The Cauchy Problem in General Relativity: local and global results from initial data

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Introduction

This mini-course will be a brief tour through certain parts of mathematical relativity. Results will be presented mainly without proofs but we hope to present enough background to enable you appreciate some recent results in the area.

Here is a brief plan of the 4 lectures.

- **Lecture 1**: Introduction to Lorentzian geometry and causal theory.
- **Lecture 2**: The Einstein equations from the PDE perspective. The constraint equations and the local existence theorem of Choquet-Bruhat.
- **Lecture 3**: Solving the constraint equations via the conformal method
- **Lecture 4**: Extensions of the Penrose singularity theorem and an initial data version of the Gannon-Lee theorem. Topological censorship from the initial data point of view.

The final lecture is based on joint work with Michael Eichmair and Greg Galloway, cf. arXiv:1204.0278.

Lorentzian Manifolds

We start with an $(n+1)$-dimensional Lorentzian manifold (M, g). (M, g) is thus a psuedo-Riemannian manifold such that the metric

$$g : T_pM \times T_pM \longrightarrow \mathbb{R}$$

is a scalar product of signature $(-1, +1, \ldots, +1)$. With respect to a Lorentzian orthonormal basis $(e_0, e_1, \ldots, e_n)$, as a matrix,

$$[g_{ij}] = \mathrm{diag}(-1, +1, \ldots, +1).$$

Example: Minkowski space $\mathbb{M}^{n+1}$ is the Lorentzian analogue of Euclidean space. For vectors $X, Y \in T_p\mathbb{R}^{n+1}$ given in Cartesian coordinates on $\mathbb{R}^{n+!}$ by

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}$$

we define the Minkowski metric η by

$$\eta(X, Y) = -X^0 Y^0 + \sum_{i=1}^{n} X^i Y^i = \eta_{ij} X^i Y^i,$$

where $\eta_{ij} = \varepsilon_i \delta_{ij}$ and $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = (-1, 1, \ldots, 1)$.

Lorentzian Manifolds and basic causal theory

For any $p \in M$, a Lorentz manifold, we have a classification of vectors $X \in T_pM$ into timelike, null or spacelike, as follows

$$X \quad \text{is} \quad \begin{cases} \text{timelike} & \text{if} \quad g(X,X) < 0 \\ \text{null} & \text{if} \quad g(X,X) = 0 \\ \text{spacelike} & \text{if} \quad g(X,X) > 0 \end{cases}$$

We extend this notion to smooth curves $\gamma : (a,b) \longrightarrow M$ as follows

$$\gamma \quad \text{is} \quad \begin{cases} \text{timelike} & \text{if} \quad \gamma'(t) \text{ is timelike,} & \forall t \in (a,b) \\ \text{null} & \text{if} \quad \gamma'(t) \text{ is null,} & \forall t \in (a,b) \\ \text{spacelike} & \text{if} \quad \gamma'(t) \text{ is spacelike,} & \forall t \in (a,b) \end{cases}$$

We say that γ is **causal** if $\gamma'(t)$ is either timelike or null, $\forall t \in (a,b)$.

The world lines of particles follow causal curves, with light traveling on null curves (null geodesics) and massive particles traveling on timelike curves. At each point

$p \in M$ the set of timelike vectors form two disjoint open cones, which we'll denote as V_p^+ and V_p^+.

Lorentzian Manifolds and basic causal theory

We'll say that the Lorentzian manifold (M^{n+1}, g) is **time orientable** if it admits a timelike vector field. This allows us to make a continuous choice of a **future** light cone V_p^+ at each point of M.

Definition

A **spacetime** (M^{n+1}, g) *is a connected, time-oriented Lorentzian manifold.*

Let T denote a timelike vector field defining the time orientation on M. For any nonzero causal vector $v \in T_pM$, $g(v, T)$ is either positive or negative. If $g(v, T)$ is negative we say that v is **future pointing** (since v then lies in V_p^+) and if $g(v, T)$ is positive we say that v is **past pointing** (since v then lies in V_p^+).

A causal (timelike, null) curve γ is said to be **future pointing** if γ' is future pointing at each point along γ.

Future and Past sets

We say $p << q$ if there is a future pointing timelike curve in M from p to q, and $p < q$ if there is a future pointing causal curve in M from p to q. $p \leq q$ means that either $p = q$ or $p < q$.

Definition

Let A be a subset of M

$$\begin{aligned} I^+(A) &= \{p \in M : q << p \quad \textit{for some} \quad q \in A\} \\ J^+(A) &= \{p \in M : q \leq p \quad \textit{for some} \quad q \in A\} \end{aligned}$$

$I^+(A)$ is called the **chronological future** of A and $J^+(A)$ is called the **causal future** of A. The past sets $I^-(A)$ and $J^-(A)$ are similarly defined.

The sets $I^+(A)$ and $I^-(A)$ are always open, but for $J^+(A)$ and $J^-(A)$ no general statement holds without further assumption (remove a point from Minkowski spacetime to see that these need not be closed). However, we do have

$$I^+(A) = I^+(I^+(A)) = I^+(J^+(A)) = J^+(I^+(A)) \subseteq J^+(J^+(A)) = J^+(A).$$

Strong Causality

We need to impose a reasonable causality condition on our spacetimes in order to prohibit pathologies (such as closed timelike curves) and make them amenable to analysis.

The **strong causality condition** holds at $p \in M$ if, given any neighborhood U of p, there is a neighborhood $V \subseteq U$ of p such that every causal curve segment with endpoints in V lies entirely in U. A spacetime M is said to be **strongly causal** if strong causality holds at each point $p \in M$.

Strong causality prohibits the existence of closed causal curves, but is much stronger:

Lemma

Suppose that strong causality holds in a compact subset $K \subset M$. If $\gamma : [0, b) \to M$ is a future inextensible causal curve that starts in K, then it eventually leaves K and does not return, i.e., $\exists t_0 \in [0, b)$ such that $\gamma(t) \notin K \ \forall t \in [t_0, b)$.

So a future inextensible causal curve can not be contained forever within a compact set on which strong causality holds.

Global hyperbolicity

Definition

(M, g) is **globally hyperbolic** *if it is* **strongly causal** *and for every pair $p < q$, the set*

$$J(p, q) = J^+(p) \cap J^-(q)$$

is compact ("internal compactness").

Mathematically, global hyperbolicity often plays a role analogous to geodesic completeness in Riemannian geometry, but as the name suggests (and as we will see), it is also related to the solvability of hyperbolic PDE. Global hyperbolicity is also connected to the (strong) cosmic censorship conjecture introduced by Roger Penrose, which says that, generically (globally hyperbolic) solutions to the Einstein equations do not admit *naked singularities* (singularities visible to some observer).

Consequences of Global hyperbolicity

The following are some consequences of global hyperbolicity:

Theorem

Let (M, g) be a globally hyperbolic spacetime. Then

1. *The sets $J^{\pm}(A)$ are closed, for all compact subsets $A \subset M$.*
2. *The sets $J^{+}(A) \cap J^{-}(B)$ are compact, for all compact subsets $A, B \subset M$.*
3. *If $p < q$, then there is a maximal future directed causal geodesic from p to q (no causal curve from p to q can have greater length).*
4. *If we have convergent sequences on M; $p_n \to p$ and $q_n \to q$ and $p_n \leq q_n$, then $p \leq q$ (i.e. the causality relation $\leq$ is closed on M).*

In the way that for Riemannian manifolds **completeness** insures the existence of minimizing geodesics between points (recall the Hopf-Rinow theorem), global hyperbolicity is the condition which insures the existence of maximal causal geodesic segments (cf. 3. above).

Domains of Dependence

$A \subset M$ is called **achronal** if there is no pair of points $p, q \in A$ that can be connected by a timelike curve. Let $A \subset M$ be achronal, we define the **future and past domains of dependence** (also called *Cauchy developments*) of A as follows

$$\begin{aligned} D^+(A) &= \{p \in M : \text{every past inextendible causal curve from } p \text{ meets } A\}, \\ D^-(A) &= \{p \in M : \text{every future inextendible causal curve from } p \text{ meets } A\}. \end{aligned}$$

(p is a *future endpoint* of a causal curve γ if for any Lipschitz parametrization $\gamma : [0, \infty) \to M$, we have that for any neighborhood U of p, $\exists T = T(U)$ such that $\gamma(t) \in U, \forall t \geq T$. γ is future inextendible if it does not have a future endpoint.)

The **domain of dependence** of A is

$$D(A) = D^+(A) \cup D^-(A)$$

Since information travels along causal curves, $D(A)$ consist of the set of points in spacetime which are (potentially) influenced by *every point* in the set A, to either the past or the future. If physics is be deterministic then initial data on A should completely determine the state of the theory on all of D(A).

Domains of dependence and global hyperbolicity

Domains of dependence are tied to global hyperbolicity because the interior of the domain of dependence (viewed itself as a spacetime) is globally hyperbolic:

Proposition

Let $A \subset M$ be achronal.

(1) *Strong causality holds on* $\mathrm{int}\ \mathrm{D(A)}$.

(2) *Internal compactness holds on* $\mathrm{int}\ \mathrm{D(A)}$, *i.e., for all $p, q \in D(A)$, $J^+(p) \cap J^-(p)$ is compact.*

We wish to find a condition on an achronal subset A that will insure that the domain of dependence of A is all of M.

$$D(A) = M.$$

This will insure that the entire spacetime is deterministic relative to A, so that we can try to approach an analytical theory (namely the Einstein field equations) via an evolutionary perspective by prescribing initial data on A, determining the spacetime metric by solving a system of PDE in $D(A)$

Cauchy surfaces

Definition

A **Cauchy surface** *S is an achronal subset of M which is met by every inextendible causal curve in M.*

If S is a Cauchy surface for M then $S = \partial I^+(S) = \partial I^-(S)$, from this one can show that S is a closed C^0 hypersurface. The existence of Cauchy surfaces and global hyperbolicity for the **entire spacetime** are closely connected.

Theorem (Geroch)

Let M be a spacetime.

1. *If M is globally hyperbolic then it admits a Cauchy surface.*
2. *If S is a Cauchy surface for M then M is homeomorphic to $\mathbb{R} \times S$.*

Thus we see that for globally hyperbolic spacetimes, the topology of a Cauchy surface S determines the topology of the entire spacetime. At the end of this lecture we will make some remarks regarding the strengthening of this result to the smooth category.

Cauchy surfaces (cont.)

Sketch of Proof: For 1. let μ be a probability measure on M so μ is a positive measure with $\mu(M) = 1$. Let $f^-(p) = \mu[J^-(p)]$ and $f^+(p) = \mu[J^+(p)]$ and using these, define a positive function $f : M \to \mathbb{R}$ by

$$f(p) = \frac{f^-(p)}{f^+(p)} = \frac{\mu[J^-(p)]}{\mu[J^+(p)]}.$$

One can show that f is continuous, and strictly increasing along future directed causal curves. The claim is that the level sets of f are each Cauchy surfaces. This is demonstrated by showing that both (1) $f^-(p) \to 0$ along every *past* inextensible causal curve, and (2) $f^+(p) \to 1$ along every *future* inextensible causal curve. This shows that f attains all values of $(0, \infty)$ along *every* inextensible causal curve, and therefore each such curve intersects each level set precisely once.

To prove 2. one introduces a future directed timelike vector field X scaled so that the time parameter of each integral curve of X extends from $-\infty$ to ∞ with $t = 0$ corresponding to S. The flow of X provides the desired homeomorphism.

The topology of globally hyperbolic spacetimes

Proposition

If a spacetime has a Cauchy surface S than

$$D(S) = M$$

<u>*Sketch of Proof:*</u> Let $p \in M$ and let γ be an inextendible timelike geodesic through p. The γ intersects S in exactly one point. So p is in one of the sets S, $I^+(S)$ and $I^-(S)$. Since S is a Cauchy surface these sets are disjoint. Also $J^\pm(S)$ and $I^\mp(S)$ are disjoint. This shows

$$J^\pm(S) = M \setminus I^\mp(S),$$

so $J^\pm(S)$ are closed sets. Since $p \in I^-(S)$ implies $p \notin D^+(S)$, we have $D^+(S) \subset= J^+(S)$. One the other hand one can see that $J^+(S) = S \cup I^+(S) \subset D^+(S)$ so we see that $J^+(S) = D^+(S)$. Reversing time orientation above we see that $J^-(S) = D^-(S)$. These together show that $D(S) = M$.

Concluding remarks Lecture 1

In summary we have seen that a spacetime M is globally hyperbolic if and only if it admits a Cauchy surface S, it's global topology is $\mathbb{R} \times S$ and $D(S) = M$.

A **time function** on a Lorentzian manifold (M, g) is a function that is strictly increasing along any future directed causal curve. In the sketch of the proof of Geroch's Theorem we introduced the time function f.

The existence of a *smooth* Cauchy surface and a *smooth* time function leading to a splitting $M = \mathbb{R} \times S$ (as a diffeomorphism) was only rigorously established in a series of papers from 2003 – 2006 by Bernal and Sánchez.

Lecture 2: The Einstein Field Equations

We wish to view the Einstein field equations as an (evolutionary) PDE system for an Lorentzian space-time M:

$$Ric(g) - \frac{1}{2}R(g)g + \Lambda g = 8\pi \frac{G}{c^4} T \qquad (1)$$

- unknown: $g = g_{ab}$ is a Lorentz metric
- $Ric(g) = R_{ab}$ is the Ricci curvature
- $R(g)$ is the scalar curvature
- $T = T_{ab}$ is the stress-energy-momentum tensor. This encodes the non-gravitational physics (e.g. electromagnetic fields)
- We will work in units where $G = c = 1$. The factor of 8π comes from considering the Newtonian limit, so that these equations reduce to Newtonian gravity at speeds much slower than the speed of light.
- Λ is the "cosmological constant" which may be zero (but is currently thought to be positive in our Universe)

The Einstein Equations as a system of PDE

For simplicity, we will consider the vacuum Einstein equations, where $T \equiv 0$. By taking traces, one sees that here the Einstein equations reduce to

$$Ric(g) = 0. \tag{2}$$

Let's begin to understand this as a system of PDEs for the unknown metric g by trying to understand equation (2) as an equation consisting of derivatives for the metric components relative to a coordinate basis $\{\partial_\mu\}$ of T_pM. First recall that the Christoffel symbols are defined by

$$\nabla_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma,$$

Where ∇ is the Levi-Cevita connection (covariant derivative). (We employ the summation convention that we sum over repeated indices.)

In terms of the metric

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta}).$$

The Einstein Equations as a system of PDE (cont.)

The Riemann curvature is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components of the Riemann curvature tensor, relative to the coordinate basis, are defined by

$$R^{\alpha}_{\beta\gamma\delta} = \langle dx^{\alpha}, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle.$$

This leads to the formula

$$R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma},$$

with the components of the Ricci tensor given by the sum

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}.$$

This shows that the Ricci tensor is linear in the second derivatives of the metric, with coefficients which are rational in the components of the metric, and quadratic in the first derivatives of the metric, again with coefficients which are rational in g.

The Einstein Equations as a system of PDE (cont.)

Thus the vacuum Einstein equations are a second order system of quasi-linear (linear in the highest order derivatives) partial differential equations for the unknown metric g.

Systems of (nonlinear) PDE with *good algebraic properties* of the terms involving the highest order derivatives have been well studied and there are many methods to approach solving these types of equations. However (2) does not fall into any of these classes (e.g. elliptic, parabolic and hyperbolic systems of equations) in an arbitrary coordinate system.

The most significant difficulty with (2) from the PDE point of view is the high degree of non-uniqueness. This is due to the naturality of the equation. which leads to the coordinate or diffeomorphism invariance: if $\phi : M \to M$ is a diffeomorphism then

$$\phi^*(Ric(g)) = Ric(\phi^* g).$$

Thus, if g is a solution to (2) on M, so is $\phi^* g$.

The Einstein Equations as a system of PDE (cont.)

Another way to express this, locally, is as follows. Suppose that, with respect to coordinates $\{x^\mu\}$, we have a matrix of functions $g_{\mu\nu}(x)$ satisfying the quasi-linear system of PDE arising from (2) as previously described. If we perform a coordinate change $x^\mu \to y^\alpha(x^\mu)$, then the matrix of functions $\tilde{g}_{\alpha\beta}(y)$ defined by

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\alpha\beta}(y) = g_{\mu\nu}(x(y)) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

will also solve (2) (where the resulting x-derivatives are replaced by y-derivatives).

In the language of physics, one says that the diffeomorphism group expresses the gauge freedom of the Einstein field equations. Remarkably, in 1952, Yvonne Choquet-Bruhat proved that there is an underlying system of *hyperbolic* PDE governing the behavior of (2). This involves the introduction of a special set of coordinates (which in particular, breaks the diffeomorphism invariance) and the exploitation of the Bianchi identity together with the **Einstein constraint equations** to obtain a solution of the geometric equation.

The Geometry of Spacelike Hypersurfaces

Let (M, g) be a spacetime and let

$$i : V \hookrightarrow M$$

be an embedded *spacelike* hypersurface. This means that the induced metric $h = i^*(g)$ on V is Riemannian (i.e. has positive definite signature). Let η denote the timelike future-pointing unit normal vector field to V. If we let ∇ be the Levi-Cevita connection on (M, g) and ∇^V be the Levi-Cevita connection on (V, h) then recall that the second fundamental form, K on V, is defined by considering vector fields X and Y tangent to V and setting

$$\nabla_X Y = \nabla_X^V Y + K(X, Y)\eta,$$

so that, for each $p \in V$

$$K : T_pV \times T_pV \longrightarrow \mathbb{R}.$$

Note that, using the fact that ∇ is torsion free and compatible with g, one can see that

$$K(X, Y) = g(\nabla_X \eta, Y) \implies K(X, Y) = K(Y, X),$$

so K is symmetric.

The Geometry of Spacelike Hypersurfaces (cont.)

A time function t on (M, g) is *adapted to* V if V is a level set of t. If $x = \{x^i\}$ are local coordinates on V then (x, t) form adapted local coordinates for M near V. With respect to such a coordinate system, the *lapse-shift* form for the vector field η is

$$\eta = N^{-1}\left(\frac{\partial}{\partial t} - X^i \frac{\partial}{\partial x^i}\right)$$

where N is call the lapse function and $X = X^i \frac{\partial}{\partial x^i}$ is called the shift vector field.

Fact

In terms of h, N and X the ambient metric on M is expressed in these coordinates by

$$g = -N^2 dt^2 + h_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$$

and the second fundamental form is given by

$$K_{ij} = K\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{1}{2} N^{-1}\left(\frac{\partial h_{ij}}{\partial t} - \mathcal{L}_X h_{ij}\right),$$

where $\mathcal{L}_X h_{ij}$ is the Lie derivative of the spatial metric h in the direction X.

The Geometry of Spacelike Hypersurfaces (cont.)

In particular, this gives a formula for the time derivative of the spatial metric

$$\frac{\partial}{\partial t} h_{ij} = 2NK_{ij} + \mathcal{L}_X h_{ij},$$

so in the special case when $N \equiv 1$ and $X \equiv 0$ (i.e. $\frac{\partial}{\partial t} = \eta$) we have

$$\frac{\partial}{\partial t} h_{ij} = 2K_{ij}.$$

Let's return to the general Einstein field equations (1) with $\Lambda = 0$, writing them with respect to an adapted local frame for $V \hookrightarrow M$. We define the Einstein tensor $G_{\mu\nu}$ via the left hand side of the equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3}$$

Recall that the vacuum equations simplify to vanishing Ricci curvature:

$$G_{\mu\nu} = 0 \Longleftrightarrow R_{\mu\nu} = 0.$$

The Einstein Constraint Equations

The Gauss and Codazzi equations for $V \hookrightarrow M$ tell us that the ambient Einstein equations (3) on M impose relationship on the intrinsic and extrinsic curvatures of $(V, h) \hookrightarrow (M, g)$ and the components of the stress-energy-momentum tensor $T_{\mu\nu}$ in a local adapted frame.

Proposition (The Einstein Constraint Equations)

If (M, g) is a spacetime satisfying the Einstein field equations (3), and $V \hookrightarrow M$ is a spacelike hypersurface with induced Riemannian metric h and second fundamental form K then

$$R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 = 16\pi T_{00} = 2G_{00} = 2\rho \tag{4}$$

$$\mathrm{div} K - \nabla(\mathrm{tr}_h K) = 8\pi T_{0i} = G_{0i} = J \tag{5}$$

where $\mathrm{div} K = \nabla_j K^j{}_i$

The scalar function ρ is called the *local mass density* and the vector field J is called the *local current density* of the set (V, h, K). Equation (4) is called the *Hamiltonian constraint equation* and Equation (5) is call the *momentum constraint equation*.

Wave Equations

For a scalar function ϕ on a spacetime (M, g) we define the *wave operator* associated to the metric g to be the linear operator given by the trace of the Hessian:

$$\begin{aligned} \Box_g \phi &\equiv \nabla_\mu \nabla^\mu \phi \\ &= \frac{1}{\sqrt{-\det g_{\alpha\beta}}} \partial_\mu (\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} \partial_\nu \phi). \end{aligned}$$

We will make use of the following result

Theorem

Given an open set $U \in V$ and smooth functions f_1, f_2 on U, there exists a unique smooth solution ϕ defined on $D(U)$ for the problem

$$\Box g \phi = 0, \qquad \phi|_U = f_1, \qquad \frac{\partial \phi}{\partial t}|_U = f_2.$$

We will will actually make use of a non-trivial generalization of this result, namely that we have existence and uniqueness for solutions of a class of **second order quasi-linear hyperbolic equations**.

Wave Coordinates

The idea is that, despite the fact the the vacuum Einstein equations are not hyperbolic, we can identify a portion of the top order part of the operator which looks like the wave operator applied to the metric.

For the moment suppose that we already know the metric g in a spacetime neighborhood $\mathcal{O}(V)$ of a spacelike hypersurface V. We introduce "wave" or "harmonic" coordinates $\{x^\alpha\}$ by setting

$$H^\alpha \equiv \Box_g x^\alpha = 0 \qquad \text{in} \qquad \mathcal{O}(V)$$

$$x^0 = 0, \quad x^i = \bar{x}^i, \quad \text{and} \quad \frac{\partial x^\alpha}{\partial t} = 0 \qquad \text{on} \qquad V$$

(Greek letters vary among spacetime indices, Roman letters are used for spatial indices only, with 0 being the "time" coordinate.)

By specifying a coordinate system we have a good chance of "breaking" the Guage symmetry imposed by the diffeomorphism invariance of the geometric equations. However this is only useful if (1) the new equation takes a form that can be analyzed in these coordinates, and (2) we have a way of propagating a set of coordinates off of the initial spacelike hypersurface V so that they are indeed harmonic relative to the evolved metric (which is **not** known a priori!). The remarkable fact is that both of these conditions are indeed satisfied.

The Reduced Einstein Equations

The reason this is useful is that one can show that the Ricci curvature can be written as

$$R_{\alpha\beta} = R^H_{\alpha\beta} - H_{(\alpha,\beta)} \tag{6}$$

where

$$\begin{aligned} R^H_{\alpha\beta} &= -\frac{1}{2} g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + Q(g, \partial g) \\ &= -\frac{1}{2} \Box_g g_{\alpha\beta} + Q(g, \partial g) \end{aligned}$$

is the "harmonic" part and $H_{(\alpha,\beta)}$ vanishes in wave coordinates.

The **reduced vacuum Einstein Equations** are

$$R^H_{\alpha\beta} = 0.$$

This is a second order quasi-linear hyperbolic system for the metric g, so we can solve this provided we specify Cauchy data

$$g_{\alpha\beta} \quad \text{and} \quad \frac{\partial g_{\alpha\beta}}{\partial t} \qquad \text{on} \qquad V.$$

The Initial Data Set

Definition

An **initial data set** *for the* $(n+1)$*-dim'l vacuum Einstein Equations is a set* (V, h, K) *where* (V, h) *is an n-dim'l Riemannian manifold and* K *is a symmetric* $(0,2)$ *tensor on* V*.*

We need to define the Cauchy data for the reduced Einstein equations from a given initial data set as above. First define

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & h_{ij} \end{pmatrix} \qquad \text{at} \qquad t = 0$$

This forces (recall the lapse and shift discussion)

$$\frac{\partial g_{ij}}{\partial t} = 2K_{ij} \qquad \text{at} \qquad t = 0.$$

We are still free to choose $\dfrac{\partial g_{0\beta}}{\partial t}$. We will do this so that

$$H_\alpha = 0 \qquad \text{initially on} \qquad V.$$

Propogating the wave coordinates

We have not yet made use of the vacuum constraint equations!

The contracted second Bianchi identity (see Ratzkin's lecture) implies that the Einstein tensor is divergence free

$$\nabla^{\beta} G_{\alpha\beta} = 0.$$

This implies an evolution equation for H^{α}

$$\Box_g H_{\alpha} + \text{l.o.t.} = 0 \tag{7}$$

where l.o.t. indicates a collection of lower order terms which are linear in H_{α}.

Since we have choosen $\dfrac{\partial g_{0\beta}}{\partial t}$ so that $H_{\alpha} = 0$ initially, if we can also insure that $\dfrac{\partial H_{\alpha}}{\partial t} = 0$, then by uniqueness for solutions to (7), we must have

$$H_{\alpha} \equiv 0 \qquad \text{on} \qquad \mathcal{O}(V).$$

This implies that the solution to the reduced Einstein equations is actually a solution to the full geometric vacuum Einstein equations

$$Ric(g) = 0.$$

Completing the argument: the role of the constraint equations

Proposition

The vacuum constraint equations for (V, h, K) imply that

$$\frac{\partial H_\alpha}{\partial t} = 0.$$

Sketch of Proof: The momentum constraint equation says

$$G_{0i} = \mathrm{div} K - \nabla(\mathrm{tr}_h K) = 0$$

This implies that

$$-\frac{1}{2} H_{0,i} - \frac{1}{2} \frac{\partial H_i}{\partial t} = 0.$$

However we have $H_{0,i} = 0$ on V so that

$$\frac{\partial H_i}{\partial t} = 0 \qquad \text{for} \qquad i = 1, \dots n.$$

Completing the argument: the role of the constraint equations (cont.)

The Hamiltonian constraint equation gives

$$G_{00} = R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 = 0$$

and this shows that

$$\begin{aligned} G_{00} &= -\frac{\partial H_0}{\partial t} - \frac{1}{2}\frac{\partial H_0}{\partial t} g_{00} \\ &= -\frac{\partial H_0}{\partial t} + \frac{1}{2}\frac{\partial H_0}{\partial t} \\ &= -\frac{\partial H_0}{\partial t} = 0 \end{aligned}$$

as desired. Therefore $\frac{\partial H_\alpha}{\partial t} = 0$ so that $H_\alpha \equiv 0$ on $\mathcal{O}(V)$. i.e. the coordinates we obtain are actually wave coordinates for the spacetime metric evolved from h on V by solving the reduced Einstein equations. This metric therefore satisfies the vacuum Einstein equations.

Initial data and local well posedness

In 1952, Yvonne Choquet-Bruhat established the existence of a local in time solution of the vacuum Einstein equations, $Ric(g) = 0$.

- The constraint equations, together with the second Bianchi identity, ensures that the wave coordinate gauge is evolved in time as one solves the reduced Einstein equations, yielding a solution of the full geometric equations.

Theorem (Choquet-Bruhat 1952)

Given an initial data set $(V; h, K)$ satisfying the vacuum constraint equations there exists a spacetime (M, g) satisfying the vacuum Einstein equations $Ric(g) = 0$ where $V \hookrightarrow M$ is a spacelike surface with induced metric h and second fundamental form K.

This is a local existence result. Nothing is claim about the "size" of (M, g) (one thinks of it simply as a "thickening of (V, h) into a small spacetime neighborhood). One would like a more global solution.

Maximal, globally hyperbolic developments

Theorem (Choquet-Bruhat & Geroch, 1969)

Given an initial data set $(V; h, K)$ *satisfying the vacuum constraint equations there exists a unique, globally hyperbolic, maximal, spacetime* (M, g) *satisfying the vacuum Einstein equations* $Ric(g) = 0$ *where* $V \hookrightarrow M$ *is a Cauchy surface with induced metric* h *and second fundamental form* K*. Moreover any other such solution is a subset of* (M, g)*.*

This is a much more satisfactory result, but it still leaves open the most difficult questions concerning global existence.

As pointed out yesterday, in both of these results, a central role is played by the **existence of initial data sets solving the Einstein constraint equations**. We now turn our attention to this question.

Solving the Einstein constraint equations

The initial data for the Einstein field equations for (M^{n+1}, g) consist of specifying on an n-dimensional manifold Σ

- a Riemannian metric $\bar{\gamma}$
- a symmetric 2-tensor $\bar{K}$
- $\mathcal{F}$ a collection of initial data for the non-gravitational fields.

The choices of initial data are constrained by the Gauss and Codazzi equations, which gives rise to the " Einstein constraint equations.

In terms of the data $(\bar{\gamma}, \bar{K}, \mathcal{F})$ above, the Einstein constraint equations are

$$
\begin{aligned}
\mathrm{div}_{\bar{\gamma}} \bar{K} - \nabla(\mathrm{tr} \bar{K}) &= J(\bar{\gamma}, \mathcal{F}) && \text{(Momentum constraint)} \\
R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\mathrm{tr} \bar{K})^2 &= 2\rho(\bar{\gamma}, \mathcal{F}) && \text{(Hamiltonian constraint)} \\
C(\bar{\gamma}, \mathcal{F}) &= 0 && \text{(Non-gravitational constraints)}
\end{aligned}
$$

Concrete example: Einstein-Maxwell in 3+1 dimensions

The non-gravitational fields consist of an electric vector field $\bar{E}$ and a magnetic vector field $\bar{B}$.

- current density: $J(\bar{\gamma}, \mathcal{F}) = (\bar{E} \times \bar{B})_{\bar{\gamma}}$
- energy density: $\rho(\bar{\gamma}, \mathcal{F}) = \frac{1}{2}(|\bar{E}|^2_{\bar{\gamma}} + |\bar{B}|^2_{\bar{\gamma}})$

The non-gravitational constraints are

$$
\begin{aligned}
\mathrm{div}_{\bar{\gamma}} \bar{E} &= 0 \\
\mathrm{div}_{\bar{\gamma}} \bar{B} &= 0
\end{aligned}
$$

Einstein-scalar fields

The vacuum Einstein equations come from a variational principle; the Einstein-Hilbert action

$$\mathcal{S}(g,\Psi) = \int_M R(g)\,dv_g.$$

Thein Einstein-scalar field action is:

$$\mathcal{S}(g,\Psi) = \int_M [R(g) - \frac{1}{2}|\nabla\Psi|_g^2 - V(\Psi)]\,dv_g,$$

Where Ψ is a scalar valued function on M. From this we obtain the Einstein-scalar field equations:

$$\boxed{\begin{gathered} G_{\alpha\beta} = T_{\alpha\beta} = \nabla_\alpha\Psi\nabla_\beta\Psi - \tfrac{1}{2}g_{\alpha\beta}\nabla_\mu\Psi\nabla^\mu\Psi - g_{\alpha\beta}V(\Psi) \\ \nabla_\mu\nabla^\mu\Psi = V'(\Psi). \end{gathered}}$$

(where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R(g)\,g_{\alpha\beta}$ is the Einstein curvature tensor, and we have deliberately neglected the factor of 8π for simplicity).

Einstein-scalar field Constraint Equations

Initial data on Σ^n consists of

- $\bar{\gamma}$ (the spatial metric)
- $\bar{K}$ (the second fundamental form, or extrinsic curvature)
- $\bar{\psi}$ (the scalar field restricted to Σ)
- $\bar{\pi}$ (the normalized time derivative of Ψ restricted to Σ).

The constraint equations are then

$$\boxed{\begin{aligned} \mathrm{div}_{\bar{\gamma}} \bar{K} - \nabla(\mathrm{tr} \bar{K}) &= -\bar{\pi} \nabla \bar{\psi} \\ R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\mathrm{tr}\bar{K})^2 &= \bar{\pi}^2 + |\nabla \bar{\psi}|^2_{\bar{\gamma}} + 2V(\bar{\psi}). \end{aligned}}$$

(Note: the scalar field does not introduce any new constraint equations.)

The conformal method (aprés Lichnerowicz, Choquet-Bruhat and York)

The constraint equations are highly underdetermined (in the $(3+1)$-dimensional vacuum case, they consist locally of 4 equations for the 12 unknowns represented by the symmetric tensors $\bar{\gamma}$ and $\bar{K}$).

Split the initial data into two parts

- "conformal data": regard as being freely chosen.
- "determined data": found by solving the constraint equations, reformulated as a determined system of elliptic PDE.

General Criteria: For constant mean curvature (CMC) initial data, where $\tau = \mathrm{tr}_{\bar{\gamma}} \bar{K}$ is constant, we want the equations to be "semi-decoupled":

- First solve the nongravitational constraints.
- Then solve the conformally formulated momentum constraint.
- These solutions enter into the conformally formulated Hamiltonian constraint, which we solve for the remaining piece of determined data.

The conformal and determined data (vacuum case)

For the gravitational (vacuum) data, the free "conformal data" consists of

- γ, a Riemannian metric on Σ, representing a chosen *conformal class of metrics* $[\gamma] = \{\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma : \theta > 0\}$.
- $\sigma = \sigma_{ab}$, a symmetric tensor which is divergence-free and trace-free w.r.t. γ (σ is a transverse-traceless or TT-tensor).
- τ, a scalar function representing the mean curvature of the Cauchy surface Σ in the resulting spacetime.

The "determined data" consists of

- ϕ, a positive function
- $W = W^a$, a vector field

Reconstructed data (vacuum case)

Use (ϕ, W) to reconstruct an initial data set $(\bar{\gamma}, \bar{K})$ from the conformal data set (γ, σ, τ) via:

$$\begin{aligned} \bar{\gamma} &= \phi^{\frac{4}{n-2}}\gamma \\ \bar{K} &= \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma \end{aligned}$$

here the operator $\mathcal{D}$ is the conformal Killing operator relative to γ, $(\mathcal{D}W)_{ab} := \nabla_a W_b + \nabla_b W_a - \frac{2}{n}\gamma_{ab}\nabla_m W^m$, whose kernel consists of conformal Killing fields.

$(\bar{\gamma}, \bar{K})$ satisfy the vacuum constraint equations if and only if (ϕ, W) satisfy

$$\begin{cases} \operatorname{div}(\mathcal{D}W) = \frac{n}{n-1}\phi^{\frac{2n}{n-2}}\nabla\tau \\ \\ c_n^{-1}\Delta_\gamma\phi - R(\gamma)\phi + \left(|\sigma + \mathcal{D}W|^2_\gamma\right)\phi^{-\frac{3n-2}{n-2}} - \dfrac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} = 0 \end{cases}$$

where $c_n = \frac{n-2}{4(n-1)}$.

Results (vacuum, CMC case with $\Lambda = 0$)

For cosmological (spatially compact) vacuum spacetimes and CMC initial data this approach yields a complete understanding of the question of the existence of solutions to the constraints equations.

Let $\mathcal{Y}([\gamma])$ denote the Yamabe invariant of the conformal class of metrics determined by γ.

Theorem (Choquet-Bruhat, York, Ó Murchadha, Isenberg)

Given a conformal initial data set (γ, σ, τ) the existence of a solution to the Lichnerowicz equation (and therefore, taking $W = 0$, to the vacuum constraints) is indicated as follows

	$\sigma \equiv 0, \tau = 0$	$\sigma \equiv 0, \tau \neq 0$	$\sigma \not\equiv 0, \tau = 0$	$\sigma \not\equiv 0, \tau \neq 0$
$\mathcal{Y}([\gamma]) < 0$	*No*	*Yes*	*No*	*Yes*
$\mathcal{Y}([\gamma]) = 0$	*Yes*	*No*	*No*	*Yes*
$\mathcal{Y}([\gamma]) > 0$	*No*	*No*	*Yes*	*Yes*

Concrete example II: Einstein-Maxwell

When we couple a Maxwell electromagnetic field in to the Einstein field equations the net effect, on the level of the constraint equations, is not very disruptive. Now the conformal initial data consists of $(\gamma, \sigma, \tau, E, B)$ where

- (γ, σ, τ) are the gravitational initial data
- E and B are divergence free vector fields providing the initial data for the electric and magnetic fields.

The Einstein-Maxwell Lichnerowicz equation is then

$$\Delta\phi - \frac{1}{8}R(\gamma)\,\phi + \frac{1}{8}(|\sigma + \mathcal{D}W|^2_\gamma)\phi^{-7} + \frac{1}{8}(|E|^2_\gamma + |B|^2_\gamma)\phi^{-3} - \frac{1}{12}\tau^2\phi^5 = 0.$$

Note that the Maxwell field contributes a term with a positive coefficient and a negative power of ϕ. This allows it to be treated in the analysis in exactly the same way at the $|\sigma + \mathcal{D}W|^2_\gamma$ term.

Conformally formulated momentum constraint equation for the Einstein-scalar field system

Under the conformal rescaling $\bar{\gamma} = \phi^{\frac{4}{n-2}}\gamma$, we rescale the scalar field initial data as follows:

$$\bar{\psi} = \psi \qquad \text{and} \qquad \bar{\pi} = \phi^{\frac{2n}{n-2}}\pi.$$

The momentum constraint then becomes

$$\boxed{\mathrm{div}_\gamma(\mathcal{D}W) = \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi.}$$

- When τ is constant, the conformally formulated momentum constraint equation does not involve the conformal factor ϕ.
- $\mathrm{div}_\gamma \circ \mathcal{D}$ is a self-adjoint, second order, elliptic operator. On a compact manifold, $\ker(\mathrm{div}_\gamma \circ \mathcal{D}) = \{\text{conformal Killing vector fields}\}$. (If there are no conformal Killing vector fields, this equation has a unique solution for any choice of (ϕ, τ, ψ, π).)

Conformally formulated Hamiltonian constraint equation for the Einstein-scalar field system

Let

$$\mathcal{R}_{\gamma,\psi} = c_n \left(R(\gamma) - |\nabla\psi|^2_\gamma \right), \qquad \mathcal{A}_{\gamma,W,\pi} = c_n \left(|\sigma + \mathcal{D}W|^2_\gamma + \pi^2 \right)$$

and

$$\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n}\tau^2 - 4V(\psi) \right).$$

The Hamiltonian constraint equation for the Einstein-scalar conformal data $(\gamma, \sigma, \tau, \psi, \pi)$ (with a given vector field W satisfying the conformally formulated momentum constraint equation) is

$$\boxed{\Delta_\gamma \phi - \mathcal{R}_{\gamma,\psi}\,\phi + \mathcal{A}_{\gamma,W,\pi}\,\phi^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\,\phi^{\frac{n+2}{n-2}} = 0.}$$

This is the Einstein-scalar field Lichnerowicz equation.

Analysis of the Einstein-scalar field Lichnerowicz equation

This equation differs from other matter/field Lichnerowicz equations (e.g. for vacuum, Maxwell, Yang-Mills, fluids) in two very significant ways:

- coefficient of linear term is $\mathcal{R}_{\gamma,\psi} = c_n \left(R(\gamma) - |\nabla\psi|^2_\gamma \right)$ vs. $R(\gamma)$.
- $\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n}\tau^2 - 4V(\psi) \right)$ may not, in general, have a fixed sign.

However

- The Lichnerowicz equation is conformally covariant: set

$$\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma \qquad \tilde{\sigma} = \theta^{-2}\sigma \qquad \tilde{\tau} = \tau$$
$$\tilde{\psi} = \psi \qquad \tilde{\pi} = \theta^{\frac{2n}{n-2}}\pi$$

  ϕ solution w.r.t. $(\gamma, \sigma, \tau, \psi, \pi) \Leftrightarrow \frac{\phi}{\theta}$ solution w.r.t. $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\tau}, \tilde{\psi}, \tilde{\pi})$.
- $\mathcal{A} \equiv 0 \Rightarrow$ a solution to the Lichnerowicz equation corresponds to a solution of the prescribed scalar curvature-scalar field equation

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

The Yamabe-scalar field conformal invariant

Definition

The Yamabe-scalar field conformal invariant is defined by

$$\mathcal{Y}_\psi([\gamma]) = \inf_{u \in H^1(\Sigma)} \frac{c_n^{-1} \int_\Sigma [|\nabla u|_\gamma^2 + c_n \left(R(\gamma) - |\nabla \psi|_\gamma^2\right) u^2] \, d\eta_\gamma}{\left(\int_\Sigma u^{\frac{2n}{n-2}} \, d\eta_\gamma\right)^{\frac{n-2}{n}}}.$$

- Hölder's inequality $\Rightarrow \mathcal{Y}_\psi([\gamma]) > -\infty$.
- $\mathcal{Y}_\psi([\gamma])$ is independent of the choice of background metric in the conformal class used to define it. It therefore defines an invariant of the conformal class and scalar field.

The conformal information from $\mathcal{Y}_\psi([\gamma])$

Define the *conformal scalar-field Laplace operator* $L_{\gamma,\psi}$ by

$$L_{\gamma,\psi} u = \Delta_\gamma u - c_n \left(R(\gamma) - |\nabla \psi|^2_\gamma \right) u.$$

(scalar-field analog of the conformal Laplace operator).

Proposition

The following conditions are equivalent:

(i) $\mathcal{Y}_\psi([\gamma]) > 0$ *(respectively* $= 0, < 0$*).*

(ii) *There exists a metric* $\tilde\gamma \in [\gamma]$ *which satisfies* $(R(\tilde\gamma) - |\tilde\nabla\psi|^2_{\tilde\gamma}) > 0$ *everywhere on* Σ *(respectively* $= 0, < 0$*).*

(iii) *For any metric* $\tilde\gamma \in [\gamma]$*, the first eigenvalue,* λ_1*, of the self-adjoint, elliptic operator* $-L_{\tilde\gamma,\psi}$ *is positive (respectively zero, negative).*

Solving the Lichnerowicz equation I

On a compact manifold, in joint work with Y. Choquet-Bruhat and J. Isenberg (2007) we were able to establish the following

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	Y	N	N
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	PR	N	N	N

Table 1: Results for $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

- Y The Lichnerowicz equation can be solved for that class of conformal data
- N The Lichnerowicz equation has no positive solution
- N&S There is a necessary and sufficient condition which needs to be checked
- PR We have partial results

Solving the Lichnerowicz equation II

We also have

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	N	Y	Y
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	NR	Y	Y	Y

Table 2: Results for $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

- Y The Lichnerowicz equation can be solved for that class of conformal data.
- N The Lichnerowicz equation has no positive solution.
- N&S There is a necessary and sufficient condition which needs to be checked.
- PR We have partial results.
- NR We have no results indicating existence or non-existence.

Remarks on the Proofs I: Non-existence

We assume (via conformal invariance and Proposition 1) that $\operatorname{sign}(\mathcal{R}_{\gamma,\psi}) = \operatorname{sign}(\mathcal{Y}_\psi([\gamma]))$ and write the Lichnerowicz equation as

$$\Delta_\gamma \phi = \mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi)$$

where

$$\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) = \mathcal{R}_{\gamma,\psi}\,\phi - \mathcal{A}_{\gamma,W,\pi}\,\phi^{-\frac{3n-2}{n-2}} + \mathcal{B}_{\tau,\psi}\,\phi^{\frac{n+2}{n-2}}.$$

- All of the "N" entries in Tables 1 & 2 correspond to the situation where if ϕ were a positive solution on Σ then either $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \leq 0$ or $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \geq 0$ (but not identically zero). Integration then leads to an immediate contradiction.

Remarks on the Proofs II: Existence

We again assume that $\operatorname{sign}(\mathcal{R}_{\gamma,\psi}) = \operatorname{sign}(\mathcal{Y}_\psi([\gamma]))$. All the "Y" existence results are obtained by the method of sub- and super-solutions.

- Y's correspond to where one can directly find constant sub- and super-solutions.
- Y's correspond to where we first conformally transform the data via the positive solution to an well chosen linear equation, and then find constant sub- and super-solutions.
- The $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ case (with $\mathcal{Y}_\psi([\gamma]) < 0$ and $\mathcal{B}_{\tau,\psi} \geq 0$) listed as "N&S Cond." may be reduced to the $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ case, where this is the prescribed scalar curvature-scalar field problem

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

  The necessary and sufficient condition for solving this problem in the pure scalar curvature case is due to A. Rauzy. We believe that his argument generalizes to our setting.

Remarks

- We obtain similar results when (Σ, γ) is asymptotically flat.
- Our constructions allow for rough (low regularity) initial data (cf. work of Y. Choquet-Bruhat and D. Maxwell).
- For existence cases we can also show the uniqueness of the solution within the conformal class.
- The most challenging and important area of current investigation for the (spacelike) constraint equations is the question of what happens (vacuum case) when the mean curvature is not constant and the equations are fully coupled. Important recent work in this area has been done by (Holst, Nagy, and Tsogtgrel; Maxwell; and Dahl, Gicquaud and Humbert)

Lecture 4 Introduction

- Topological Censorship: results asserting the topological simplicity (at the fundamental group level) of the *Domain of Outer Communications* (DOC). These are *global spacetime* results.
- Singularity theorems are a precursor to topological censorship. In GR this means results implying geodesic incompleteness.
- We will explore the extent to which singularity theorems and topological censorship can be obtained from constraints placed solely on an initial data set, avoiding the very difficult questions of global evolution.
- This work (joint with Michael Eichmair and Greg Galloway, arXiv:1204.0278) relies on recent developments in the theory of marginally outer trapped surfaces (MOTS) and our understanding of the topology of three-manifolds.

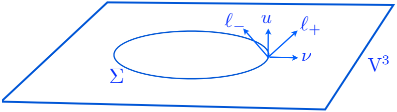
Initial data singularity theorems

Question: What constitutes an initial data singularity theorem?

Penrose: Certainly, conditions on an initial data set that imply the existence of a **trapped surface** should qualify.

$$(M^4, g) = \text{4-dim spacetime}$$
$$(V, h, K) = \text{3-dim initial data set in } (M^4, g)$$
$$\Sigma = \text{closed 2-sided surface in } V$$

Σ admits a smooth unit normal field ν in V.



$\ell_+ = u + \nu$ future directed outward null normal

$\ell_- = u - \nu$ future directed inward null normal

Initial data singularity theorems

With respect to ℓ_+ and ℓ_- we define the null expansion scalars, θ_+, θ_-:

$$\theta_\pm = \mathrm{tr}\chi_\pm = \mathrm{div}_\Sigma \ell_\pm$$

Σ is called a **trapped surface** if both $\theta_- < 0$ and $\theta_+ < 0$. This signals the presence of a strong gravitational field.

- Focusing on the outward null normal, Σ is called a **marginally outer trapped surface** (MOTS) if $\theta_+ = 0$.
- In terms of initial data (V, h, K),

  $$\theta_\pm = \mathrm{tr}_\Sigma K \pm H\,,$$

  where $H =$ mean curvature of Σ within V. Thus, we see that in the time-symmetric (or "Riemannian") case, when $K \equiv 0$, MOTS correspond to minimal surfaces.

Initial data singularity theorems

Theorem (Penrose singularity theorem, 1965)

Consider a spacetime (M, g). Suppose:

(i) *M is globally hyperbolic with non-compact Cauchy surface V.*

(ii) *M obeys the Null Energy Condition (NEC): $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X.*

(iii) *V contains a trapped surface Σ.*

Then at least one of the future directed null normal geodesics to Σ must be incomplete.

Conditions on an initial data set that imply the existence of a trapped surface should therefore be viewed as an initial data singularity theorem. Cf., Beig and Ô'Murchadha, '91.

Schoen and Yau '83 have given criteria for the existence of a MOTS in an initial data set. We claim that conditions on an initial set data that imply the existence of a MOTS should also be viewed as an initial data singularity theorem.

Initial data singularity theorems

A Penrose-type singularity theorem holds for MOTS:

Theorem (Eichmair, Galloway and P., 2012)

Consider a spacetime (M, g). Suppose:

(i) *M is globally hyperbolic with non-compact Cauchy surface V.*
(ii) *M obeys the NEC.*
(iii) *V contains a MOTS Σ.*
(iv) *The generic condition holds on each future and past inextendible null normal geodesic η to Σ.*

Then at least one of the null normal geodesics to Σ must be future or past incomplete.

We must take things one step further. There is a more general object in an initial data set that gives rise to a Penrose-type singularity theorem, which we refer to as an immersed MOTS.

Immersed MOTS

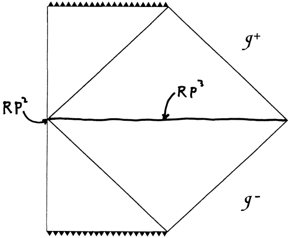
Definition

A subset $\Sigma \subset V$, in an initial data set (V, h, K), is an immersed MOTS if there exists a finite cover $p : \tilde{V} \to V$ and a closed, embedded MOTS $\tilde{\Sigma}$ in $(\tilde{V}, p^*h, p^*K)$ such that $p(\tilde{\Sigma}) = \Sigma$.

A natural example is given by the RP^2 in the so-called RP^3 geon. This is obtained from the extended Schwarzschild solution:

$$M_{Sch} = \mathbb{R}^2 \times S^2 \quad \text{with} \quad ds^2 = (-32m^3/r)e^{r/2m}(-dT^2 + dX^2 + r^2 d\Omega^2)$$

by making the identifications, $X \to -X\,, \quad p \in S^2 \to -p \in S^2$



Initial data singularity theorem for immersed MOTS

Theorem (Eichmair, Galloway and P., 2012)

Consider a spacetime (M, g). Suppose:

(i) *M is globally hyperbolic with non-compact Cauchy surface V.*
(ii) *M obeys the NEC.*
(iii) *V contains an immersed MOTS Σ.*
(iv) *The generic condition holds on each future and past inextendible null normal geodesic η to Σ.*

Then at least one of the null normal geodesics to Σ must be future or past incomplete.

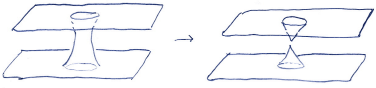
Proof: If Σ is an immersed MOTS in a Cauchy surface V then pass to the covering spacetime to obtain a MOTS $\tilde{\Sigma}$ in a Cauchy surface $\tilde{V}$. Applying our previous "Penrose for MOTSs" Theorem and projecting back down produces the required incomplete geodesic.

Conclusion *Conditions on an initial data set that imply the existence of an immersed MOTS should be viewed as an initial data singularity theorem.*

Gannon-Lee singularity theorem

An important precursor to topological censorship is the Gannon-Lee singularity theorem.

Familiar examples suggest that nontrivial topological structures tend to pinch off and form singularities.



Theorem (Gannon 1975, Lee 1976)

Let (M, g) be a globally hyperbolic spacetime which satisfies the null energy condition (NEC), and which contains a Cauchy surface S which is regular near infinity. If S is not simply connected then M is future null geodesically incomplete.

Principle of Topological Censorship

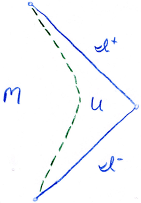
Weak Cosmic Censorship Conjecture: generically the process of gravitational collapse leads to the formation of an event horizon which hides the singularities from view.

Topological Censorship: nontrivial topology is hidden behind the event horizon, and the DOC - the region exterior to all black holes (and white holes) - should have simple topology.

This notion was formalized by the Topological Censorship Theorem of Friedman, Schleich and Witt ('93). Their theorem applies to AF spacetimes, i.e. spacetimes admitting a regular null infinity

$$\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-, \qquad \mathscr{I}^\pm \approx \mathbb{R} \times S^2$$

a neighborhood U of which is simply connected.



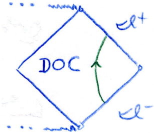
Topological Censorship Theorem of FSW

Theorem (Friedman, Schleich and Witt 1993)

Let (M, g) be a globally hyperbolic, AF spacetime satisfying the NEC. Then every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ can be deformed (with endpoints fixed) to a curve lying in the simply connected neighborhood U of $\mathscr{I}$.

This is a statement about the Domain of Outer Communications:

$$D = DOC = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$$



Topology of the DOC

The FSW Topological Censorship theorem does not give direct information about the topology of the DOC.

In '94 Chruściel and Wald used FSW to prove that for stationary black hole spacetimes, the DOC is simply connected.

Subsequent to that Galloway ('95) showed that the simple connectivity of the DOC holds in general.

Theorem (Galloway, 1995)

Let (M, g) be an asymptotically flat spacetime such that a neighborhood of $\mathscr{I}$ is simply connected, Suppose that the DOC is globally hyperbolic and satisfies the NEC. Then the DOC is simply connected.

Thus, topological censorship can be taken as the statement that the Domain of Outer Communications is simple connected.

Initial data version of Gannon-Lee singularity theorem

Theorem (Eichmair, Galloway and P., 2012)

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set. If V is not diffeomorphic to $\mathbb{R}^3$ then V contains an immersed MOTS.

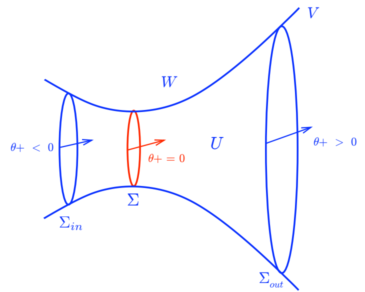
Thus, if V is not $\mathbb{R}^3$, spacetime is singular, from the initial data point of view.

- This theorem may be viewed as a non-time-symmetric version of a theorem of Meeks-Simon-Yau (1982), which implies that an asymptotically flat 3-manifold (no curvature conditions!) that is not diffeomorphic to $\mathbb{R}^3$ contains an *embedded* stable minimal sphere or projective plane.
- The proof relies on deep existence results for MOTSs together with our understanding of the topology of three-manifolds.

Existence of MOTSs

Theorem (Schoen; Andersson & Metzger; Eichmair)

Let W be a connected compact manifold-with-boundary in an initial data set (V, h, K). Suppose, $\partial W = \Sigma_{in} \cup \Sigma_{out}$, such that Σ_{in} is outer trapped ($\theta_+ < 0$) and Σ_{out} is outer untrapped ($\theta_+ > 0$). Then there exists a smooth compact MOTS, Σ, in W that separates Σ_{in} from Σ_{out}.



The proof is based on inducing blow-up of Jang's equation, cf., survey article by Andersson, Eichmair, Metzger, arXiv:1006.4601.

Jang's equation

Given the initial data set (V^n, h, K), Jang's equation is the equation,

$$\hat{h}^{ij}\left(\frac{D_i D_j f}{\sqrt{1+|Df|^2}} - K_{ij}\right) = 0 \tag{8}$$

where f is a function on V, D is the Levi-Civita connection of h, and $\hat{h}^{ij} = h^{ij} - \frac{f^i f^j}{1+|Df|^2}$.

As observed by Schoen and Yau, (2) admits the geometric interpretation,

$$H(f) + \mathrm{tr}_{h_f} \bar{K} = 0\,, \tag{9}$$

where $H(f) =$ the mean curvature of $\Sigma_f = \mathrm{graph}\, f$ in $\mathbb{R} \times V$ and $h_f =$ induced metric on Σ_f.

Thus one sees that Jang's equation is closely related to the MOTS equation.

Topology of Three-dimensional Manifolds

Consider a compact, orientable 3-manifold N.

- Case 1: If N is simply connected then by the resolution of the Poincaré conjecture N is diffeomorphic to the 3-sphere, S^3.
- Case 2: If N is not simply connected, by work of Hempel, together with the positive resolution of the geometrization conjecture it is known that $\pi_1(N)$ is residually finite.

A group is said to be residually finite if for each non-identity element in the group there is a normal subgroup of finite index not containing that element. Corresponding to any such proper normal subgroup there is a finite sheeted covering manifold $p : \tilde{N} \to N$.

The upshot of these very deep (analytic) results are that either N is S^3 or N has a finite sheeted cover.

Initial data version of Gannon-Lee singularity theorem

Sketch of proof: Assume V contains no immersed MOTSs and show it is diffeomorphic to $\mathbb{R}^3$.

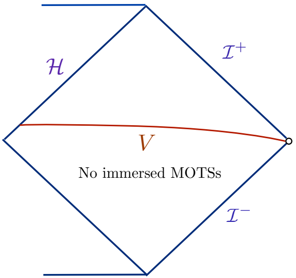
- V must have only one end (else it would contain a MOTS ※).
- V must be orientable (else, by working in the oriented double cover, it would contain an immersed MOTS ※).

  Hence, $V = \mathbb{R}^3 \# N$, where N is a compact orientable 3-manifold.

- N must be simply connected.
  - If N were not simply connected, since $\pi_1(N)$ is residually finite, N admits a finite nontrivial cover. Hence V would admit a finite cover $\tilde{V}$ with more than one end, again leading to the presence of an immersed MOTS in V ※.
- Thus, by the resolution of the Poincaré conjecture, N is diffeomorphic to S^3. Consequently, V is diffeomorphic to $\mathbb{R}^3$. $\square$

Initial data version of topological censorship

Consider the initial data setting for topological censorship:



- We regard the initial data manifold V as representing an asymptotically flat spacelike slice in the DOC whose boundary ∂V corresponds to a cross section of the event horizon.
- This cross section is assumed to be represented by a MOTS.
- We assume further that there are no immersed MOTSs in $V \setminus \partial V$.

Initial data version of topological censorship

Theorem (Eichmair, Galloway and P., 2012)

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set such that V is a manifold-with-boundary, whose boundary ∂V is a compact MOTS. If all components of ∂V are spherical and if there are no immersed MOTS in $V \setminus \partial V$, then V is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.

If the spacetime dominant energy condition holds in a spacetime neighborhood of ∂V then each component of ∂V is necessarily spherical, cf. Galloway ('08)

The proof is very similar to the proof of the initial data version of Gannon-Lee, but added care is needed in dealing with the presence of the MOTS boundary components.

Thank you very much for your attention!