in the sense that if the first Dirichlet eigenvalue is equal to that of the corresponding ball, then the domain must actually be a ball.

Proof. Recall variational characterization of the first eigenvalue:

$$\lambda_1(D) = \inf\left\{\frac{\int_D |\nabla u|^2 dV}{\int_D u^2 dV} | u \in C_0^2(D)\right\} \tag{a}$$

By the Courant nodal domain theorem, we can take a test function for the Rayleigh quotient be non-negative. Let u be a test function, and for $0 \leq t \leq \hat{u} = max(u)$. Let $D_t = \{u > t\}$.

Now we define a comparison function $u_* : B \to [0, \infty)$ as follows. First let B_t be the ball centered at the origin with $Vol(B_t) = Vol(D_t)$. Then let u_* be the radially symmetric function such than $B_t = \{u_* > t\}$. By the co-area formula,

$$\int_t^{\hat{u}} \int_{\partial D_\tau} \frac{dA}{|\nabla u|} d\tau = Vol(D_t) = Vol(B_t) = \int_t^{\hat{u}} \int_{\partial B_\tau} \frac{dA}{|\nabla u_*|} d\tau.$$

(The co-area formula expresses the integral of a function over an open set in Euclidean space in terms of the integral of the level sets of another function.)
Differentiating with respect to t gives us

$$\int_{\partial D_t} \frac{dA}{|\nabla u|} = \int_{\partial B_t} \frac{dA}{|\nabla u_*|} \tag{b}$$

for all t. Then

$$\begin{aligned}\int_D u^2 dV &= \int_0^{\hat{u}} \int_{\partial D_t} \frac{u^2 dA}{|\nabla u|} dt = \int_0^{\hat{u}} t^2 \int_{\partial D_t} \frac{dA}{|\nabla u|} dt \\ &= \int_0^{\hat{u}} t^2 \int_{\partial B_t} \frac{dA}{|\nabla u_*|} dt = \int_B u_*^2 dV.\end{aligned} \tag{c}$$

Now, for $0 \leq t \leq \hat{u}$ let

$$\psi(t) = \int_{D_t} |\nabla u|^2 dV, \qquad \psi_*(t) = \int_{B_t} |\nabla u_*|^2 dV.$$

by the co-area formula

$$\psi' = -\int_{\partial D_t} |\nabla u| dA, \qquad \psi_*' = -\int_{\partial B_t} |\nabla u_*| dA.$$

We use the Cauchy-Schwarz inequalty, the isoperimetric inequality, and the fact that the derivative of u_* is constant on ∂B_t (*since u_* is radial*) to see

$$\begin{aligned}\left(\int_{D_t} |\nabla u| dA\right)\left(\int_{\partial D_t} \frac{dA}{|\nabla u|}\right) &\geq \left(\int_{\partial D_t} dA\right)^2 = (\text{Area}(\partial D_t))^2 \\ &\geq (\text{Area}(\partial B_t))^2 = \left(\int_{B_t} |\nabla u_*| dA\right)\left(\int_{\partial B_t} \frac{dA}{|\nabla u_*|}\right).\end{aligned}$$

(The isoperimetric inequality is a geometric inequality involving the square of the circumference of a closed curve in the plane and the area of a plane region it encloses, as well as its various generalizations.)
We use equation (b) to cancel the common factor of

$$\int_{\partial D_t} \frac{dA}{|\nabla u|} = \int_{\partial B_t} \frac{dA}{|\nabla u_*|},$$

and so

$$-\psi' = \int_{D_t} |\nabla u| dA \geq \int_{B_t} |\nabla u_*| dA = -\psi_*'.$$

Integrating this last differential inequality and using $\psi(\hat{u}) = 0 = \psi_*(\hat{u})$ we see

$$\int_D |\nabla u|^2 dV = \psi(0) \geq \psi_*(0) = \int_B |\nabla u_*|^2 dV.$$

Combine this inequalty with (c) and (a) to give the desired inequality on the eigenvalues:

$$\lambda_1(D) \geq \lambda_1(B).$$

Moreover, equality of the eigenvalues forces the level sets ∂D_t to be all be spheres centered at the origin. Also, the equality case of the Cauchy-Schwarz inequality forces $|\nabla u|$ to be constant on the level set ∂D_t. Thus u must be radially symmetric, and so in this case $u = u_*$. □

Alternatively the Faber-Krahn inequality can be stated as follows,
A geometric isoperimetric inequality:

$$Vol(D) = Vol(B) \Rightarrow Area(\partial D) \geq Area(\partial B),$$

where B is a ball, implies a physical isoperimetric inequality:

$$Vol(D) = Vol(B) \Rightarrow \lambda_1(\partial D) \geq \lambda_1(\partial B).$$

At this point it suffices to state Hersch's theorem[12] reinforced by the Faber-Krahn inequality as follows:
If $D \subset \mathbb{R}^n$ is convex with in-radius, $R_D = 1$ then

$$\lambda_1(D) \geq \lambda_1(\text{slab with width 1}).$$

3 Hyperbolic density of simply connected domains.

In this section we aim to show the relation between Hayman's inequality[9] and the density of the hyperbolic metric in a simply connected domain in the complex plane. We first gather all the background concepts concerning the density metric, looking at a number of essential mapping theorems and theory of conformal mappings in great detail, before computing density metrics for a few simply connected domains and finally analysing the properties of this metric therein showing the link with the first variational problem.

3.1 Conformal Mappings.

Basic theory of conformal mappings.
A mapping $f : A \to B$ is called conformal if, for each $z_0 \in A$, f rotates tangent vectors to curves through z_0 by a finite angle θ and stretches them by a definite factor r. Simply put a mapping which preserves magnitudes and direction of angles.
The **conformal mapping theorem** states that *if $f : A \to B$ and $f'(z_0) \neq 0$ for each $z_0 \in A$ then f is conformal.*
Points where $f'(z_0) = 0$ form singular points and at such points angles cease to be preserved and hence f ceases to be conformal.
The conformal property may be described in terms of the Jacobian derivative matrix of a coordinate transformation. If the Jacobian matrix of the transformation is everywhere a scalar times a rotation matrix, then the transformation is conformal. So in real coordinates

$$\begin{aligned} f &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \\ &= \sqrt{a^2+b^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$

Inverse Function Theorem. *Let $f : A \to C$ be analytic (with f'continous) and assume that $f'(z_0) \neq 0$. Then there exists a neighbourhood V of $f(z_0)$ such that $f : U \to V$ is a bijection and its inverse function f^{-1} is analytic with derivative given by*

$$\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \qquad \textit{where } w = f(z).$$

The above only allows us to conclude the existence of a local inverse for f.

Proposition 1. *1. If $f : A \to B$ is conformal and bijective then $f^{-1} : B \to A$ is also conformal.*

2. If $f : A \to B$ and $f : B \to C$ are conformal and bijective, then $g \circ f : A \to C$ is conformal and bijective.

Proof. 1. Since f is bijective, the mapping f^{-1} exists. By the inverse function theorem, f^{-1} is analytic with

$$\frac{d}{dw}f^{-1}(w) = \frac{1}{\frac{d}{dz}f(z)} \qquad \text{where } w = f(z) \text{ so } \frac{d}{dw}f^{-1}(w) \neq 0.$$

$\therefore f^{-1}$ is also conformal.

2. Certainly $g \circ f$ is bijective and analytic, since g and f are. (inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.) The derivative of $g \circ f$ at z is $g'(f(z)) \circ f'(z) \neq 0 \therefore g \circ f$ is conformal by definition.

□

Conformal maps we have discussed thus far focus on regions which are open connected sets but an extension to closed sets can be made via the following theorem.

Osgood-Caratheodory Theorem. *If A_1 and A_2 are bounded simply connected regions whose boundaries γ_1 and γ_2 are simple continuous closed curves, then any conformal map of A_1 one-to-one onto A_2 can be extended to a continuous map of $A_1 \cup \gamma_1$ one-to-one onto $A_2 \cup \gamma_2$*

Fractional Linear Transformations.
The simplest kind of a conformal mapping is the fractional linear transformation *(known also as Möbius or bilinear transformation)* which is a mapping of the form

$$T(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are fixed complex numbers for $ad - bc \neq 0$ otherwise T would be constant.

Proposition 2. *The map T defined by the fractional linear transformation is bijective and conformal from*

$$A = \left\{ z | cz + d \neq 0, \ i.e.\ z \neq \frac{-d}{c} \right\} \ \text{onto } B \ = \left\{ w | w \neq \frac{a}{c} \right\}.$$

Inverse is also a fractional linear transformation given by

$$T^{-1}(w) = \frac{-dw+b}{cw-a}.$$

Proof. Indeed T is analytic on A and $S(w) = \frac{(-dw+b)}{(cw-a)}$ is analytic on B.
T is bijcetive if we can show that $T \circ S$ and $S \circ T$ are identities because T will

have S as its inverse. We proceed as follows;

$$\begin{aligned} T(S(w)) &= \frac{a(\frac{-dw+b}{cw-a})+b}{c(\frac{-dw+b}{cw-a})+d} \\ &= \frac{-adw+ab+bcw-ab}{-cdw+bc+dcw-da} \\ &= \frac{(bc-ad)w}{bc-ad} \\ &= w. \end{aligned}$$

Similarly, $ST(z)=z$.
Finally, $T'(z)\neq 0$ because

$$\frac{d}{dz}S(T(z)) = \frac{d}{dz}z = 1$$

and so

$$S'(T(z)) \centerdot T'(z) = 1$$

$\therefore T'(z)\neq 0$. □

Proposition 3. *Any conformal map of $D=\{z : |z|<1\}$ onto itself is a fractional linear transformation of the form.*

$$T(z) = e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z} \qquad \textit{fixed } z_0\in D \textit{ and } \theta\in[0,2\pi].$$

T of the above form is a conformal map of D onto D.

Proof. We first verify that for T of this form, $|z|=1$ implies that $|T(z)|=1$. Now,

$$|T(z)| = \left|\frac{z-z_0}{1-\bar{z}_0 z}\right| = \frac{z-z_0}{|z||z^{-1}-\bar{z}_0|}.$$

But $|z|=1$ and so $z^{-1}=\bar{z}$. Hence we get

$$|T(z)| = \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|} = 1.$$

Only singularity of T is at $z=\bar{z}_0^{-1}$ which lies outside the unit circle.
Thus by the *maximum modulus theorem*, T maps D onto D. But by *proposition 2*

$$T^{-1}(w) = e^{i\theta}\left[\frac{w-(-e^{i\theta}z_0)}{1-(-e^{i\theta}\bar{z}_0)w}\right]$$

which, since it has the same form as T is also a map from D to D. Thus T is conformal from D onto D.
Let $R : D\to D$ be any conformal map. Let $z_0=R^{-1}(0)$ and let $\theta=$arg$R'(z_0)$.

The map T defined in the proposition also has $T(z_0) = 0$ and $\theta =$arg $T'(z_0)$; indeed,

$$T'(z) = e^{i\theta}\left[\frac{1-|z_0|^2}{(1-\bar{z}_0 z)^2}\right]$$

which at $z = z_0$, equals

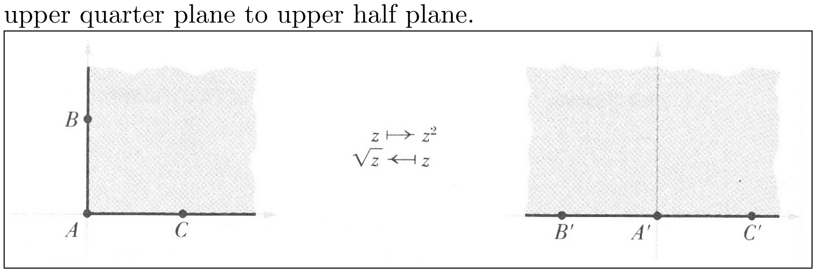
$$e^{i\theta}\left(\frac{1}{1-|z_0|^2}\right)$$

a real constant times $e^{i\theta}$. Thus by uniqueness of conformal maps from the Riemann mapping theorem, $R = T$. □

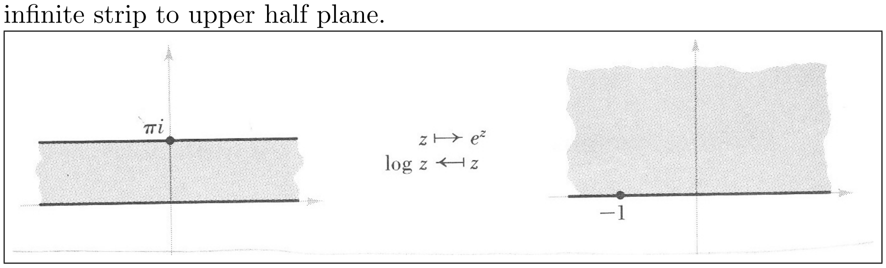
From the above, we conclude that the only way to map a disk onto itself conformally is by means of a fractional linear transformation.
(The maximum modulus theorem states that if $f(z)$ is analytic inside and on simple closed curve $\mathcal{C}$ and is not identically equal to a constant , then the maximum value of $|f(z)|$ occurs on $\mathcal{C}$.)

Some common transformations.

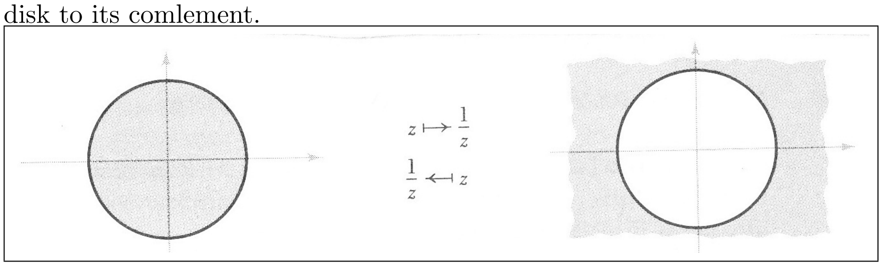
1. upper quarter plane to upper half plane.



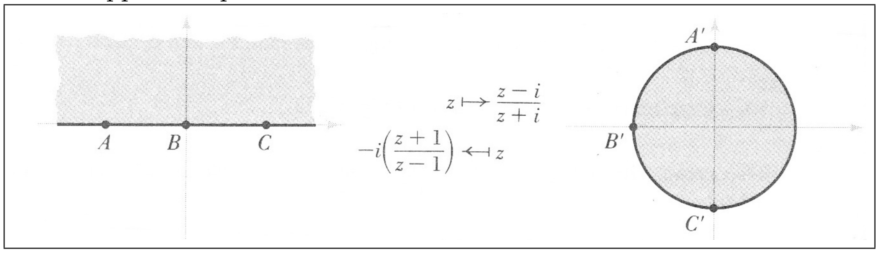
2. infinite strip to upper half plane.



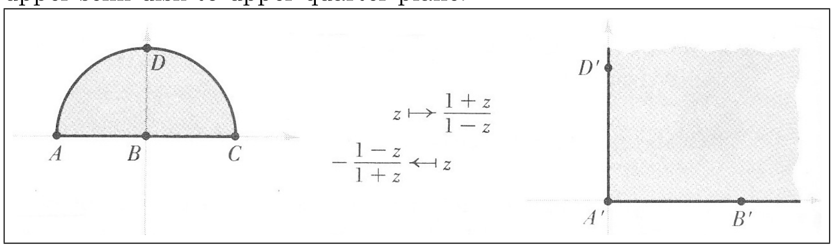
3. disk to its comlement.



4. disk to upper half plane.



5. upper semi-disk to upper quarter plane.



Conformal map of a strip into a unit circle.

$$z \mapsto e^z \mapsto \frac{e^z - i}{e^z + i}.$$

(strip $\mapsto$ upper half plane $\mapsto$ unit circle)
The preceeding mapping theorems and more on applications of conformal mappings can be found in Marsden and Hoffman[5].

Solution of Dirichlet problems by Conformal mapping.
The idea is to map a Dirichlet problem,

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{for} \quad (x,y) \in D \\ u|_{\partial D} = 0 \end{cases}$$

to a Dirichlet problem for the unit disk in a different plane say $w - plane$. We can solve the problem for the unit disk, then map back this solution the $z - plane$ for the original problem in D[4]. This process appears as a change of variables in the integral solution for the disk. The conformal mapping can be regarded as a change of variables that preserves the property of harmonicity. If is u harmonic in a region B and $f : A \to B$ is analytic, then $u \circ f$ is harmonic in A because the Laplacian in 2 dimensions is conformally invariant.
If values of u are prescribed on the boundary circle γ, say $u(\zeta) = g(\zeta)$ for a given function $g : D \to \mathbb{D}$ is conformal, then for $|z| < 1$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma g(\zeta) \left(\frac{\zeta + z}{\zeta - z}\right) \frac{1}{\zeta} d\zeta.$$

This integral formula determines $f(z)$ at points in the open disk, given values of $\text{Re}[f(z)]$ on the boundary unit circle. The solution of the Dirichlet problem for the unit disk, which asks for a harmonic function taking values given by g on the boundary unit circle, is retrieved from this formula as

$$u(x, y) = \mathrm{Re}[f(x + iy)].$$

3.2 Mapping Theorems.

Riemann Mapping Theorem. *Let A be a simply connected region such that $A \neq \mathbb{C}$. Then there exists a bijective conformal map $f : A \to D$ where $D = \{z : |z| < 1\}$. Furthermore, for any fixed $z_0 \in A$ we can find f such that $f(z_0) = 0$ and $f'(z_0) > 0$. With such a specification, f is unique.*

If A and B are two simply connected regions with $A \neq \mathbb{C}$, $B \neq \mathbb{C}$ then there is a bijective conformal map $g : A \to B$. Indeed, if $f : A \to D$ and $h : B \to D$ are conformal, we can set $g = h^{-1} \circ f$. Two regions A and B are called conformal if there is a bijective conformal map from A to B thus *Riemann mapping theorem* implies two simply connected regions (unequal to $\mathbb{C}$) are conformal.

Proof. We prove the uniqueness in the theorem.

Suppose f and g are bijective conformal maps of A onto D with $f(z_0) = g(z_0) = 0$, $f'(z_0) > 0$ and $g'(z_0) > 0$. We want to show that $f(z) = g(z)$ for all $z \in A$.

We define h on D by $h(w) = g(f^{-1}(w))$ for $w \in D$. Then $h : D \to D$ and $h(0) = g(f^{-1}(0)) = g(z_0) = 0$. By Schwarz's lemma $|h(w)| \leq |w|$ for all $w \in D$. For $h^{-1} = f \circ g^{-1}$ also applies so $|h(\zeta)| \leq |\zeta|$ for all $\zeta \in D$

With $\zeta = h^{-1}(w)$, this gives us $|w| \leq |h(w)|$. Combining the above two inequalities, we get $|h(w)| = |w|$ for all $w \in D$. The Schwarz's lemma now tells us that $h(w) = cw$ for a constant c with $|c| = 1$ thus $cw = g(f^{-1}(w))$. With $z = f^{-1}(w)$ we obtain $cf(z) = g(z)$ for all $z \in A$. In particular $cf'(z_0) = g'(z_0)$.

Since both $f'(z_0)$ and $g'(z_0)$ are positive real numbers so is c thus $c = 1$ and so $f(z) = g(z)$ i.e. unique.

The proof of existence breaks into parts that use different ideas. Consider a family of functions,

$$\mathcal{F} = \{\text{analytic, injective } f : \Omega \to D \text{ such that} f(z_0) = 0\}.$$

The argument shows the following;

1. $\mathcal{F}$ is nonempty.
2. If some $f \in \mathcal{F}$ satisfies

   $$|f'(z_0)| \geq |g'(z_0)| \text{ for all } g \in \mathcal{F}$$

   then f is surjective.
3. $\mathcal{F}$ is equicontinuous.

So the Arzela-Ascoli theorem, Weierstrass theorem and Hurwitz theorem complete the argument of existence.

□

Corollary to Riemann mapping Theorem. *Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$ then is an analytic function f on G such that*

1. $f(a) = 0$ *and* $f'(a) > 0$.
2. f *is one-to-one.*
3. $f(G) = D = \{z : |z| < 1\}$.

Koebe quarter Theorem. *A one-to-one analytic function $f : D \to \mathbb{C}$ from the unit disk $\mathbb{D}$ onto a subset of the complex plane contains a disk whose center is $f(0)$ and whose radius is $\frac{f'(0)}{4}$. The Koebe function $f(z) = \frac{z}{(1-z)^2}$ shows that the constant $\frac{1}{4}$ in the theorem cannot be improved.*

If a univalent function (one-to-one and analytic) on the unit disk maps 0 to 0 and has derivative 1 at 0 then, the image of the unit disk contains the ball of radius $\frac{1}{4}$. So for any $w \notin f(\mathbb{D})$ we have that $|w| > \frac{1}{4}$.

Bieberbach's conjecture. *For a holomorphic function of the form*

$$f(z) = z + \sum_{n \geq 2} a_n z^n$$

which is defined and injective on the open unit disk then

$$|a_n| \leq n \qquad \textit{for } n \geq 2.$$

The above gives a necessary condition on a holomorphic function in order for it to map the open unit disk of the complex plane injectively to the complex plane i.e. coefficient inequality for univalent functions.
(Bieberbach's conjecture was proved by Louis de Branges in 1984.)

Proof. *(Koebe quarter theorem)*
Applying an affine map, it can be assumed that

$$f(0) = 0, \qquad f'(0) = 1,$$

so that

$$f(z) = z + a_2 z^2 + \dots$$

(An affine map preserves linearity of points and ratios distances between points lying on a straight line but not angles or length.)
If w is not in $f(\mathbb{D})$ then

$$h(z) = \frac{wf(z)}{w - f(z)} = z + (a_2 + w^{-1})z^2 + \dots$$

is one-to-one in$|z| < 1$.
Applying the coefficient inequality from Bieberbach's conjecture (i.e. $a_2 = 2$) to f and h gives

$$|w|^{-1} \leq |a_2| + |a_2 + w^{-1}| \leq 4$$

so that

$$|w| > \frac{1}{4}.$$

□

Koebe Distortion Theorem. *Let $f(z)$ be a univalent function on $|z| < 1$ normalized so that $f(0) = 0$ and $f'(0) = 1$ and let $r = |z|$ then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$
$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$
$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}$$

with equality if and only if f is a Koebe function $f(z) = \frac{z}{(1-e^{i\theta}z)^2}$.

The Koebe Distortion Theorem gives a series of bounds for a univalent function and its derivatives. The term "distortion" comes from the geometric interpretation of $f'(0)$ as the infnitesimal magnification factor of arclength under f.

3.3 Hyperbolic metric.

The parallel postulate says that if a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles. Hyperbolic geometry is a non-Euclidean geometry in which the parallel postulate is replaced by the assumption that through any point in a plane there are two or more lines that do not intersect a given line in the plane. A characteristic property of hyperbolic geometry is that the angles of a triangle add to less than a straight angle.
The Poincaré disk is a model for hyperbolic geometry in which a line is represented as an arc of a circle whose ends are perpendicular to the disk's boundary (and diameters are also permitted). Two arcs which do not meet correspond to parallel rays, arcs which meet orthogonally correspond to perpendicular lines, and arcs which meet on the boundary are a pair of limits rays [13].
For the Poincaré hyperbolic disk, the hyperbolic metric is invariant under Mőbius transformations, which are all the conformal mappings conformal mappings from the disk onto itself.
The hyperbolic density $\rho(\zeta)$ in the unit disk $\mathbb{D}$ is defined for all $\zeta \in \mathbb{D}$ by the formula

$$\rho(\zeta) = \frac{1}{1-|\zeta|^2}.$$

The hyperbolic metric which is defined from this density by integration is given by

$$\rho(\zeta_1, \zeta_2) = \inf \int_\gamma \rho(\tau)|d\tau|.$$

where infimum is taken over all paths in $\mathbb{D}$ joining ζ_1 and ζ_2.
Definition of our metric: $\sigma(z_0, \mathbb{D}) = |f'(z_0)| \quad (= \rho(\tau))$

- $z_0 \in \mathbb{D}$
- $f(z)$ is a map of $\mathcal{D}$ onto $\mathbb{D}$ so that $f(z_0) = 0$

Schwarz lemma
Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk centered at the origin and let $f : \mathcal{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathcal{D}$ then $|f(z)| \leq |z|$ for all $z \in \mathcal{D}$ and $|f'(0)| < 1$.
Moreover, if $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$ then

$$f(z) = az \text{ for } a \in \mathbb{C} \text{ with } |a| = 1$$

Examples of hyperbolic metrics for various domains.
Using our knowledge of conformal mappings onto a unit disk we can now show the metrics for different domains.

1. If $\mathbb{D}$ is the unit disk then

$$\sigma(z_0, \mathbb{D}) = \frac{1}{1 - |z_0|^2}.$$

Consider $f(z) - \frac{z_0 - z}{1 - \bar{z}_0 z}$ the conformal map of $\mathbb{D}$ onto itself.

$$f'(z) = \frac{(1 - \bar{z}_0 z)(-1) + (z_0 - z)\bar{z}_0}{(1 - \bar{z}_0 z)^2}$$
$$f'(z_0) = \frac{(1 - \bar{z}_0 z_0)(-1)}{(1 - \bar{z}_0 z_0)^2}$$
$$|f'(z_0)| = \frac{1}{1 - |z_0|^2} = \sigma(z_0, \mathbb{D}).$$

2. If H is the upper half-plane $y > 0$, where $z = x + iy$ then

$$\sigma(z, H) = \frac{1}{2y}.$$

3. If S is the strip $y_1 < y < y_2$, where $z = x + iy$ then

$$\sigma(z, S) = \frac{\pi}{2(y_2 - y_1)} \operatorname{cosec} \frac{\pi(y - y_1)}{y_2 - y_1}.$$

Principle of the Hyperbolic metric. *Suppose $w = f(z)$ maps a domain $\mathcal{D}_1$ into a domain $\mathcal{D}_2$ so that $f(z_0) = w_0$ then*

$$\sigma(w_0, \mathcal{D}_2)|f'(z_0)| \leq \sigma(z_0, \mathcal{D}_1)$$

equality holds if and only if $f(z)$ maps $\mathcal{D}_1$ onto $\mathcal{D}_2$.

The following theorem aids in proving the above principle:
Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two domains whose complements contain at least two points. Suppose also that $z_0 \in \mathcal{D}_1$ and that $w_0 \in \mathcal{D}_2$, then there exists a unique map $w = f(z)$ from $\mathcal{D}_1$ onto $\mathcal{D}_2$ such that $f(z_0) = w_0$ and $f'(z_0) > 0$.

*Proof. (*Principle of Hyperbolic metric)
suppose that $f_1(\zeta)$ maps the unit disk $\mathbb{D}$ onto $\mathcal{D}_1$ so that $f_1(0) = z_0$ and that$f_2(\zeta)$ maps the unit disk $\mathbb{D}$ onto $\mathcal{D}_2$ so that $f_1(0) = w_0$ then

$$F(\zeta) = f_2^{-1} \circ f \circ f_1(\zeta)$$

maps $\mathbb{D}$ into itself so that $F(0) = 0$.
By Schwarz's lemma

$$|F'(0)| = |f_2'(0)|^{-1}|f'(z_0)||f_1'(0)| \leq 1 \tag{a}$$

$$\text{i.e.} \qquad |f'(z_0)| \leq \frac{|f_2'(0)|}{|f_1'(0)|} = \frac{\sigma(z_0, \mathcal{D}_1)}{\sigma(w_0, \mathcal{D}_2)}$$

equality holds in (a) if and only if $F(\zeta) = e^{i\theta}\zeta$. This is a case of f mapping $\mathbb{D}_1$ onto $\mathbb{D}_2$ thus done. □

The above theorem tells us that maps of $\mathcal{D}_1$ into $\mathcal{D}_2$ decreases the hyperbolic lengths unless map is reversible i.e. from $\mathcal{D}_1$ onto $\mathcal{D}_2$. After applying this to the idenity map we see that if $\mathcal{D}_1 \subset \mathcal{D}_2$,

$$\sigma(z_0, \mathcal{D}_1) \geq \sigma(z_0, \mathcal{D}_2)$$

when $z \in \mathcal{D}_1$, with strict inequalty unless $\mathcal{D}_1 = \mathcal{D}_2$. The hyperbolic metric decreases strictly with expanding domain.
All the properties of the hyperbolic metric we have shown are found in Hayman[2].

Conformal Radius and relation to inradius.
The conformal radius is a way to measure the size of a simply connected planar domain $\mathcal{D}$ viewed from a point z in it. It is well-suited to use in complex analysis, with regards to conformal maps and conformal geometry[14].
Given a simply connected domain $\mathcal{D} \subset \mathbb{C}$ in the complex plane, and a point $z \in \mathcal{D}$, by the Riemann mapping theorem there exists a unique conformal map $f : \mathcal{D} \to \mathbb{D}$ onto the unit disk with $f(z) = 0 \in \mathbb{D}$ and derivative $f'(z) > 0$. The conformal radius of $\mathbb{D}$ from z is then defined as

$$rad(z, \mathcal{D}) = \frac{1}{f'(z)}.$$

If $\psi : \mathcal{D} \to \mathcal{D}'$ is a conformal bijection and $z \in \mathcal{D}$, then

$$rad(z', \mathcal{D}') = |\psi'(z)| rad(z, \mathcal{D})$$

the above shows the conformal radius is well behaved under conformal maps. As a consequence of the Schwarz lemma and the Koebe 1/4 theorem: for $z \in \mathcal{D} \subset \mathbb{C}$,

$$\frac{rad(z, \mathcal{D})}{4} \leq dist(z, \partial D) \leq rad(z, \mathcal{D}).$$

We note that the lower bound in the inequality is as a result of the Koebe theorem while the upper bound results from the Cauchy-Schwarz inequality. It is clear that the conformal radius is the reciprocal of our hyperbolic metric $\sigma(z, \mathcal{D})$. Optimal scenario for the inequality:

- The upper bound is clearly attained by the unit disk with the $z = 0 \in \mathbb{D}$ origin
  Comparison with the disk gives that $\sigma_D \leq \frac{1}{R_D}$.
- The lower bound is attained by the following 'slit domain': $\mathcal{D} = \mathbb{C} \backslash \mathbb{R}_+$ and $z = -r \in \mathbb{R}_-$.

It is clear from the above inequality that the hyperbloic metric has an inverse relation with the inradius much like the first Dirichlet eigenvalue, λ_1. It follows from the Koebe Theorem, with $c = \frac{1}{4}$, that there is a universal constant c such that

$$\sigma_D = \inf_{z \in D} \sigma(z; D) \geq \frac{c}{R_D}$$

the best value of c above is known as the schlicht Bloch-Landau constant. The conformal mapping for which c is attained is the extremal function and the related domain the extremal domain. A disk of radius c in this domain is then called the extremal disk.

The key property of the hyperbolic metric is that it decreases with increasing domain and this mirrors the domain monoticity of the first Dirichlet eigenvalue clearly showing the parallels between variational problem 1 and the hyperbolic metric in a simply connected domain.

4 Expected lifetime of Brownian motion in a domain.

Let B_t be the Brownian motion in domain D. Let

$$\tau_d(z) = \inf\{t > 0 : B_t \notin D\},$$

where z is the initial starting point, be the first exit time of B_t from D. We will denote by $E(\tau_D(z))$ the expectation of τ_D under the measure of the Brownian motion starting at the point z in D. It is known that, whenever D is a planar simply connected domain, then

$$\sup_{z \in D} E(\tau_D(z)) \leq bR_D^2.$$

[11]. Problems from the above inequality revolve around finding the best value of the constant b and the extremal domain for the inequalty.
If U denotes the disk of radius R_D, then $\sup_{z\in U} E(\tau_U(z)) = \frac{R_D^2}{2}$ and the domain monoticity of the expected times gives $\sup_{z\in D} E(\tau_D(z)) \geq \frac{R_D^2}{2}$ reiterating that the extremal domain has to contain an extremal disk. The larger the domain the larger the lifetime.
In this section we wish to verify the link between the expected first exit time of the Brownian motion and the the first two variational problems.

4.1 Properties of Brownian Motion.

A stochastic process is a random variable for which response variable Z_t is indexed by t which is time of observation. Standard Brownian Motion is a stochastic process $\{B_t, t \geq 0\}$ with state space, $\mathbb{R}$, with the following properties

1. independent increments i.e. $B_t - B_s$ is independent of B_r for $r \leq s$ *(memoryless property)*

2. stationary increments i.e. $B_t - B_s$ and $B_{t+x} - B_{s+x}$ have same distributions for all s, t, x

3. gaussian increments: $B_t - B_s \sim N(0, t-s)$ for $(s < t)$
(N(0,t-s) means Gaussian distribution wtih mean parameter, 0 and variance parmeter, t-s.)

$$f_{B_t - B_s}(z) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{z^2}{2(t-s)}}$$

4. continuous sample paths

5. $B_0 = 0$

Probabilistic Intepretation.
Recall the first exit time $\tau_D(z)$. A standard Brownian motion in the plane departs from a point in D and runs until it exits D in a time τ_D that depends on the path. This is a stochastic process in D whose transition probabilties *(one step jump probabilties)* are denoted by $p_D(t,x,y)$ so that the probabilty that the process that initially departs from $x \in A \subset D$, where A is a Borel set, at time t is

$$\int_A p_D(t,x,y)dy$$

These transition probabilities are the fundamental solutions of the heat equation in D - the heat kernel for D - they satisfy

$$\frac{1}{2}\Delta_y p_D(t,x,y) = \frac{\partial}{\partial t} p_D(t,x,y).$$

(The heat kernel represents the evolution of temperature in a region whose boundary is held at a fixed particular temperature (typically zero) such that initial unit energy is placed at a point $t = 0$.)

The explanation of the above is that of a plate with shape D, its boundary maintained at zero temperature and one unit of heat put at x at time $t = 0$: the resulting heat density at the point y at time t is $p_D(t,x,y)$. The connection between this heat problem and the Brownian motion in D is that each is a diffusion of concentration at x at time 0 that is absorbed on reaching the boundary of the domain[10]. The phenomenon is that heat flow works by replacing the temperature at with the average value of the neighbouring temperatures in the diffusion process and this accounts for the continuous sample paths in the Brownian motion evolution.

The exit time τ_D of the diffusion depends on the particular Brownian path and as such it is a random variable, a measure function, on path space. Taking the integral relative to the Weiner measure P_x (strictly positive probability measure) on the path space we denote this by

$$E(\tau_D(x)) = \int_0^\infty P(\tau_D(x) > 0)dt.$$

$P(\tau_D(x) > 0)$ is the probability that a Brownian motion that departs from x has not, at time t, already been absorbed at the boundary of the domain so it is the probability that this Brownian motion is still in D at time t and therefore equals $\int_D p_D(t,x,y)dy$[10]. This gives

$$E(\tau_D(x)) = \int_D \left(\int_0^\infty p_D(t,x,y)dt \right) dy.$$

We see that

$$f(x,y) = \int_0^\infty p_D(t,x,y)dt$$

is a function of x and y alone. For fixed x, the function $f(x,y)$ is positive, harmonic and symmetric in $D\backslash\{x\}$ as a function of y and it vanishes on the boundary D. It exhibits the Markov property i.e. the conditional probability distribution of future states of the process depends only upon the present state. $f(x,y)$ is the Green's function $G_D(x,y)$ for D, and the expected lifetime of Brownian motion in D starting from x is

$$E_x(\tau_D) = \int_D G_D(x,y)dy.$$

(Green's function is an integral representation for the solution of the Dirichlet problem)
Above is the probabilist's normalization of the Green's function (*making it a probabilty density or mass function*) for one half of the Dirichlet Laplacian[10]. In the unit disk $U = \{y : |y| < 1\}$ it is given by

$$G_D(0,y) = \frac{1}{\pi}\log\frac{1}{y}.$$

At this point we may conclude that

$$\Delta(E_x(\tau_D)) = \Delta\left(\int_D G_D(x,y)dy\right) = -2$$

since the Green's function provides the solution

$$v(x) = \int_D G_D(x,y)f(y)dy$$

to the Poisson problem $\frac{1}{2}\Delta v + f = 0$ in D and $v = 0$ on the boundary of D.

To further explore the relationship between Green's function and the expected lifetime we can compute explicit formulae for the expected lifetime of Brownian motion certain domains:

- for the unit disk $\mathbb{D} = \mathbb{D}(0,R)$ one has $E_x(\tau_D) = \frac{1}{2}(R^2 - |x|^2)$
- for the strip $S = \{(x_1,x_2) : |x_2| < R\}$ one has $E_x(\tau_S) = R^2 - x_2^2)$

4.2 Torsional rigidity.

Torsional rigidity is the torque required per unit angle of twist per unit length of beam.A round bar offers the most resistance to any twist because it has the greatest torsional rigidity. From the problem raised by St. Venant we now know that among all simply connected domains of given area, a disk of that area has the greatest torsional rigidity[10].
We consider functions $f(x,y)$ defined in the interior of a given plane domain D and on the boundary ∂D which satisfy the boundary condition

$$f = 0 \qquad \text{on } \partial D$$

then for P, the torsional rigidity of domain D as presented by Póyla and Szegő[1], necessarily,

$$\frac{4}{P} := \inf \left\{ \frac{\int\int |\nabla f|^2 dxdy}{\int\int |f|^2 dxdy} \right\} \tag{a}$$

with equality if and only if

$$f = cv$$

- c is a constant different from 0
- v is characterized by the following

$$\begin{cases} v_{xx} + v_{yy} + 2 = 0 \text{ in } D \\ v = 0 \text{ on } \partial D \end{cases} \tag{b}$$

In fact, v is the stress function, the partial derivatives of which determine the components of stress[1]. The stress function can also be expressed as follows

$$v = \Phi - \frac{1}{2}(x^2 + y^2)$$

From the characterization of the stress function it is equivalent to having Φ harmonic and taking prescribed boundary values:

$$\begin{cases} \Phi_{xx} + \Phi_{yy} = 0 \text{ in } D \\ \Phi = \frac{1}{2}(x^2 + y^2) \text{ on } \partial D \end{cases}$$

Φ exists and is uniquely determined by the Poisson integral formula.
From the theory of elasticity it is known that

$$P = 2 \int\int v dxdy \tag{c}$$

This we can see directly by integration by parts of (a) via

$$\begin{aligned} 4 \left(\int_D v dV \right)^2 &= P \int |\nabla v|^2 dV \\ &= P \left(- \int_D v \Delta v dV \right) \\ &= 2P \left(\int_D v dV \right) \end{aligned}$$

So it follows that

$$P = 2 \int\int v dxdy$$

We must now prove that the definition of P is equiavlent to that given in connection with the variational form above.

We now consider a function f satisfying (a). By the differential equation (b)

$$\begin{aligned} 2\int\int f dxdy &= -\int\int f(v_{xx}+v_{yy})dxdy \\ &= \int\int (f_x v_x + f_y v_y)dxdy - \int\int \left[\frac{\partial(f v_x)}{\partial y} + \frac{\partial(f v_y)}{\partial y}\right] dxdy \end{aligned}$$

Due to boundary conditions on f, it follows that

$$2\int\int f dxdy = \int\int (f_x v_x + f_y v_y)dxdy \tag{d}$$

The function v, saitisfying (b), is a particular function f. Applied to this case (d) yields in conjunction with (c)

$$P = 2\int\int v dxdy = \int\int (v_x^2 + v_y^2)dxdy \tag{e}$$

Hence it follows that equality is attained in (a) when $f = v$. For a general f, it follows from (d) that

$$\begin{aligned} \left(2\int\int f dxdy\right)^2 &\leq \left(\int\int (v_x^2+v_y^2)^{\frac{1}{2}}(f_x^2+f_y^2)^{\frac{1}{2}} dxdy\right)^2 \\ &\leq P\int\int (f_x^2+f_y^2)dxdy \end{aligned} \tag{f}$$

We used in succession Cauchy-Schwarz inequality and equation (e) to obtain our earlier assertion.
If the case of equality is attained in both inequalities under (f), there exists a constant c such that

$$f_x^2 + f_y^2 = c^2(v_x^2+v_y^2), \qquad f_x = cv_x, \qquad f_y = cv_y$$

and we conclude finally that $f = cv$ is the only form for which equality is attained.

4.3 Link between the value of the first exit time of a Brownian particle and the torsion function.

A uniqueness result for the heat equation on a finite interval.
Solutions to the inhomogeneous heat equation

$$u_t - ku_{xx} = f(t,x) \tag{Heat}$$

are unique under Dirichlet conditions.

Proof. Assume we have two solutions to the heat equation above with specifed Cauchy (initial value) and Dirichlet data. Then by subtracting them and calling the difference w; we get another solution w satisfying

$$\begin{cases} w_t - w_{xx} = 0; \qquad (t;x) \in [0;T] \times [0;L]; \\ w(0;x) = 0; \qquad x \in [0;L]; \\ w(t;0) = 0 = w(t;L) \qquad t \in [0;T]. \end{cases}$$

We want to show that $w(t;x) = 0$ for $(t;x) \in [0;T] \times [0;L]$. We multiply both sides of the new heat equation by w and integrate dx over the interval $[0;L]$ to derive

$$\int_{[0,L]} ww_t dx = \int_{[0,L]} ww_{xx} dx$$

differentiate under the integral,

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_{[0,L]} w^2(t,x)dx &= \int_{[0,L]} ww_t dx \\ &= \int_{[0,L]} ww_{xx} dx \\ &= \underbrace{-\int_{[0,t]} (w_x(t,x))^2 dx}_{\leq 0} + \underbrace{w(t,x)w_x(t,x)|_{x=0}^{x=L}}_{= 0 \text{ by boundary cond.}} \\ &\leq 0 \end{aligned}$$

So if we define the *energy* by

$$H(t) := \underbrace{\int_{[0,L]} w^2(t,x)dx,}_{\geq 0}$$

then we have shown that

$$\frac{d}{dt}H(t) \leq 0.$$

But $H(0) = 0$ by the initial conditions of w. Therefore, $H(t) = 0$ for $t \in [0;T]$. But since $w^2(t,x)$ is continuous and non-negative, it must be that $w^2(t;x) = 0$ for $(t,x) \in [0,T] \times [0,L]$ thus uniqueness is shown. □

We have shown that the expected lifetime is the solution to the boundary value problem

$$\begin{cases} \Delta u = -2 \text{ in } D, \\ u = 0 \text{ on the boundary of } D \end{cases} \tag{****}$$

which describes the torsion or stress for a beam of cross section D. The torsional rigidity P_D of a beam of cross section D has the form

$$P_D = \int_D |\nabla u(z)|^2 dm(z)$$

where $u(z) = E_z(\tau_D)$ is the solution of (****) and $dm(z)$ is the Lebesgue measure in D. We have also integrated by parts in the equation of torsional rigidity to see that

$$P_D = 2\int_D E_z(\tau_D)dm(z)$$

where dm=volume element for the usual Lebesgue measure.

The expected lifetime of Brownian motion in D and the torsion function from elasticity are one and the same. This is also verified by the uniqueness of the heat kernel. From the probabilistic view of the torsion function it follows that if x is a point in D_1 and if the domain D_1 is contained in the domain D_2 then $u_1(x) \leq u_2(x)$ where u_1 and u_2 are the torsion functions for D_1 and D_2 respectively[10].

4.4 Existence of extremal domains for the expected lifetime of Brownian motion.

We restate the first inequality at the beginning of this section as a theorem proved by Carroll[11].

Theorem L. *There exists a simply connected domain D of finite inradius R_D and a point $z \in D$ for which*

$$E_x(\tau_D) = \mathbb{A}R_D^2 \tag{1}$$

$\mathbb{A}$ is the best possible b from $\sup_{z\in D} E(\tau_D(z)) \leq bR_D^2$ and represents a universal constant We make use of the following proposition and lemma with a view to proving the above theorem.

Proposition 1. *Suppose that the sequence of conformal maps $\{f_n\}_1^\infty$, $f_n : \mathbb{D} \to \mathcal{D}$ normalized so that $f_n(0) = 0$ and $f_n'(0) > 0$, converges to a non-constant function f, uniformly on compact subsets of D. Suppose that*

$$M = \sup_n R_{D_n} < \infty$$

Then

$$\lim_{n\to\infty} E_0(\tau_{D_n}) = E_0(\tau_D).$$

Loosely speaking the proposition states that the convergence of a sequence of simply connected domains of uniformly bounded inradius implies the convergence of the expected lifetime of Brownian motion for these domains.

Lemma 1. *Let h be a conformal mapping of the unit disk $\mathbb{D}$ with $h(0) = 0$. Let $R_{h(\mathbb{D})} \leq M < \infty$ and $E_0(\tau_{h(\mathbb{D})}) \geq L > 0$. There is a positive number η, depending only on M and L, such that*

$$|h'(0)| \geq 0$$

We have to assume that $h(D)$ has finte radius otherwise the lemma does not hold. In fact, $h(z) = \frac{\eta z}{(1-z)^2}$ conformally maps the unit disk onto the plane slit along the negative real axis from $-\infty$ to $\frac{-\eta}{4}$.

Proof of theorem L. By definition of $\mathbb{A}$ as the supremum of the quantity $\frac{E_z(\tau_D)}{R_D^2}$ over all simply connected domains D and over all points z in D, we may choose a sequence of simply connected domains D_n, and a point z_n in each D_n, such that

$$\frac{E_{z_n}(\tau_{D_n})}{R_D^2} \geq \mathbb{A} - \frac{1}{n}, \qquad n \geq 1.$$

By applying a suitable translation, we may assume that $z_n = 0$ in each case and then, by applying a suitable scaling, we may further assume that $E_0(\tau_{D_n}) = 1$. We write f_n for the conformal mapping of the unit disk $\mathbb{D}$ onto D_n for which $f_n(0) = 0$ and $f_n'(0) > 0$. Then, $n \geq 0$,

$$\frac{1}{R^2_{f_n(D)}} \geq \mathbb{A} - \frac{1}{n}, \qquad \text{that is } R_{f_n(D)} \leq \frac{1}{\sqrt{\mathbb{A} - \frac{1}{n}}} \tag{i}$$

Thus since $\mathbb{A} \geq 1.584$ first part of the proposition is satisfied by the sequence $\{D_n\}$, with $M = \frac{1}{\sqrt{0.584}}$.
Another expression for the expected lifetime $E_0(\tau_{h(D)})$ is $\frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2$, where h is a conformal in the unit disk and $h(z) = \sum_{n=1}^{\infty} a_n z^n$. To obtain the expression for the expected lifetime we use conformal invariance, solution in the disk; $u_0 = \frac{1}{2}(1 - |z|^2)$ and that $|\nabla u_0|^2 = 2(E_0)$ in the torsional rigidity expressions. Since $E_0(\tau_{f_n(D)}) = 1$, we deduce that $f_n'(0) \leq \sqrt{2}$.
By Lemma 1, there is a positive η for which $f_n(0) \geq \eta$ for each n. Hence $f'(0) \leq n$ and is therefore positive, so that f is not constant. We may therefore assume that the sequence of conformal maps $\{f_n\}_1^{\infty}$ converges uniformly on compact subsets of the unit disk to a conformal mapping f of the unit disk onto a simply connected domain D. By Proposition 1, $E_0(\tau_D) = 1$.
We lastly show that $\frac{1}{R^2_{f(D)}} = \mathbb{A}$.
By definition of $\mathbb{A}$, $\frac{1}{R^2_{f(D)}} \leq \mathbb{A}$. Suppose, if possible, that $\frac{1}{R^2_{f(D)}} < \mathbb{A}$, so that $R_{f(D)} > \frac{1}{\mathbb{A}}$. There is, then, a disk $D(a, r)$ contained in D with radius $r = \frac{1}{\sqrt{\mathbb{A}}} + 3\epsilon$ for some positive ϵ. It follows that the closed disk $\overline{D(a, r-\epsilon)}$ is contained in D. We denote by γ the simple closed curve in D which is the preimage under f of the circle $C(a, r-\epsilon)$. Then $f_n \to f$ uniformly on γ. In particular, there is a natural number $\mathbb{N}$ such that $|f_n - f| \leq \epsilon$ on γ for $n \geq \mathbb{N}$. But then $|f_n - a| \geq r - 2\epsilon = \frac{1}{\sqrt{\mathbb{A}}} + \epsilon$ on γ, so that $f_n(\gamma)$ encloses the disk $D(a, \frac{1}{\sqrt{\mathbb{A}}} + \epsilon)$. We have found that $R_{f_n(D)} \geq \frac{1}{\sqrt{\mathbb{A}}} + \epsilon$, which violates (i) for sufficiently large n. Hence D is an extremal domain for the inequality (1), proving Theorem L.

□

Since we use conformal mappings of the unit disk to several domains under the normalization $f(0) = 0$ and $f'(0) > 0$ the variational problem

in this section leads us to that the constant b represents the schlicht Bloch-Landau constant also. This is verified further by that the extremal domain, from the domain monoticity of the expected lifetimes, must necessarily contain an extremal disk.

5 Conclusion.

Up to now we have studied the three seemingly different variational problems and managed to obtain various characterizations of them. The first variational problem deals with simply connected domains namely convex domains as the base case such that the maximum inscribed disk bears the best resamblance to original domain. From Hersh's theorem the first Dirichlet eigenvalue is inversely proportional to the inradius of the domain and this is mirrored with the hyperbolic metric which decreases with expanding domain. So we see that the problem of finding the lower bound for the first Dirichlet eigenvalue is related to the problem of schlicht Bloch-Landau constant of the hyperbolic metric.

From the hyperbolic metric we are trying to find the schlicht Bloch-Landau constant i.e the value of c for which the conformal map in relation to the inradius is an extremal function. While in the third variational problem we use conformal mappings of the unit disk to ascertain the existence of an extremal domain for the expected exit time of the Brownian motion which leads us to that the constant b in $\sup_{z\in D} E(\tau_D(z)) \leq bR_D^2$ really is the schlicht Bloch-Landau constant and the second variational problem and the third are clearly related.

Also the similarity of the variational characterization of the torsional rigidity to that of the first Dirichlet eigenvalue shows a clear relationship between variational problem 1 and variationa problem 3 albeit an inverse one. This inverse relation is shown further by the domain monoticity of the expected lifetimes which increase with expanding domain while the first Dirichlet eigenvalue decreases with expanding domain.

We have shown that the three variational problems are similar hence it is our belief that the extremal domain is the same for the three variational problems. However to this day the problem of finding the extremal domain has not been solved but studies have leaned towards a slit domain as the extremal domain. One difficulty in ascertaining this extremal domain is that the candidate slit domains offer very low regularity. The confines of this project were the study of simply connected domains. From a combination of Hersch's theorem on first Drichlet eigenvalue, R.Sperb in calculating b in $\sup_{z\in D} E(\tau_D(z)) \leq bR_D^2$ to be 1 and Szegő in calculating c in $\sigma(z;D) \geq \frac{c}{R_D}$. Concerning convex domains the strip is the extremal domain for all three problems [15].

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