Moser's Theorem & Differential Forms
Seminar Report

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Abstract

This report aims to subsidize the seminar given on the 2^{nd} of August 2013 on *"Moser's Theorem and Differential Forms"*. In doing so, a short summary of differential forms and other concepts of differential geometry which are necessary to understand *Moser's Theorem* will be given. Next, *Moser's Theorem* will be stated and it's utility will be motivated. Lastly, a proof of *Moser's Theorem* will be given (which unfortunately had to be omitted in the seminar representation due to time constraints).

1 Introduction

The main focus of this report is *Moser's Theorem* (see **Theorem 3.1**) and it's proof. However, to fully appreciate *Moser's Theorem*, some background of differential geometry is required. **Section 2** will give all the necessary definitions and results to understand the main Theorem. It should however be noted that, unlike the seminar presentation, this report does not intend to teach said background-material – it is aimed at readers who are already familiar with the basic concepts of differential geometry but may need a reminder of said concepts.
For a more in depth treatment of the material discussed in **section 2**, readers are referred to [4].

Next, *Moser's Theorem* as well as an interpretation and the proof following [5] are given in **section 3**. This proof presupposes various results from differential and algebraic geometry. For the convenience of the reader, all these results are stated (without proof) in **Appendix A**.

2 Differential Forms – An Overview

As mentioned in the Introduction, in order to fully appreciate *Moser's Theorem*, some background knowledge is required. This section aims to give a short overview of the most important concepts needed for the later sections of the report.
All the information provided in this section is sourced from [4], unless stated otherwise.

In the following, $\mathcal{M}$ will denote a smooth, differentiable, (not necessarily Riemannian), m-dimensional manifold, unless otherwise specified. As it is usually done, $T(\mathcal{M})$ and $T^*(\mathcal{M})$ will denote the *tangent* and *cotangent bundles* of $\mathcal{M}$, respectively. For any $P \in \mathcal{M}$, the fibres of $T(\mathcal{M})$ (resp. $T^*(\mathcal{M})$) will be denoted by $T_P(\mathcal{M})$ (resp. $T^*_P(\mathcal{M})$), such that:

$$T(\mathcal{M}) = \bigsqcup_{P \in \mathcal{M}} T_P(\mathcal{M}) \qquad \text{and} \qquad T^*(\mathcal{M}) = \bigsqcup_{P \in \mathcal{M}} T^*_P(\mathcal{M}) \tag{1}$$

where $\bigsqcup$ denotes a disjoint union.

Given a particular coordinate representation, the basis elements of $T_P(\mathcal{M})$, for some $P \in \mathcal{M}$, will be denoted by:

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots \frac{\partial}{\partial x^m}$$

while the basis elements of $T^*_P(\mathcal{M})$ will be denoted by:

$$\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m$$

Definition 2.1. *If $\omega_P \in T^*_P(\mathcal{M})$, for some $P \in \mathcal{M}$, then ω_P is said to be a 1-form.*

From now on, the explicit dependence of ω_P on P will b suppressed, i.e. ω_P will simply be denoted by ω.

We will denote $(T^k)^*(\mathcal{M})$ to be the k^{th} tensor product of $T^*(\mathcal{M})$:

$$(T^k)^*(\mathcal{M}) = \underbrace{T^*(\mathcal{M}) \otimes T^*(\mathcal{M}) \otimes \ldots \otimes T^*(\mathcal{M})}_{k \text{ times}} \tag{2}$$

The set $\Lambda^k(\mathcal{M}) \subseteq (T^k)^*(\mathcal{M})$ will then be defined as the set of all alternating k-tensors in $(T^k)^*(\mathcal{M})$.

Definition 2.2. *If $\omega \in \Lambda^k(\mathcal{M})$, then ω is said to be a k-form.*

We would now like to combine the various spaces $\Lambda^k(\mathcal{M})$ to an algebra. To do so, we first need to define an operation on these sets that respects the anti-symmetry property of their elements. This operation is called the *wedge-product*:

Definition 2.3. *Let $\alpha \in \Lambda^p(\mathcal{M})$ and $\beta \in \Lambda^q(\mathcal{M})$. The wedge product $\wedge$,*

$$\wedge : \Lambda^p(\mathcal{M}) \times \Lambda^q(\mathcal{M}) \longrightarrow \Lambda^{p+q}(\mathcal{M})$$

is a map whose action on (α, β) is defined as follows:

$$\wedge(\alpha, \beta) := \alpha \wedge \beta = \frac{(p+q)!}{p!q!} \mathrm{Alt}(\alpha \otimes \beta)$$

where, for ω a p-form, $(v_1, v_2, \ldots, v_p) \in T^p(\mathcal{M})$ and S_p the symmetric group on p elements, we have that:

$$(\mathrm{Alt}(\omega))(v_1, v_2, \ldots, v_p) = \frac{1}{p!} \sum_{\sigma \in S_p} (\mathrm{sgn}(\sigma))\, \omega\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(p)}\right)$$

It should be noted that due to this definition, the *wedge product* is anti-symmetric in the exchange of two of its components, for example:
Let $\alpha = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^p \in \Lambda^p(\mathcal{M})$ and $\beta = \mathrm{d}x^{p+1} \wedge \ldots \wedge \mathrm{d}x^{p+q} \in \Lambda^q(\mathcal{M})$, in some coordinate representation of $\mathcal{M}$. Then:

$$\alpha \wedge \beta = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^i \wedge \mathrm{d}x^{i+1} \wedge \ldots \wedge \mathrm{d}x^{p+q} = -\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{i+1} \wedge \mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^{p+q} \tag{3}$$

Due to (3), in particular, we have that

$$\mathrm{d}x^i \wedge \mathrm{d}x^i = 0 \tag{4}$$

for every $\mathrm{d}x^i \in \Lambda^1(\mathcal{M})$. In general, it then follows that, for every $\omega \in \Lambda^k(\mathcal{M})$, where k is odd,

$$\omega \wedge \omega = 0 \tag{5}$$

Readers are encouraged to convince themselves of this fact.

One should observe that $\omega \wedge \omega$ is not necessarily true if ω has rank $2k$. An example for this would be:

$$\begin{aligned}\left(\mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}x^3\mathrm{d}x^4\right) \wedge \left(\mathrm{d}x^1\mathrm{d}x^2 + \mathrm{d}x^3\mathrm{d}x^4\right) &= \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}x^4 + \mathrm{d}x^3\mathrm{d}x^4\mathrm{d}x^1\mathrm{d}x^2 \\ &= 2\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}x^4 \neq 0\end{aligned}$$

Equation (4) furthermore implies that:

$$\Lambda^k(\mathcal{M}) = \{0\} \qquad \text{for all } k > m \tag{6}$$

where m is as before the dimension of $\mathcal{M}$. The above is true because of the following: In a particular coordinate representation, the basis 1-forms are $\mathrm{d}x^1, \ldots, \mathrm{d}x^m$, and the basis m-form is $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$. Thus, to build a basis $(m+1)$-form γ, one necessarily has to do the following:

$$\begin{aligned}\gamma &= \left(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^m\right) \mathrm{d}x^i \\ &= (-1)^{m-i} \left(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^i \wedge \mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^m\right) \\ &= (-1)^{m-i} \left(\mathrm{d}x^1 \wedge \ldots \wedge 0 \wedge \ldots \wedge \mathrm{d}x^m\right) \\ &= 0\end{aligned}$$

We may now define the *graded algebra*:

Definition 2.4. *We define $\Lambda(\mathcal{M})$ such that:*

$$\Lambda(\mathcal{M}) = \bigsqcup_{k\in\{0,\ldots,m\}} \Lambda^k(\mathcal{M})$$

where m is the dimension of the manifold $\mathcal{M}$. Then $\Lambda(\mathcal{M})$ together with the wedge product is called the graded algebra*.*

Besides the wedge product, one can define other operations on the elements of the graded algebra. Two of these operations, namely the *exterior derivative*[1] and *interior multiplication* will be introduced below.

Definition 2.5. *The exterior derivative* d *is a linear map:*

$$\mathrm{d} : \tilde{\Lambda}^p(\mathcal{M}) \longrightarrow \tilde{\Lambda}^{p+1}(\mathcal{M})$$

where $\tilde{\Lambda}^p(\mathcal{M})$ is understood to be a particular section of $\Lambda^p(\mathcal{M})$, and d *satisfies the following conditions:*
Let α and β be a p- and q-form respectively. Then:

1. $\mathrm{d}\,(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$
2. $\mathrm{d}\,(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^p \alpha \wedge (\mathrm{d}\beta)$
3. *For every $\omega \in \Lambda(\mathcal{M})$, $\mathrm{d}\,(\mathrm{d}\omega) = \mathrm{d}^2\omega = 0$*
4. *For every function f,*
$$\mathrm{d}f = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \mathrm{d}x^i$$

Another important **property of the exterior derivative** is:[2]

Property 2.6. *Let ω be a $(p-1)$-form that, for a particular choice of coordinates, can be represented as:*

$$\omega = f(x^1)\mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^p$$

Then:

$$\mathrm{d}\omega = \mathrm{d}\left(f(x^1)\right) \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^p$$

At this point, the following terminology will be introduced:

Definition 2.7. *Let ω be a p-form. Then:*

1. *ω is said to be* closed *if:*
$$\mathrm{d}\omega = 0$$

[1] The definition for the exterior derivative is taken from [2].
[2] A proof for this property can be found in [4].

2. ω *is said to be* exact *if there exists a* $(p-1)$*-form* β *such that:*

$$\omega = \mathrm{d}\beta$$

It should be noted that due to the second property of the exterior derivative (*cf.* **Definition 2.6**), all exact forms are also closed.

Next, *interior multiplication* will be defined:

Definition 2.8. *The interior multiplication* i_X *of a vector field* X*, where* X *is a given section of* $T(\mathcal{M})$*, and a* p*-form* ω *is a linear map:*

$$i_X : \Lambda^p(\mathcal{M}) \longrightarrow \Lambda^{p-1}(\mathcal{M})$$

such that:

$$\underbrace{i_X\omega(\cdot,\cdot,\ldots,\cdot)}_{p\text{ unassigned slots}} = \underbrace{\omega(X,\cdot,\ldots,\cdot)}_{p-1\text{ unassigned slots}}$$

Lastly, some more terminology needs to be introduced to fully understand the main theorem of this seminar report:

Terminology 2.9. 1. *A* p*-form* ω *is said to be* of odd kind *if, under a change of variables from* $(x^1,\ldots,x^p)$ *to* $(y^1,\ldots,y^p)$*,* ω *behaves as follows:*

$$\omega_{(x^1,\ldots,x^p)} \longmapsto \mathbf{J}\omega_{(y^1,\ldots,y^p)}$$

where $\omega_{(x^1,\ldots,x^p)}$ *denotes* ω *represented in the* $(x^1,\ldots,x^p)$*-coordinates, and similarly for* $\omega_{(y^1,\ldots,y^p)}$*, and* $\mathbf{J}$ *is the* Jacobian *of this coordinate transformation.*

2. *A* p*-form* ω *is said to be* non-degenerate*, if it is non-vanishing everywhere on* $\mathcal{M}$*. In particular, this implies that* $\mathbf{J} \neq 0$ *for any coordinate transformation on* $\mathcal{M}$*.*

Readers are advised that the terminology *of odd kind* is non-standard, but since Moser makes use of this terminology in [5], it is also adapted here.

3 Moser's Theorem

With all the background material of **section 2** in mind, we are finally able to present *Moser's Theorem*.[3]

Theorem 3.1. *Let τ_t be a family of closed 2- or n- forms of odd kind which are non-degenerate for $0 \leq t \leq 1$ and with fixed periods, i.e.:*

$$\int_{\mathcal{C}} \tau_t = \int_{\mathcal{C}} \tau_0 \tag{7}$$

for every 2- or n-dimensional cycle $\mathcal{C}$ on $\mathcal{M}$, where a cycle *is a closed (i.e. compact and without boundary) sub-manifold of $\mathcal{M}$. Then there exists an automorphism ϕ_t such that*

$$\phi_t^* \tau_t = \tau_0 \tag{8}$$

and ϕ_0 is the identity mapping.

Before proving the above Theorem, we will look at what *Moser's Theorem* says quantitatively. Thus, we introduce some more terminology: Let $\mathcal{M}$ be an m-dimensional manifold.

1. A *volume form* is a m-dimensional non-degenerate form. It should be noted that a volume form is automatically closed.
   As intuitively expected, a *volume form* is used o measure the volume of $\mathcal{M}$, see [4].

2. A *symplectic form* is a 2-dimensional, closed, non-degenerate form. It should be noted that only even-dimensional manifolds allow for symplectic forms on them, see [4].
   Symplectic forms are often used in classical mechanics, see [2].

Now, we look at *Moser's Theorem* quantitatively. The below interpretation is based on [7].

Quantitative Statement:
Considers a m-dimensional manifold $\mathcal{M}$. Suppose that on this manifold, there exists a family of *volume forms* (resp. *symplectic forms*) such that (7) is fulfilled. What **Theorem 3.1** tells us is that the fact that the integrals on the LHS and the RHS of (7) are equal is not a coincidence, but is only possible if the forms in the aforementioned family are in fact related by a change of variables (as the map ϕ_t in the theorem is essentially a change of variables). The remarkable implication of this is not only that there exists a standard volume form τ_0 (resp. standard symplectic form τ_0) on the manifold $\mathcal{M}$, but that once this standardized form is found, it is legitimate to work with τ_0 rather than any

[3]This Theorem is taken from [5].

other form τ_t as they are related by (8). In practise, this is very useful, as in a particular problem, the form τ_t might be complicated, but the standard form τ_0 might be easy to work with.

Now, **Theorem 3.1** will be proven. As mentioned earlier, this proof requires a familiarity with various other results, such as for example the *Hodge Decomposition Theorem.* All the results necessary to understand the proof of **Theorem 3.1** are given in **Appendix A**.

Proof. This proof strongly follows [5].
The strategy is to determine the integral curves X of the mapping ϕ_t, instead of determining ϕ_t directly, which we know exist by the *Fundamental Theorem of Flows* (see **Theorem A.1.1** in **Appendix A.1**). This strategy is now commonly referred to as *Moser's Trick.* Once X_t is determined, ϕ_t can be recovered by solving the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t = X_t \circ \phi_t \tag{9}$$

We now need to find an expression for X_t. First, we notice that *"Cartan's Magic Formula"*[4] reads as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_t^* \tau_t\right) = \phi_t^* \left[\dot{\tau}_t + \mathrm{d}\left(i_{X_t}\tau_t\right) + i_{X_t}\left(\mathrm{d}\tau_t\right)\right] \tag{10}$$

where $\dot{\tau}_t = \frac{\mathrm{d}}{\mathrm{d}t}\tau_t$.
By assumption of the Theorem, τ_t is closed. Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_t^* \tau_t\right) = \phi_t^* \left[\dot{\tau}_t + \mathrm{d}\left(i_{X_t}\tau_t\right)\right] \tag{11}$$

Next, we set the left hand side of (11) to zero:

$$0 = \phi_t^* \left[\dot{\tau}_t + \mathrm{d}\left(i_{X_t}\tau_t\right)\right] \tag{12}$$

This is done such that the solution ϕ_t of the resulting equation satisfies (8), *by construction.*

Then, by the *Picard-Lindelöf Theorem*[5], we know that solutions to (12) exist in an interval $(t-\varepsilon, t+\varepsilon)$ for some small ϵ. We can then apply the *Escape Lemma* (*cf.* **Lemma A.2.1** in **Appendix A.2**) to see that these solutions must in fact exist in the whole manifold $\mathcal{M}$, as, by assumption, $\mathcal{C}$ is closed, so in particular, $\mathcal{C}$ is compact.

We will now proceed to solve equation (12) to obtain an expression for X_t as a function of τ_t.

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[4] This formula is sourced from [5].
[5] The *Picard-Lindelöf Theorem* is a standard result from the theory of differntial equations. Readers are referred to [3] and references therein.

We can use the *Hodge Decomposition Theorem* (see **Theorem A.3.5** in **Appendix A.3**) to express $\dot{\tau}_t$ as a sum of an exact form, $\mathrm{d}\alpha_t$, and a harmonic form, h_t:

$$\dot{\tau}_t = \mathrm{d}\alpha_t + h_t \tag{13}$$

Now, we recall that, by assumption in the Theorem, the periods over closed cycles of the τ_t are fixed, hence:

$$\int_{\mathcal{C}} \tau_t = \text{constant} \quad \Longrightarrow \quad \int_{\mathcal{C}} \dot{\tau}_t = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{C}} \tau_t = 0 \tag{14}$$

It is important to note that, since $\mathcal{C}$ is compact, τ_t is bounded. Therefore, by the *Lebesque Dominated Convergence Theorem* (see **Theorem A.4.1** in **Appendix A.4**), we were able to exchange the derivative and the integral sign in (14).

By a corollary of *Stoke's Theorem* (see **Corollary A.5.2** in **Appendix A.5**), the integral over a cycle of an exact form vanishes. Hence, we find:

$$0 = \int_{\mathcal{C}} \dot{\tau}_t = \int_{\mathcal{C}} \mathrm{d}\alpha_t + h_t = \int_{\mathcal{C}} \mathrm{d}\alpha_t + \int_{\mathcal{C}} h_t = 0 + \int_{\mathcal{C}} h_t = \int_{\mathcal{C}} h_t \tag{15}$$

Since (15) is required to hold for all t, it follows that $h_t = 0$. Thus, $\dot{\tau}_t$ is an exact form:

$$\dot{\tau}_t = \mathrm{d}\alpha_t \tag{16}$$

Substituting result (16) into equation (12) yields the following:

$$0 = \phi_t^* \left[\dot{\tau}_t + \mathrm{d}\left(i_{X_t}\tau_t\right)\right] = \phi_t^* \left[\mathrm{d}\alpha_t + \mathrm{d}\left(i_{X_t}\tau_t\right)\right] \quad \Longrightarrow \quad \mathrm{d}\alpha_t + \mathrm{d}\left(i_{X_t}\tau_t\right) = 0$$

as $\phi_t \neq 0$ by assumption (as otherwise, in particular, $\phi_0 = 0$, which contradicts the fact that ϕ_0 is assumed to be the identity mapping). Therefore, integrating the above once, we have obtained the following linear equation for X_t:

$$i_{X_t}\tau_t = -\alpha_t \tag{17}$$

We note that this equation has in fact a unique solution for X_t, since, by assumption, τ_t is non-degenerate.
To illustrate that X_t is uniquely determined by (17), the quantity $i_{X_t}\tau_t$ will be expressed in coordinate representation. We first denote X_t as follows:

$$X_t = \sum_{i=1}^{n} \mathcal{X}_{k_i}(t) \frac{\partial}{\partial k_i} \tag{18}$$

We now consider the cases for τ_t a 2-form and τ_t an n-form separately.

First, if τ_t is a 2-form, one can represent τ_t in particular coordinates as follows:

$$\tau_t = \sum_{i=1}^{n} a_{k_i l_i}(t,x) \mathrm{d}x_{k_i} \mathrm{d}x_{l_i} \tag{19}$$

We observe that, by *Darboux's Theorem* (see **Theorem A.6.1** in **Appendix A.6**), there always exist local coordinates such that (19) holds. Furthermore, by antisymmetry of the wedge product, we require that $a_{k_i l_i} = -a_{l_i k_i}$ in the above.

Then, we find that:

$$i_{X_t}\tau_t = \sum_{i=1}^{n} a_{k_i l_i}(t,x)\mathcal{X}_{k_i}(t)\mathrm{d}x_{l_i} \tag{20}$$

On the other hand, for τ_t a n-form, τ_t can be represented as follows:

$$\tau_t = b(t,x)\mathrm{d}x_{k_1}\ldots\mathrm{d}x_{k_n} \tag{21}$$

In this case, the interior multiplication between X_t and τ_t becomes:

$$i_{X_t}\tau_t = b(t,x)\sum_{i=1}^{n}(1)^{i-1}\,\mathcal{X}_{k_i}(t)\mathrm{d}x_{k_1}\ldots\mathrm{d}\hat{x}_{k_i}\ldots\mathrm{d}x_{k_n} \tag{22}$$

where $\mathrm{d}\hat{x}_{k_i}$ denotes that $\mathrm{d}x_{k_i}$ is omitted in the i^{th} term or of the sum.

So, for both τ_t a 2-form and for τ_t and n-form, due to the non-degeneracy of τ_t, the components of α_t (and thus of X_t) are uniquely determined.

Now that we have obtained an expression for X_t (via equation (17)), we can substitute it into (9) and then integrate the resulting equation, which is possible since integration on a compact manifold such as $\mathcal{C}$ is well-defined. Its solution ϕ_t satisfies by construction equation (8) and thus proves the Theorem. $\square$

A Assorted Theorems

This appendix serves to give various theorems, without proof, which are essential to understanding the proof of *Moser's Theorem*. Unless stated otherwise, all results are sourced from [4], where the according proofs can be found as well.

A.1 Fundamental Theorem of Flows

Theorem A.1.1. *Let X be a smooth vector field on a smooth manifold $\mathcal{M}$. Then there exists a unique maximal flow $\phi : \mathcal{D} \longrightarrow \mathcal{M}$, where $\mathcal{D}$ denotes the flow-domain, $\mathcal{D} \subset \mathbb{R} \times \mathcal{M}$, such that ϕ is generated by X. ϕ has the following properties:*

1. *For each point $P \in \mathcal{M}$, the curve $\phi^{(P)} : \mathcal{D}^{(P)} \longrightarrow \mathcal{M}$ is the unique maximal integral curve of X. The notation $\phi^{(P)}$ indicates that ϕ starts at the point P.*
2. *If $s \in \mathcal{D}(P)$, then $\mathcal{D}^{\phi(s,P)}$ is the interval $\mathcal{D}^{(P)} - s := \{t - s : t \in \mathcal{D}^{(P)}\}$.*
3. *For each $t \in \mathbb{R}$, the set $\mathcal{M}_t := \{Q \in \mathcal{M} : (t, Q) \in \mathcal{D}\}$ is open in $\mathcal{M}$ and $\phi_t : \mathcal{M}_t \longrightarrow \mathcal{M}_{-t}$ is a diffeomorphism with inverse ϕ_{-t}.*

A.2 Escape Lemma

The *Escape Lemma* is a result describing the behaviour of integral curves on a manifold:

Lemma A.2.1. *Let $\mathcal{M}$ be a smooth manifold and let X be a smooth vector field over $\mathcal{M}$. If $\phi : \mathcal{I} \longrightarrow \mathcal{M}$ is a maximal integral curve of X, whose domain $\mathcal{I}$ has a finite least upper bound b, then for any $a \in \mathcal{I}$, $\phi([a, b))$ is **not** contained in any compact subset of $\mathcal{M}$.*

A.3 Hodge Decomposition Theorem

All the definitions and results of this section are sourced from [6], unless explicitly specified otherwise. However, the notation used for the *dual of the exterior derivative* ([6] calls this the *adjoint exterior derivative*) is taken from [5], as, according to [7], [6] uses a non-standard notation.
To understand the *Hodge Decomposition Theorem*, some more background knowledge is required:

We define the *Hodge Star Operator*:

Definition A.3.1. *Let $\mathcal{M}$ be an m-dimensional Riemannian manifold with metric g. The Hodge $*$ operator is a linear map defined as follows: For every $p \in \mathbb{N}$,*

$$* : \Lambda^p(\mathcal{M}) \longrightarrow \Lambda^{m-p}(\mathcal{M})$$

where the action of $*$ *on a basis vector* $\mathrm{d}x^{k_1} \wedge \ldots \wedge \mathrm{d}x^{k_p} \in \Lambda^p(\mathcal{M})$ *is given by:*

$$* \left(\mathrm{d}x^{k_1} \wedge \ldots \wedge \mathrm{d}x^{k_p}\right) = \frac{\sqrt{\mid g \mid}}{(m-p)!} \varepsilon^{k_1 \ldots k_p}{}_{k_{p+1} \ldots k_m} \mathrm{d}x^{k_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{k_m}$$

where $\varepsilon^{k_1 \ldots k_p}{}_{k_{p+1} \ldots k_m}$ *is the* m*-dimensional alternating tensor and* $\mid g \mid$ *is the determinant of the metric* g*.*

Using this operator, one can define a *dual exterior derivative* or *codifferential* as follows:

Definition A.3.2. *For and* m*-dimensional Riemannian manifold* $(\mathcal{M}, g)$*, the dual exterior derivative operator* $\delta : \Lambda^p(\mathcal{M}) \longrightarrow \Lambda^{p-1}(\mathcal{M})$ *is defined a follows:*

$$\delta = (-1)^{mp+m+1} * \mathrm{d}*$$

where $*\mathrm{d}*$ *is understood to be the composition of the various operators.*

Definition A.3.3. *The Laplacian Operator* $\Delta : \Lambda^p(\mathcal{M}) \longrightarrow \Lambda^p(\mathcal{M})$ *is defined by:*

$$\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}$$

We are finally in a position to define a *harmonic form*:

Definition A.3.4. *A* p*-form* ω *is said to be harmonic, if:*

$$\Delta\omega = 0$$

The *Hodge Decomposition Theorem*[6] then reads as follows:

Theorem A.3.5. *Let* $(\mathcal{M}, g)$ *be a compact, orientable Riamannian manifold without boundary. Then every closed* p*-form* ω *can be represented in the form:*

$$\omega = \mathrm{d}\alpha + h$$

where h *is a harmonic* p*-form and* α *is a* $(p-1)$*-form.*

It should be noted that, in the above Theorem, α is not necessarily unique. As however a unique choice of α is required for the proof of *Moser's Theorem*, we will furthermore require that $\alpha = \delta\beta$ for some p-form β.[7]

Readers should be advised that the *Hodge Decomposition Theorem* is a big and difficult Theorem, unlike the other results in this section, for which proofs can be found in most standard textbooks. A proof of the *Hodge Decomposition Theorem* is available in [8].

[6] This Theorem is taken from [5].
[7] *cf.* [5].

A.4 Lebesgue Dominated Convergence Theorem

This Theorem is sourced from [1].

Theorem A.4.1. *Suppose that $\{f_n\}$ is a sequence of measurable functions, that $f_n \longrightarrow f$ point-wise almost everywhere as $n \longrightarrow \infty$, and that $\mid f_n \mid \leq h$ for all n, where h is an integrable function. Then f is integrable, and:*

$$\int f \mathrm{d}\mu = \int \lim_{n\to\infty} f_n \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \mathrm{d}\mu$$

where $\mathrm{d}\mu$ is the Lebesgue measure.

In the context of *Moser's Theorem*, this result applies to equation (14) since the derivative can be expressed as a limit.

A.5 Stoke's Theorem

Theorem A.5.1. *Let $\mathcal{M}$ be an oriented smooth m-dimensional manifold with boundary $\partial\mathcal{M}$, where it is understood that $\partial\mathcal{M}$ has the orientation induced by $\mathcal{M}$, and let ω be a smooth $(m-1)$-form with compact support on $\mathcal{M}$. Then:*

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega$$

It should be noted that ω on the right hand side of the above equation is interpreted as $\iota^*_{\partial\mathcal{M}}\omega$, where $\iota_{\partial\mathcal{M}} : \mathcal{M} \longrightarrow \partial\mathcal{M}$ is the restriction map from $\mathcal{M}$ onto $\partial\mathcal{M}$.

From *Stoke's Theorem* follows a Corollary:

Corollary A.5.2. *If $\mathcal{M}$ is as in the above Theorem, but $\partial\mathcal{M} = \emptyset$, then the integral of every exact form $\alpha = \mathrm{d}\beta$ over $\mathcal{M}$ is zero:*

$$\int_{\partial\mathcal{M}} \alpha = 0$$

A.6 Darboux's Theorem

Darboux's Theorem is a result about symplectic forms:

Theorem A.6.1. *Let $(\mathcal{M}, \omega)$ be a 2n-dimensional sympletic manifold, ω a symplectic form on $\mathcal{M}$. For any $P \in \mathcal{M}$, there exist smooth coordinates:*

$$(x^1, \ldots, x^n, y^1, \ldots, y^n)$$

centered at P in which ω has the following coordinate representation:

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

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