ON THE CONSTRAINTS PROBLEM FOR THE EINSTEIN-YANG-MILLS-HIGGS SYSTEM

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  2. whenever the associated evolution system is satisfied in the space-time and the gauge conditions are fulfilled on the initial null hypersurfaces (that carry the initial data), then these gauge conditions and the complete EYMH system are satisfied in the space-time.

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  1. The evolution problem, which amounts to solving the reduced nonlinear hyperbolic system obtained from the EYMH system thanks to the choice of the gauge conditions.
  2. The constraints problem.

The Problem

- **What is the constraints problem ?**
  Due to the gauge conditions the data for the reduced EYMH system can not be given freely. It is necessary to construct, from arbitrary choice of some components of the gravitational potentials (called free data) on the initial null hypersurfaces, all the initial data such that the solution of the reduced EYMH system with those initial data satisfies the gauge conditions on the initial null hypersurfaces. The construction of such data is referred to as the resolution of the constraints problem.

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- Here we use harmonic gauge and Lorentz gauge conditions to solve the constraints problem on two intersecting smooth null hypersurfaces.

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- Characteristic initial value problems are fundamental in the theory of black holes formation in GR (cf. Christodoulou 2009).
- The EYMH system is a physically interesting model in GR and Gauge field Theory (cf. A. Balakin et al (2008) for review an basic references).

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- The order of differentiability of the initial data is considerably higher than that of the solution.
- For the Einstein equations, the constraints reduce to explicit ordinary differential equations instead of the elliptic PDEs which arise when the classical Cauchy problem is considered (cf. Lecture 3 of Pollack).

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- In 2010 Dossa and Tadmon generalized the results of Müller Zum Hagen by solving, in weighted Sobolev spaces, the Goursat problem for a class of quasilinear hyperbolic systems of second order. They applied the obtained result to the Einstein-Yang-Mills-Higgs equations.

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- Some other works on characteristic initial value problem have been achieved by Christodoulou, Choquet-Bruhat, Chrusciel, Martin-Garcia, Cabet, Houpa, Seifert, Lefloch, Stewart, etc.

Goal and methods

- Goal
  - Resolution of the constraints problem associated to the EYMH system, i.e., the construction, from arbitrary choice of some components of the unknown gravitational field and Yang-Mills potential (called free data) on the initial null hypersurfaces, of the complete set of initial data for the reduced EYMH system such that the harmonic gauge and the Lorentz gauge conditions are satisfied on the initial null hypersurfaces.

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- The main tools
  - Hierarchical process developed by Rendall to solve the constraints problem for the Einstein equations in vacuum and with perfect fluid source.
  - Known existence and uniqueness theorems for ordinary differential equations.

The outline

- Section 2 : The Einstein-Yang-Mills-Higgs System.
- Section 3 : The Constraints Problem for the EYMH System.
- Section 4 : Conclusion and future challenges.

Notations

- Roman indices vary from 1 to 4.
- Standard convention of summing over repeated indices is used, i.e., $u_i v^i = \sum_{i=1}^{4} u_i v^i$.

Geometric framework

L denotes a compact domain of $\mathbb{R}^4$ with a piecewise smooth boundary ∂L, G^1 and G^2 are two 3-dimensional surfaces such that $G^\omega \subset L$ for $\omega = 1, 2$. We assume that G^ω are defined by

$$G^\omega = \{x \in L : x^\omega = 0\}, \qquad \omega = 1,\ 2, \tag{2.1}$$

where $x = (x^a) = \left(x^1, ..., x^4\right)$ is the global canonical coordinates system of $\mathbb{R}^4$. In addition we suppose that $G^1 \cup G^2 \subset \partial L$, and set

$$\tau(x) = x^1 + x^2, \quad T_0 = \sup_{x \in L} \tau(x). \tag{2.2}$$

For $t \in [0, T_0]$, define

$$L_t = \{x \in L : 0 \le \tau(x) \le t\}, \quad G^\omega_t = \{x \in G^\omega : 0 \le \tau(x) \le t\}. \tag{2.3}$$

Remark *The initial data will be constructed on* $G^1_T \cup G^2_T$*, for* $T \in (0, T_0]$.

The Yang-Mills potential

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- A Yang-Mills potential is usually represented by a $1-$form A defined on $\mathcal{M}$ with values in the Lie algebra $\mathcal{G}$ of a Lie group G.
- We assume that the Lie group G admits a non-degenerate bi-invariant metric (it is the case if G is the product of Abelian and semi-simple groups).

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- We assume that the Lie group G admits a non-degenerate bi-invariant metric (it is the case if G is the product of Abelian and semi-simple groups).
- The Lie algebra $\mathcal{G}$ admits then an Ad-invariant non degenerate scalar product, denoted by a dot ".", which enjoys the following property

$$f.[k,l] = [f,k].l, \ \forall f,k,l \in \mathcal{G}. \tag{2.4}$$

The Yang-Mills potential

- Here $[,]$ denote the Lie brackets of the Lie algebra $\mathcal{G}$.
- It is assumed that $\mathcal{G}$ is an N-dimensional $\mathbb{R}$-based Lie algebra.
- For simplicity $\left(x^i\right)_{i=1,\ldots,4}$ also denote the local coordinates in $\mathcal{M}$ and $\left(\varepsilon_I\right)_{I=1,\ldots,N}$ denotes an orthogonal basis of $\mathcal{G}$.
- Then the Yang-Mills potential is locally defined as follows

$$A = A^I_i dx^i \otimes \varepsilon_I, \text{ with } A^I_i : \mathcal{M} \to \mathbb{R}.$$

The Yang-Mills and Higgs fields

- The Yang-Mills field is the curvature of the Yang-Mills potential. It is represented by a $\mathcal{G}$-valued antisymmetric 2−form F defined on $\mathcal{M}$ by

$$F = dA + \frac{1}{2}[A, A]. \tag{2.5}$$

In the local coordinates (x^i) and basis (ε_I) the above equality (2.5) reads

$$F^I_{ij} = \nabla_i A^I_j - \nabla_j A^I_i + [A_i, A_j]^I = \nabla_i A^I_j - \nabla_j A^I_i + C^I_{JK} A^J_i A^K_j,$$

or in the summary form

$$F_{ij} = \nabla_i A_j - \nabla_j A_i + [A_i, A_j].$$

- The Higgs field is represented by a $\mathcal{G}$-valued function Φ defined on $\mathcal{M}$. In the local basis (ε_I), Φ is defined as follows

$$\Phi = \Phi^I \varepsilon_I, \text{ with } \Phi^I : \mathcal{M} \to \mathbb{R}.$$

The complete form of EYMH system

- In the local coordinates (x^i) on $\mathcal{M}$, the EYMH system reads as follows :

$$\begin{aligned} & R_{ij} - \tfrac{1}{2} R g_{ij} = \rho_{ij}, \\ & \widehat{\nabla}_i F^{ij} = J^j, \\ & \widehat{\nabla}_i \widehat{\nabla}^i \Phi = H. \end{aligned} \tag{2.6}$$

- Here R_{ij} and R are, respectively, the Ricci curvature and the scalar curvature of the metric g.
- F^{ij} are the contravariant components of the Yang-Mills field (here and throughout the lecture, indices are raised or lowered w.r.t. the space-time metric g, e.g., $F^{ij} = g^{ia} g^{jb} F_{ab}$).

The complete form of EYMH system

- Φ is the Higgs field.
- ρ_{ij} is the stress-energy or the energy-momentum tensor, defined by

$$\rho_{ij} = F_{ik}.F_j{}^k - \frac{1}{4} g_{ij} F_{kl}.F^{kl} + \Phi_{ij}, \tag{2.7}$$

where

$$\Phi_{ij} = \widehat{\nabla}_i \Phi . \widehat{\nabla}_j \Phi - \frac{1}{2} g_{ij} \left(\widehat{\nabla}_k \Phi . \widehat{\nabla}^k \Phi + V\left(\Phi^2\right) \right),$$

V being a C^∞ real valued function defined on $\mathbb{R}$ (often called the self interaction potential), and $\Phi^2 = \Phi.\Phi$.

The complete form of EYMH system

- J^k is the Yang-Mills current defined by

$$J^k\left(A, \Phi, D\Phi\right) = \left[\Phi, \widehat{\nabla}^k \Phi\right], \tag{2.8}$$

where $D = \left(\frac{\partial}{\partial x^i}\right)_{i=1,\ldots,4}$.

- $\widehat{\nabla}$ is the gauge covariant derivative or the Yang-Mills operator, acting on Φ and F^{ij} as follows :

$$\widehat{\nabla}_i \Phi = \nabla_i \Phi + [A_i, \Phi], \quad \widehat{\nabla}_i F^{ij} = \nabla_i F^{ij} + \left[A_i, F^{ij}\right].$$

- H is the Higgs potential ; it is a C^∞ $\mathcal{G}$-valued function given by

$$H^I\left(\Phi\right) = V'\left(\Phi^2\right)\Phi^I, \tag{2.9}$$

where V' is the derivative of V.

The complete form of EYMH system

Remark

- (*i*) *Due to the Bianchi identities, it is easy to see that : if the EYMH system* (2.6) *is satisfied, then the stress-energy tensor* ρ_{ij} *given by* (2.7) *and the current* J^a *given by* (2.8) *satisfy the following conservation laws*

$$\nabla_i \rho^{ij} = 0, \quad \widehat{\nabla}_i J^i = 0. \tag{2.10}$$

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- (*iii*) *If the YMH system is satisfied then a direct calculation shows that the conservation laws* (2.10) *are fulfilled by the stress-energy tensor given by* (2.7) *and the current given by* (2.8), *for* $H(\Phi)$ *given by* (2.9). It results that *the EYMH system is coherent*.

The reduced form of EYMH system

- It is a well known fact that system (2.6) is not an evolution system as it stands.
- In order to reduce it to an evolution system, one needs to impose to the unknown functions (the components of the unknown metric and those of the unknown Yang-Mills potential) some supplementary conditions called gauge conditions or to choose a special or preferred system of coordinates.
- In the present lecture we will use the Lorentz gauge condition and the harmonic coordinates which were historically the first special coordinates (e.g., in 1952, Choquet-Bruhat used these special coordinates to prove the local well-posedness of the vacuum Einstein equations).

The harmonic gauge and the Lorentz gauge conditions

Definition

Let $\left(x^i\right)_{i=1,\dots,4}$ be local coordinates on a $4-d$ manifold $\mathcal{M}$ endowed with a Lorentzian metric g. $\left(x^i\right)_{i=1,\dots,4}$ are called harmonic coordinates if they satisfy the following equation

$$\Box_g x^i = 0, \;\; i = 1, 2, 3, 4, \tag{2.12}$$

where $\Box_g = \nabla_k \nabla^k$ is the geometric wave operator, ∇ representing the covariant derivative relative to the metric g.

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Let $\left(x^i\right)_{i=1,\ldots,4}$ be local coordinates on a $4-d$ manifold $\mathcal{M}$ equipped with a Lorentzian metric g. Recall the definition of the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(g_{mj,i} + g_{mi,j} - g_{ij,m}\right),$$

The harmonic gauge and the Lorentz gauge conditions

and set

$$\Gamma^k = g^{ij}\Gamma^k_{ij}, \tag{2.13}$$

where g^{ij} denotes the inverse of g_{ij}, and the subscript "," denotes partial derivative, e.g., $g_{mj,i} = \frac{\partial g_{mj}}{\partial x^i}$. It is easy to see by a simple calculation that (2.12) is equivalent to the following so-called harmonic gauge condition

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We now define the Lorentz gauge condition.

Definition

Relative to the coordinates $\left(x^i\right)_{i=1,\ldots,4}$, the Yang-Mills potential A satisfies the Lorentz gauge condition if

$$\Delta \equiv \nabla_i A^i = 0. \tag{2.15}$$

Reduction of the EYMH system

Let us now consider the complete Einstein-Yang-Mills-Higgs equations (2.6) in arbitrary local coordinates $\left(x^i\right)_{i=1,\dots,4}$. The following proposition provides the reduction of the EYMH equations, with unknowns (g_{ij}, A_p, Φ), modulo the harmonic gauge and the Lorentz gauge conditions.

Proposition 2.1. Let (g_{ij}, A_p, Φ) be such that the complete EYMH equations (2.6) are satisfied together with the harmonic gauge condition (2.14) and the Lorentz gauge condition (2.15). Then (g_{ij}, A_p, Φ) solves the following system of reduced EYMH equations :

$$\begin{aligned} &\widetilde{R}_{ij} = \tau_{ij}\left(g, A, \Phi, Dg, DA, D\Phi\right), \\ &LA_p = J_p\left(A, \Phi, D\Phi\right), \\ &S\Phi = H\left(\Phi\right). \end{aligned} \tag{2.16}$$

Reduction of the EYMH system

Here

$$
\begin{aligned}
\widetilde{R}_{ij} &\equiv R_{ij} - \frac{1}{2}\left(g_{ki}\Gamma^k_{,j} + g_{kj}\Gamma^k_{,i}\right) \\
&= -\frac{1}{2}g^{km}g_{ij,mk} + Q_{ij}(g, Dg), \\
\tau_{ij} &= F_{ik}F_j{}^k - \frac{1}{4}g_{ij}F_{kl}F^{kl} + \widehat{\nabla}_i\Phi.\widehat{\nabla}_j\Phi + \frac{1}{2}g_{ij}V\left(\Phi^2\right), \\
LA_p &\equiv g_{jp}\widehat{\nabla}_i F^{ij} + \left(\Delta_{,p} + \Gamma^l_{,p}A_l + \Gamma^l A_{l,p}\right) \\
&= g^{ik}A_{p,ik} + g^{ki}_{,p}A_{k,i} + g^{ik}\left[A_k, A_p\right]_{,i} \\
&+ g_{jp}\left(g^{ik}g^{jl}\right)_{,i}\left[A_{l,k} - A_{k,l} + \left[A_k, A_l\right]\right] \\
&+ g_{jp}\Gamma^i_{im}F^{mj} + g_{jp}\Gamma^j_{im}F^{im} + g_{jp}\left[A_i, F^{ij}\right], \\
S\Phi &\equiv \widehat{\nabla}_i\widehat{\nabla}^i\Phi + \Gamma^l\Phi_{,l} - \left[\Delta, \Phi\right] \\
&= g^{ij}\Phi_{,ij} + 2\left[A_i, \nabla^i\Phi\right] + \left[A_i, \left[A^i, \Phi\right]\right],
\end{aligned}
\tag{2.17}
$$

Reduction of the EYMH system

and Q_{ij} is a rational function of its arguments depending quadratically on Dg, given by

$$\begin{aligned} Q_{ij}(g, Dg) = {} & \tfrac{1}{2}(g_{ki,j} + g_{kj,i})\Gamma^k + \tfrac{1}{2}g^{km}g^{nl}(g_{nk,j}g_{im,l} + g_{nk,i}g_{jm,l}) \\ & - \tfrac{1}{4}g^{km}g^{nl}g_{kn,i}g_{lm,j} - \tfrac{1}{2}g^{km}g^{nl}g_{mn,k}(g_{lj,i} + g_{li,j} - g_{ij,l}) \\ & + \tfrac{1}{4}g^{km}g^{nl}g_{km,l}(g_{in,j} + g_{jn,i} - g_{ij,n}) - \tfrac{1}{2}g^{km}g^{nl}g_{ki,n}(g_{lj,m} - g_{mj,l}) \end{aligned}$$

Reduction of the EYMH system

Remark

- (*i*) *Thanks to* (2.17), *any solution* (g_{ij}, A_p, Φ) *of the reduced EYMH system* (2.16) *that satisfies the constraints* $\Gamma^k \equiv g^{ij}\Gamma^k_{ij} = 0$ *and* $\Delta \equiv \nabla_i A^i = 0$ *is also a solution of the complete EYMH system* (2.6).

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- (*ii*) *For the constraints* $\Gamma^k = 0$ *and* $\Delta = 0$ *to be satisfied everywhere, it is enough that they are satisfied on* $G^1 \cup G^2$: *one uses the Bianchi identities to show that* $\left(\Gamma^k, \Delta\right)$ *solves a second order homogeneous linear system.*

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- (iii) *The reduced EYMH system* (2.16) *constitutes the evolution system associated to the EYMH system* (2.6).
- (iv) *Solving the constraints problem consists in constructing, from arbitrary choice of some components of the gravitational potentials and Yang-Mills potential (called free data) on* $G^1 \cup G^2$*, all initial data for the reduced EYMH such that the constraints* $\Gamma^k = 0$ *and* $\Delta = 0$ *are satisfied on* $G^1 \cup G^2$ *for the solution of the corresponding evolution problem*

The objective and assumptions on free data

- The goal here is to construct C^∞ initial data for the reduced EYMH system such that the constraints $\Gamma^k = 0$ and $\Delta = 0$ are satisfied on $G^1 \cup G^2$ for the solution of the corresponding evolution problem, where G^1 and G^2 are defined in (2.1).
- The problem is addressed in three main steps through a judicious adaptation of the hierarchical method set up by Rendall to construct, for the Einstein equations in vacuum and with perfect fluid source, C^∞ data satisfying the harmonic gauge conditions $\Gamma^k = 0$ on $G^1 \cup G^2$.
- The construction of the data is done fully on G^1 and it will be clear that data on G^2 are constructed in quite a similar way.
- The novelty here is that the data are constructed for the EYMH model whereas those of Rendall were constructed either for the vacuum Einstein or Einstein-perfect fluid models. Moreover all calculations, though very tedious and lengthy, are performed in details.

The objective and assumptions on free data

- The result presented in this lecture constitutes an important step towards the global resolution, by adapting recent methods developed by Lindblad and Rodnianski, and Svedberg to study ordinary (spacelike) Cauchy problems for Einstein equations in vacuum and Einstein-Maxwell-Scalar field system respectively, of the Goursat problem associated to the EYMH equations in spaces of functions of finite differentiability order.
- The construction will be made in a standard harmonic coordinates system.
- For the sake of completeness we recall the implementation of the method of Rendall to construct C^∞ initial data on G^1_T for the characteristic EYMH system.
- Proofs, that were missing, of some key propositions used in previous works are given with details (see Propositions 3.1, 3.5, and 3.8). Those proofs contribute to enlighten the aforementioned previous works and make them more accessible to a larger audience.

First step

From now on, Greek indices vary from 3 to 4, unless otherwise stated. We assume the following conditions for the free data

$$\begin{array}{l} g_{22} = g_{23} = g_{24} = 0, \quad A_2 = 0 \text{ on } G^1_T, \\ \Phi,\ A_3 \text{ and } A_4 \text{ are given } C^\infty \text{ functions on } G^1_T. \end{array} \tag{3.1}$$

Let $T \in (0, T_0]$, $(h_{\alpha\beta})$ a matrix function with determinant 1 at each point of G^1_T. Set $g_{\alpha\beta} = \Omega h_{\alpha\beta}$, where $\Omega > 0$ is an unknown function called the conformal factor. From the free data given above in (3.1) one easily sees that the following algebraic relations hold on G^1_T

$$g_{12}g^{12} = 1, \quad g^{11} = g^{1\alpha} = 0, \quad g^{2\beta}g_{\alpha\beta} = -g^{12}g_{1\alpha}, \quad g_{\lambda\beta}g^{\alpha\beta} = \delta^\alpha_\lambda. \tag{3.2}$$

First step

At this step of the construction process, we need the expression of R_{22} as well as that of τ_{22}.

Proposition 3.1. *On G^1_T the following equalities hold*

$$R_{22} = \frac{1}{4}g^{12}g^{\alpha\beta}g_{\alpha\beta,2}\left(2g_{12,2} - g_{22,1}\right) + \frac{1}{4}g^{\beta\lambda}_{,2}g_{\lambda\beta,2} - \frac{1}{2}\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2},$$
$$\tau_{22} = \Omega^{-1}h^{\alpha\beta}A_{\alpha,2}.A_{\beta,2} + \left(\Phi_{,2}\right)^2. \tag{3.3}$$

Proof of Proposition 3.1

By definition of the Ricci curvature, it holds that

$$R_{22} = \Gamma^k_{22,k} - \Gamma^k_{2k,2} + \Gamma^k_{kl}\Gamma^l_{22} - \Gamma^k_{2l}\Gamma^l_{2k}. \qquad (3.4)$$

We compute each term of the r.h.s of (3.4) on G^1 by using the conditions (3.1) and (3.2) to gain

$$\begin{aligned} 2\Gamma^1_{22,1} &= g^{1m}_{,1}\left(2g_{2m,2} - g_{22,m}\right) + g^{1m}\left(2g_{2m,21} - g_{22,m1}\right) \\ &= g^{11}_{,1}\left(2g_{21,2} - g_{22,1}\right) + g^{12}\left(g_{22,21}\right), \end{aligned}$$

$$\begin{aligned} 2\Gamma^2_{22,2} &= g^{2m}_{,2}\left(2g_{2m,2} - g_{22,m}\right) + g^{2m}\left(2g_{2m,22} - g_{22,m2}\right) \\ &= g^{21}_{,2}\left(2g_{21,2} - g_{22,1}\right) + g^{21}\left(2g_{21,22} - g_{22,12}\right), \end{aligned}$$

$$2\Gamma^\alpha_{22,\alpha} = g^{\alpha m}_{,\alpha}\left(2g_{2m,2} - g_{22,m}\right) + g^{\alpha m}\left(2g_{2m,2\alpha} - g_{22,m\alpha}\right) = 0, \quad \alpha = 3, 4.$$

Thus

$$2\Gamma^k_{22,k} = \left(g^{11}_{,1} + g^{12}_{,2}\right)\left(2g_{12,2} - g_{22,1}\right) + 2g^{12}g_{12,22}. \qquad (3.5)$$

By expanding the equalities $\left(g^{1i}g_{2i}\right)_{,1} = 0$ and $\left(g^{2i}g_{2i}\right)_{,2} = 0$ on G^1, we get the respective equalities

$$g^{11}_{,1} = -\left(g^{12}\right)^2 g_{22,1}, \tag{3.6}$$

and

$$g^{12}_{,2} = -\left(g^{12}\right)^2 g_{12,2}. \tag{3.7}$$

Then, considering (3.5), (3.6), and (3.7), we obtain

$$2\Gamma^k_{22,k} = -\left(g^{12}\right)^2 \left(g_{12,2} + g_{22,1}\right)\left(2g_{12,2} - g_{22,1}\right) + 2g^{12}g_{12,22}. \tag{3.8}$$

Similarly, we obtain

$$2\Gamma^k_{2k,2} = -2\left(g^{12}g_{12,2}\right)^2 + 2g^{12}g_{12,22} + \left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}. \tag{3.9}$$

(3.8) and (3.9) yield

$$\Gamma^k_{22,k} - \Gamma^k_{2k,2} = \frac{1}{2}\left(g^{12}\right)^2 g_{22,1}\left(g_{22,1} - g_{12,2}\right) - \frac{1}{2}\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}. \tag{3.10}$$

Proof of Proposition 3.1

It also holds that

$$4\Gamma^k_{kl}\Gamma^l_{22} = 4\left(\Gamma^k_{k1}\Gamma^1_{22} + \Gamma^k_{k2}\Gamma^2_{22} + \Gamma^k_{k\alpha}\Gamma^\alpha_{22}\right).$$

Straightforward computations on G^1 give

$$2\Gamma^1_{22} = 0, \quad 2\Gamma^2_{22} = g^{12}\left(2g_{12,2} - g_{22,1}\right), \quad 2\Gamma^\alpha_{22} = 0,$$
$$2\Gamma^1_{12} = g^{12}g_{22,1}, \quad 2\Gamma^\alpha_{2\alpha} = g^{\alpha\beta}g_{\alpha\beta,2}.$$

This implies

$$2\Gamma^k_{k2} = 2g^{12}g_{12,2} + g^{\alpha\beta}g_{\alpha\beta,2}.$$

Thus

$$4\Gamma^k_{kl}\Gamma^l_{22} = g^{12}\left(2g_{12,2} - g_{22,1}\right)\left(2g^{12}g_{12,2} + g^{\alpha\beta}g_{\alpha\beta,2}\right). \qquad (3.11)$$

Proof of Proposition 3.1

In the same way we have

$$4\Gamma^k_{2l}\Gamma^l_{2k} = 4\left(\Gamma^k_{12}\Gamma^1_{2k} + \Gamma^k_{22}\Gamma^2_{2k} + \Gamma^k_{2\alpha}\Gamma^\alpha_{2k}\right),$$

with

$$\begin{aligned}
4\Gamma^k_{12}\Gamma^1_{2k} &= 4\left(\Gamma^1_{12}\Gamma^1_{12} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^\alpha_{12}\Gamma^1_{2\alpha}\right) = \left(g^{12}g_{22,1}\right)^2, \\
4\Gamma^k_{22}\Gamma^2_{2k} &= 4\left(\Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} + \Gamma^\alpha_{22}\Gamma^2_{2\alpha}\right) = \left[g^{12}\left(2g_{12,2} - g_{22,1}\right)\right]^2, \\
4\Gamma^k_{2\alpha}\Gamma^\alpha_{2k} &= 4\left(\Gamma^1_{2\alpha}\Gamma^\alpha_{12} + \Gamma^2_{2\alpha}\Gamma^\alpha_{22} + \Gamma^\beta_{2\alpha}\Gamma^\alpha_{2\beta}\right) = \left(g^{\beta\lambda}g_{\lambda\alpha,2}\right)\left(g^{\alpha\mu}g_{\mu\beta,2}\right),
\end{aligned}$$

as simple calculation on G^1 shows that

$$\begin{aligned}
\Gamma^1_{22} &= \Gamma^1_{2\alpha} = \Gamma^\alpha_{22} = 0, \quad 2\Gamma^1_{12} = g^{12}g_{22,1}, \quad 2\Gamma^2_{22} = g^{12}\left(2g_{12,2} - g_{22,1}\right), \\
2\Gamma^\beta_{2\alpha} &= g^{\beta m}\left(g_{m\alpha,2} + g_{2m,\alpha} - g_{2\alpha,m}\right) = g^{\beta\lambda}g_{\lambda\alpha,2}.
\end{aligned}$$

Proof of Proposition 3.1

Now the following relations hold

$$\left(g^{\beta\lambda}g_{\lambda\alpha}\right)_{,2} = 0 \Leftrightarrow g^{\beta\lambda}g_{\lambda\alpha,2} + g^{\beta\lambda}_{,2}g_{\lambda\alpha} = 0 \Leftrightarrow g^{\beta\lambda}g_{\lambda\alpha,2} = -g^{\beta\lambda}_{,2}g_{\lambda\alpha}.$$

Therefore

$$4\Gamma^k_{2\alpha}\Gamma^\alpha_{2k} = -g^{\beta\lambda}_{,2}g_{\lambda\alpha}g^{\alpha\mu}g_{\mu\beta,2} = -g^{\beta\lambda}_{,2}g_{\lambda\beta,2}.$$

Thus

$$4\Gamma^k_{2l}\Gamma^l_{2k} = \left(g^{12}g_{22,1}\right)^2 + \left[g^{12}\left(2g_{12,2} - g_{22,1}\right)\right]^2 - g^{\beta\lambda}_{,2}g_{\lambda\beta,2}. \qquad (3.12)$$

(3.11) and (3.12) give

$$\Gamma^k_{kl}\Gamma^l_{22} - \Gamma^k_{2l}\Gamma^l_{2k} = \frac{1}{4}g^{12}g^{\alpha\beta}g_{\alpha\beta,2}\left(2g_{12,2} - g_{22,1}\right) + \frac{1}{2}\left(g^{12}\right)^2 g_{22,1}\left(g_{12,2} - g_{22,1}\right) + \frac{1}{4}g^{\beta\lambda}_{,2}g_{\lambda\beta,2}. \qquad (3.13)$$

Proof of Proposition 3.1

In view of (3.4), (3.10) and (3.13), we finally gain the first equality of (3.3). On the other hand, in view of (2.17), we have

$$\tau_{22} = F_{2k}.F_2{}^k - \frac{1}{4}g_{22}F_{kl}.F^{kl} + \widehat{\nabla}_2\Phi.\widehat{\nabla}_2\Phi + \frac{1}{2}g_{22}V\left(\Phi^2\right). \qquad (3.14)$$

On G^1, given that (see (3.1) and (3.2)) $g^{11} = g^{1\alpha} = 0$, $g_{22} = g_{2\alpha} = 0$, and $A_2 = 0$, (3.14) yields the second equality of (3.3) by a direct and simple calculation. This finishes the proof of Proposition 3.1.

First step

If in addition to (3.1) we assume $g_{22,1} = 2g_{12,2}$ on G^1_T, then $\Gamma^1 = 0$ is equivalent to

$$g_{12,2} = \frac{1}{2} g_{12} \frac{\Omega_{,2}}{\Omega}. \tag{3.15}$$

The equation

$$\frac{1}{4} g^{\alpha\beta}_{,2} g_{\alpha\beta,2} - \frac{1}{2}\left(g^{\alpha\beta} g_{\alpha\beta,2}\right)_{,2} = \tau_{22}, \tag{3.15a}$$

provides the following non linear second order ODE for the conformal factor Ω

$$-\left(\frac{\Omega_{,2}}{\Omega}\right)^2 + \frac{1}{2}h_{\alpha\beta,2}h^{\alpha\beta}_{,2} - 2\left(\frac{\Omega_{,2}}{\Omega}\right)_{,2} = \Omega^{-1}h^{\alpha\beta}A_{\alpha,2}.A_{\beta,2}. \qquad (3.16)$$

Setting $\Omega = e^v$ yields, in view of (3.16),

$$2v_{,22} = f\left(x, v, v_{,2}\right), \qquad (3.17)$$

where

$$f\left(x, v, v_{,2}\right) = -\left(v_{,2}\right)^2 - 2e^{-v}h^{\alpha\beta}A_{\alpha,2}.A_{\beta,2} + \frac{1}{2}h_{\alpha\beta,2}h^{\alpha\beta}_{,2} - 2\left(\Phi_{,2}\right)^2.$$

Construction of the conformal factor

The following proposition provides the construction of the conformal factor. Its proof follows directly from known local existence and uniqueness results concerning non linear ODEs (depending on parameters with C^∞ coefficients and initial data).

Proposition 3.2. *Let* $T \in (0, T_0]$ *and assume the following smoothness condition for the free data*

$$h_{33}, h_{34}, h_{44}, A_3, A_4, \Phi \in C^\infty\left(G^1_T\right).$$

Take $v_0,\ v_1 \in C^\infty(\Gamma)$, *where* $\Gamma \equiv G^1_T \cap G^2_T$. *Then there exists* $T_1 \in (0, T]$ *such that* (3.17) *has a unique solution* $v \in C^\infty\left(G^1_{T_1}\right)$ *satisfying* $v = v_0$ *and* $v_{,2} = v_1$ *on* Γ.

Construction of g_{12}

As the conformal factor is already known, we now consider the first order linear ODE (3.15) which, since $\Omega = e^\nu$, reads

$$g_{12,2} = \frac{1}{2} g_{12}\, \nu_{,2}. \tag{3.18}$$

The following proposition provides the construction of g_{12}. Its proof

follows straightforwardly from known global existence and uniqueness results concerning linear ODEs (depending on parameters with C^∞ coefficients and initial data).

Proposition 3.3. *Let* $w_0 \in C^\infty(\Gamma)$. *Then* (3.18) *has a unique solution* $g_{12} \in C^\infty\left(G^1_{T_1}\right)$ *satisfying* $g_{12} = w_0$ *on* Γ.

Arrangement of the condition $g_{22,1} - 2g_{12,2} = 0$ on $G^1_{T_1}$

We now arrange the condition $g_{22,1} - 2g_{12,2} = 0$ on $G^1_{T_1}$ in Proposition 3.4 below.

Proposition 3.4. On $G^1_{T_1}$, the reduced equation $\widetilde{R}_{22} = \tau_{22}$ is equivalent to the following homogenous ODE with unknown $g_{22,1} - 2g_{12,2}$

$$\left(g^{12}\right)^2 g_{12,2}\left(g_{22,1} - 2g_{12,2}\right) - g^{12}\left(g_{22,1} - 2g_{12,2}\right)_{,2} = 0. \qquad (3.19)$$

Assume $g_{22,1} = 2g_{12,2}$ on Γ. Then $g_{22,1} - 2g_{12,2} = 0$ on $G^1_{T_1}$ and so $\Gamma^1 = 0$ on $G^1_{T_1}$.

Proof of Proposition 3.4

In view of (3.18) and (3.2), it holds that $g^{\alpha\beta}g_{\alpha\beta,2} = 4g^{12}g_{12,2}$ on $G^1_{T_1}$. Thus on $G^1_{T_1}$ it holds that

$$
\begin{aligned}
2\Gamma^1 &= g^{ij}g^{1k}\left(2g_{ki,j} - g_{ij,k}\right) = g^{ij}g^{12}\left(2g_{2i,j} - g_{ij,2}\right) \\
&= g^{12}\left[g^{12}\left(g_{12,2}\right) + g^{21}\left(2g_{22,1} - g_{21,2}\right) + g^{\alpha\beta}\left(-g_{\alpha\beta,2}\right)\right] \\
&= g^{12}\left(2g^{12}g_{22,1} - g^{\alpha\beta}g_{\alpha\beta,2}\right) \\
&= g^{12}\left(2g^{12}g_{22,1} - 4g^{12}g_{12,2}\right) \\
&= 2\left(g^{12}\right)^2\left(g_{22,1} - 2g_{12,2}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma^1_{,2} &= \left[\left(g^{12}\right)^2\left(g_{22,1} - 2g_{12,2}\right)\right]_{,2} \\
&= 2g^{12}g^{12}_{,2}\left(g_{22,1} - 2g_{12,2}\right) + \left(g^{12}\right)^2\left(g_{22,1} - 2g_{12,2}\right)_{,2}.
\end{aligned}
$$

A simple calculation shows that $g^{12}_{,2} = -\left(g^{12}\right)^2 g_{12,2}$ on $G^1_{T_1}$, since $g^{12}g_{12} = 1$ on $G^1_{T_1}$. Hence

$$\Gamma^1_{,2} = -2\left(g^{12}\right)^3 g_{12,2}\left(g_{22,1} - 2g_{12,2}\right) + \left(g^{12}\right)^2\left(g_{22,1} - 2g_{12,2}\right)_{,2}.$$

It is easy to see that $g_{k2}\Gamma^k_{,2} = g_{12}\Gamma^1_{,2}$ on $G^1_{T_1}$, since $g_{2k} = 0$ for $k \neq 1$. In view of (3.3) and (3.15a) we have

$$R_{22} - g_{12}\Gamma^1_{,2} = \left(g^{12}\right)^2 g_{12,2}\left(g_{22,1} - 2g_{12,2}\right) - g^{12}\left(g_{22,1} - 2g_{12,2}\right)_{,2} + \tau_{22}.$$

Therefore, since $\widetilde{R}_{22} \equiv R_{22} - \frac{1}{2}\left(g_{k2}\Gamma^k_{,2} + g_{k2}\Gamma^k_{,2}\right) = -\frac{1}{2}g^{km}g_{22,mk} + Q_{22}$, the reduced equation $\widetilde{R}_{22} = \tau_{22}$ is equivalent to (3.19).

Second step

We now show how to construct the data g_{13}, g_{14}, and A_1 on $G^1_{T_1}$, and arrange the relations $\Gamma^\alpha = 0$ and $\Delta = 0$ on $G^1_{T_1}$, $\alpha = 3, 4$.

- The principle is to find a good combination of $R_{2\alpha}$, Γ^α, $\Gamma^\alpha_{,2}$, LA_2, Δ and $\Delta_{,2}$ that will provide a system of ODEs on $G^1_{T_1}$ with unknowns $g_{1\alpha}$ and A_1.
- It is at this step that the assumption $A_2 = 0$ on G^1_T, which permits to avoid to deal with g_{11} at this stage of the construction process, is needed. We have the following proposition which is of paramount importance in the lecture.
- Its proof contributes to shed more light on previous work where it was missing.

An important proposition : Proposition 3.5

Proposition 3.5. (i) On $G^1_{T_1}$, the following combinations hold

$$
\begin{aligned}
&R_{2\alpha} + \tfrac{1}{2} g_{\alpha\beta}\Gamma^{\beta}_{,2} + \left(g^{12} g_{12,2} g_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta,2}\right)\Gamma^{\beta} \\
&= g^{12} g_{1\alpha,22} + \left(g^{12}\right)^2 g_{12,2} g_{1\alpha,2} - g_{\alpha\beta,2} g^{\beta\lambda} g^{12} g_{1\lambda,2} \\
&+ \left\{ \left(g^{12}\right)^2 g_{12,2} g_{\alpha\beta} g^{\beta\lambda}_{,2} + \tfrac{1}{2}\left[g_{\alpha\beta} g^{12} g^{\beta\lambda}_{,22} - g^{12} g^{\beta\lambda} g_{\alpha\beta,22}\right]\right\} g_{1\lambda} + c_\alpha ,
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
&LA_2 - 2\Delta_{,2} - 2g^{12} g_{12,2}\Delta + 2\left(g^{12} g_{12,2} A_\nu + A_{\nu,2}\right)\Gamma^\nu + 2A_\nu \Gamma^\nu_{,2} \\
&= -2g^{12} A_{1,22} - 2\left(g^{12}\right)^2 g_{12,2} A_{1,2} + 2g^{12} g^{\alpha\lambda} A_{\alpha,2} g_{1\lambda,2} + K^\lambda g_{1\lambda} + A_g .
\end{aligned} \tag{3.21}
$$

An important proposition : Proposition 3.5

Here

$$
\begin{aligned}
c_\alpha = {} & \tfrac{1}{2}\left(g^{12}\right) g_{12,2}\left[-2g^{12}g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right] \\
& + \tfrac{1}{4} g_{\alpha\beta,2}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda} + g^{\beta\lambda}g^{\mu\theta}\left(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right] \\
& + \tfrac{1}{2}\left(g^{\lambda\beta}g_{\alpha\beta,2}\right)_{,\lambda} - 3\left(g^{12}g_{12,2}\right)_{,\alpha} - \tfrac{1}{2}\left(g^{12}\right)^2 g_{12,2}g_{12,\alpha} \\
& + \tfrac{1}{2} g^{12}\left(g_{22,1\alpha} + g_{12,2\alpha}\right) + \tfrac{1}{2} g^{\beta\lambda}_{,2}\left(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}\right) \\
& + \tfrac{1}{2} g_{\alpha\beta}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda} + g^{\beta\lambda}g^{\mu\theta}\left(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right]_{,2},
\end{aligned}
$$

$$
\begin{aligned}
K^\lambda = {} & 4\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_{\alpha,2} + \left[2g^{12}g^{\alpha\lambda}g^{\beta\mu}g_{\mu\beta,2}A_\alpha + 2g^{12}g^{\alpha\lambda}A_{\alpha,2}\right]_{,2} \\
& - 2\left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2} g^{\beta\mu}g_{\mu\beta,2} - 2g^{12}g^{\alpha\lambda}A_\alpha\left[g^{\beta\mu}g_{\mu\beta,2}\right]_{,2},
\end{aligned}
$$

$$
\begin{aligned}
A_g = {} & -2\left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2} g_{12,\lambda} - 2g^{12}g^{\alpha\lambda}A_\alpha g_{12,2\lambda} \\
& - g^{\beta\alpha}\left(\left[A_\beta, A_{\alpha,2}\right] - g^{12}g_{12,\beta}A_{\alpha,2}\right) \\
& + \left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2} g_{12}\left[g^{\beta\mu}\left(g_{\mu\lambda,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}\right)\right] \\
& + g^{12}g^{\alpha\lambda}A_\alpha\left(g_{12}\left[g^{\beta\mu}\left(g_{\mu\lambda,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}\right)\right]\right)_{,2} \\
& - \left\{g^{\alpha\beta}\left[g^{12}g_{12,\beta} + \tfrac{1}{2}g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right)\right] + g^{\alpha\beta}_{,\beta}\right\} A_{\alpha,2} \\
& + \left(2g^{12}g^{\alpha\lambda}g_{12,\lambda}A_\alpha - \left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right] A_\alpha\right)_{,2}.
\end{aligned}
$$

An important proposition : Proposition 3.5

(ii) On $G^1_{T_1}$ the system

$$\begin{aligned}
&R_{23} + \tfrac{1}{2} g_{3\beta}\Gamma^{\beta}_{,2} + \left(g^{12}g_{12,2}g_{3\beta} + \tfrac{1}{2}g_{3\beta,2} - A_\beta.A_{3,2}\right)\Gamma^\beta + A_{3,2}.\Delta = \tau_{23},\\
&R_{24} + \tfrac{1}{2} g_{4\beta}\Gamma^{\beta}_{,2} + \left(g^{12}g_{12,2}g_{4\beta} + \tfrac{1}{2}g_{4\beta,2} - A_\beta.A_{4,2}\right)\Gamma^\beta + A_{4,2}.\Delta = \tau_{24},\\
&LA_2 - 2\Delta_{,2} - 2g^{12}g_{12,2}\Delta + 2\left(g^{12}g_{12,2}A_\nu + A_{\nu,2}\right)\Gamma^\nu + 2A_\nu\Gamma^\nu_{,2} = J_2,
\end{aligned} \tag{3.22}$$

is equivalent to the following second order system of ODEs with unknown (A_1, g_{13}, g_{14})

$$\begin{aligned}
&g^{12}g_{13,22} + \kappa^\lambda_3 g_{1\lambda,2} + \varkappa_3.A_{1,2} + \chi^\lambda_3 g_{1\lambda} + F_3 = 0,\\
&g^{12}g_{14,22} + \kappa^\lambda_4 g_{1\lambda,2} + \varkappa_4.A_{1,2} + \chi^\lambda_4 g_{1\lambda} + F_4 = 0,\\
&-2g^{12}A_{1,22} - 2\left(g^{12}\right)^2 g_{12,2}A_{1,2} + a^\lambda g_{1\lambda} + b = 0,
\end{aligned} \tag{3.23}$$

where all the coefficients are known on $G^1_{T_1}$ and given as follows :

$$
\begin{aligned}
&\kappa_3^3 = \left(g^{12}\right)^2 g_{12,2} - g_{3\beta,2}g^{\beta 3}, \quad \kappa_3^4 = -g_{3\beta,2}g^{\beta 4}, \quad \kappa_4^3 = -g_{4\beta,2}g^{\beta 3}, \\
&\kappa_4^4 = \left(g^{12}\right)^2 g_{12,2} - g_{4\beta,2}g^{\beta 4}, \quad \varkappa_3 = 2g^{12}A_{3,2}, \quad \varkappa_4 = 2g^{12}A_{4,2}, \\
&\chi_\alpha^\lambda = \left(g^{12}\right)^2 g_{12,2}g_{\alpha\beta}g^{\beta\lambda}_{,2} + \frac{1}{2}\left[g_{\alpha\beta}g^{12}g^{\beta\lambda}_{,22} - g^{12}g^{\beta\lambda}g_{\alpha\beta,22}\right] \\
&\qquad -2g^{\nu\lambda}g^{12}A_{\alpha,2}.A_{\nu,2}, \\
&a^\lambda = 4\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_{\alpha,2} + \left[2g^{12}g^{\alpha\lambda}g^{\beta\mu}g_{\mu\beta,2}A_\alpha + 2g^{12}g^{\alpha\lambda}A_{\alpha,2}\right]_{,2} \\
&\qquad - 2\left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2} g^{\beta\mu}g_{\mu\beta,2} - 2g^{12}g^{\alpha\lambda}A_\alpha\left[g^{\beta\mu}g_{\mu\beta,2}\right]_{,2}, \\
&\mathcal{F}_\alpha = c_\alpha + g^{\beta\lambda}\left(A_{\alpha,\lambda} - A_{\lambda,\alpha} + [A_\lambda, A_\alpha]\right).A_{\beta,2}, \quad b = A_g - J_2.
\end{aligned}
\tag{3.24}
$$

Proof of Proposition 3.5

Proof of item (i).
Proof of (3.20)**.** By definition of the Ricci curvature, it holds that

$$R_{2\alpha} = \Gamma^k_{2\alpha,k} - \Gamma^k_{2k,\alpha} + \Gamma^k_{lk}\Gamma^l_{2\alpha} - \Gamma^k_{l\alpha}\Gamma^l_{2k}. \tag{3.25}$$

We compute each term of the r.h.s of (3.25) on G^1 by using the conditions (3.1) and (3.2) to gain

$$\begin{aligned} 2\Gamma^k_{2\alpha,k} &= \left(g^{11}_{,1} + g^{12}_{,2}\right)\left(g_{12,\alpha} + g_{1\alpha,2} - g_{2\alpha,1}\right) \\ &+ g^{12}\left(g_{22,1\alpha} + g_{12,2\alpha} + g_{1\alpha,22} - g_{2\alpha,12}\right) \\ &+ \left(g^{1\beta}_{,1} + g^{2\beta}_{,2}\right) g_{\alpha\beta,2} + g^{2\beta} g_{\alpha\beta,22} + \left(g^{\lambda\beta} g_{\alpha\beta,2}\right)_{,\lambda}. \end{aligned} \tag{3.25a}$$

Proof of Proposition 3.5

By expanding the equalities $\left(g^{i\beta}g_{2i}\right)_{,1} = 0$ and $\left(g^{i\beta}g_{1i}\right)_{,2} = 0$ on G^1, we get

$$
\begin{aligned}
g^{1\beta}_{,1} &= -g^{12}\left(2g^{2\beta}g_{12,2} + g^{\lambda\beta}g_{2\lambda,1}\right), \\
g^{2\beta}_{,2} &= -g^{12}\left(g^{2\beta}g_{12,2} + g^{\lambda\beta}g_{1\lambda,2} + g^{\lambda\beta}_{,2}g_{1\lambda}\right).
\end{aligned}
\tag{3.25b}
$$

Proof of Proposition 3.5

(3.6) , (3.7) , (3.25a) and (3.25b) yield

$$
\begin{aligned}
2\Gamma^k_{2\alpha,k} &= -3\left(g^{12}\right)^2 g_{12,2}\left(g_{12,\alpha}+g_{1\alpha,2}-g_{2\alpha,1}\right)\\
&+g^{12}\left(g_{22,1\alpha}+g_{12,2\alpha}+g_{1\alpha,22}-g_{2\alpha,12}\right)\\
&-g^{12}\left[3g^{2\beta}g_{12,2}+g^{\lambda\beta}\left(g_{1\lambda,2}+g_{2\lambda,1}\right)+g^{\lambda\beta}_{,2}g_{1\lambda}\right]g_{\alpha\beta,2}\\
&+g^{2\beta}g_{\alpha\beta,22}+\left(g^{\lambda\beta}g_{\alpha\beta,2}\right)_{,\lambda}.
\end{aligned}
\tag{3.26a}
$$

We now compute Γ^k_{2k} and $\Gamma^k_{2k,\alpha}$ on G^1 by using equalities $g^{\beta\lambda}g_{\lambda\beta,2} = 4g^{12}g_{12,2}$ and $2g_{12,2} - g_{22,1} = 0$ (see proof of Proposition 3.4), to have

$$
2\Gamma^k_{2k} = 6g^{12}g_{12,2}, \quad \Gamma^k_{2k,\alpha} = 3\left(g^{12}g_{12,2}\right)_{,\alpha}. \tag{3.26b}
$$

Proof of Proposition 3.5

(3.26a) and (3.26b) give

$$\begin{aligned}\Gamma^k_{2\alpha,k} - \Gamma^k_{2k,\alpha} = & -\frac{3}{2}\left(g^{12}\right)^2 g_{12,2}\left(g_{1\alpha,2} - g_{2\alpha,1}\right) + \frac{1}{2} g^{12}\left(g_{1\alpha,22} - g_{2\alpha,12}\right) \\ & - \frac{1}{2} g^{12}\left[3 g^{2\beta} g_{12,2} + g^{\lambda\beta}\left(g_{1\lambda,2} + g_{2\lambda,1}\right) + g^{\lambda\beta}_{,2} g_{1\lambda}\right] g_{\alpha\beta,2} \\ & + \frac{1}{2} g^{2\beta} g_{\alpha\beta,22} + \frac{1}{2}\left(g^{\lambda\beta} g_{\alpha\beta,2}\right)_{,\lambda} - 3\left(g^{12} g_{12,2}\right)_{,\alpha} \\ & - \frac{3}{2}\left(g^{12}\right)^2 g_{12,2} g_{12,\alpha} + \frac{1}{2} g^{12}\left(g_{22,1\alpha} + g_{12,2\alpha}\right). \qquad (3.26c)\end{aligned}$$

Proof of Proposition 3.5

In addition, direct calculations on G^1 give

$$\begin{aligned} 4\Gamma^k_{lk}\Gamma^l_{2\alpha} &= 6g^{12}g_{12,2}\left[g^{21}\left(g_{21,\alpha}+g_{1\alpha,2}-g_{2\alpha,1}\right)+g^{2\beta}g_{\beta\alpha,2}\right] \\ &\quad + g^{\beta\lambda}g_{\lambda\alpha,2}\left[2g^{12}g_{12,\beta}+g^{\lambda\mu}\left(g_{\mu\beta,\lambda}+g_{\lambda\mu,\beta}-g_{\beta\lambda,\mu}\right)\right], \end{aligned} \tag{3.27a}$$

and

$$\begin{aligned} 4\Gamma^k_{l\alpha}\Gamma^l_{2k} &= 2\left(g^{12}\right)^2 g_{12,2}\left(g_{12,\alpha}+g_{2\alpha,1}-g_{1\alpha,2}\right) \\ &\quad - g^{12}g_{\lambda\alpha,2}\left[2g^{2\lambda}g_{12,2}+g^{\lambda\beta}\left(g_{1\beta,2}+g_{2\beta,1}-g_{12,\beta}\right)\right] \\ &\quad + g^{\beta\theta}g_{\theta\alpha,2}\left[g^{12}\left(g_{1\beta,2}+g_{12,\beta}-g_{2\beta,1}\right)+g^{2\mu}g_{\mu\beta,2}\right] \\ &\quad + g^{\lambda\theta}g_{\theta\beta,2}\left[-g^{\beta2}g_{\lambda\alpha,2}+g^{\beta\mu}\left(g_{\lambda\mu,\alpha}+g_{\mu\alpha,\lambda}-g_{\lambda\alpha,\mu}\right)\right]. \end{aligned} \tag{3.27b}$$

Proof of Proposition 3.5

From the relation $\left(g^{\beta\lambda}g_{\alpha\beta}\right)_{,2} = 0$, (3.27$a$) and (3.27$b$) yield

$$\begin{aligned}\Gamma^k_{lk}\Gamma^l_{2\alpha} - \Gamma^k_{l\alpha}\Gamma^l_{2k} &= 2\left(g^{12}\right)^2 g_{12,2}\left(g_{1\alpha,2} - g_{2\alpha,1}\right)\\ &+ 2g^{12}g_{12,2}g^{2\beta}g_{\beta\alpha,2} + \frac{1}{2}g^{\beta\lambda}g_{\alpha\beta,2}g_{2\lambda,1}\\ &+ \frac{1}{2}g^{\beta\lambda}_{,2}\left(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}\right) + \left(g^{12}\right)^2 g_{12,2}g_{12,\alpha}. \qquad (3.27c)\end{aligned}$$

Proof of Proposition 3.5

In view of (3.25), (3.26c) and (3.27c) give

$$
\begin{aligned}
R_{2\alpha} = & \frac{1}{2}\left(g^{12}\right)^2 g_{12,2}\left(g_{1\alpha,2} - g_{2\alpha,1}\right) + \frac{1}{2} g^{12}\left(g_{1\alpha,22} - g_{2\alpha,12}\right) \\
& + \frac{1}{2} g^{12} g_{12,2} g^{2\beta} g_{\beta\alpha,2} - \frac{1}{2} g^{12}\left[g^{\lambda\beta} g_{1\lambda,2} + g^{\lambda\beta}_{,2} g_{1\lambda}\right] g_{\alpha\beta,2} \\
& + \frac{1}{2} g^{2\beta} g_{\alpha\beta,22} + \frac{1}{2}\left(g^{\lambda\beta} g_{\alpha\beta,2}\right)_{,\lambda} - 3\left(g^{12} g_{12,2}\right)_{,\alpha} \\
& - \frac{1}{2}\left(g^{12}\right)^2 g_{12,2} g_{12,\alpha} + \frac{1}{2} g^{12}\left(g_{22,1\alpha} + g_{12,2\alpha}\right) + \frac{1}{2} g^{\beta\lambda}_{,2}\left(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}\right).
\end{aligned}
\tag{3.28}
$$

Calculation of $\tau_{2\alpha}$. In view of (2.17) it holds that

$$\tau_{ij} = F_{ik}.F_j{}^k - \frac{1}{4}g_{ij}F_{kl}.F^{kl} + \widehat{\nabla}_i\Phi.\widehat{\nabla}_j\Phi + \frac{1}{2}g_{ij}V\left(\Phi^2\right). \tag{3.29a}$$

(3.29a) gives, since $g_{2\alpha} = 0$ on G^1,

$$\tau_{2\alpha} = F_{2k}.F_\alpha{}^k + \widehat{\nabla}_2\Phi.\widehat{\nabla}_\alpha\Phi. \tag{3.29b}$$

(3.29b) reads

$$\tau_{2\alpha} = F_{21}.F_\alpha{}^1 + F_{2\beta}.F_\alpha{}^\beta + \widehat{\nabla}_2\Phi.\widehat{\nabla}_\alpha\Phi, \tag{3.29c}$$

Proof of Proposition 3.5

with

$$F_{21} = A_{1,2} - A_{2,1} + [A_2, A_1], \quad F_\alpha{}^1 = g^{1i} F_{\alpha i} = g^{12} F_{\alpha 2},$$
$$F_\alpha{}^\beta = g^{\beta i} F_{\alpha i} = g^{2\beta} F_{\alpha 2} + g^{\beta\lambda} F_{\alpha\lambda} = -g^{12} g^{\beta\lambda} g_{1\lambda} F_{\alpha 2} + g^{\beta\lambda} F_{\alpha\lambda}. \tag{3.30}$$

(3.29c) and (3.30) give

$$\tau_{2\alpha} = -g^{12} F_{2\alpha}.(A_{1,2} - A_{2,1} + [A_2, A_1]) + g^{12} g^{\beta\lambda} F_{2\alpha}.F_{2\beta} g_{1\lambda} - g^{\beta\lambda} F_{\lambda\alpha}.F_{2\beta}$$
$$+ (\Phi_{,2} + [A_2, \Phi]).(\Phi_{,\alpha} + [A_\alpha, \Phi]). \tag{3.31}$$

Proof of Proposition 3.5

Now

$$F_{2\alpha} = A_{\alpha,2} - A_{2,\alpha} + [A_2, A_\alpha], \; F_{\lambda\alpha} = A_{\alpha,\lambda} - A_{\lambda,\alpha} + [A_\lambda, A_\alpha]. \quad (3.32)$$

In view of (3.32), the assumption $A_2 = 0$ on G^1 implies that $F_{2\alpha} = A_{\alpha,2}$ and $[A_2, A_1] = [A_2, \Phi] = 0$ on G^1. It therefore follows from (3.31) and (3.32) that

$$\begin{aligned}\tau_{2\alpha} = &-g^{12}A_{\alpha,2}.\left(A_{1,2} - A_{2,1}\right) + g^{12}g^{\beta\lambda}A_{\alpha,2}.A_{\beta,2}g_{1\lambda} \\ &- g^{\beta\lambda}A_{\beta,2}.\left(A_{\alpha,\lambda} - A_{\lambda,\alpha} + [A_\lambda, A_\alpha]\right) + \left(\Phi_{,2}\right).\left(\Phi_{,\alpha} + [A_\alpha, \Phi]\right).\end{aligned} \quad (3.33)$$

Proof of Proposition 3.5

Similarly, the following equality holds on G^1

$$\Gamma^\beta = g^{\beta\lambda} g^{12} \left(g_{\lambda 1,2} + g_{\lambda 2,1}\right) + g^{\beta\lambda}_{,2} g^{12} g_{1\lambda} + \frac{1}{2}\left[-2g^{\beta\lambda} g^{12} g_{12,\lambda} + g^{\beta\lambda} g^{\mu\theta}\left(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right]. \quad (3.34)$$

Thus

$$\Gamma^\beta_{,2} = g^{\beta\lambda}_{,2} g^{12} \left(g_{\lambda 1,2} + g_{\lambda 2,1}\right) + g^{\beta\lambda}\left[g^{12}_{,2}\left(g_{\lambda 1,2} + g_{\lambda 2,1}\right) + g^{12}\left(g_{\lambda 1,22} + g_{\lambda 2,12}\right)\right] + g^{\beta\lambda}_{,22} g^{12} g_{1\lambda} + g^{\beta\lambda}_{,2}\left[g^{12}_{,2} g_{1\lambda} + g^{12} g_{1\lambda,2}\right] + \frac{1}{2}\left[-2g^{\beta\lambda} g^{12} g_{12,\lambda} + g^{\beta\lambda} g^{\mu\theta}\left(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right]_{,2}. \quad (3.35)$$

Proof of Proposition 3.5

Replacing $g^{12}_{,2}$ by its expression given by (3.7) gives

$$\begin{aligned}
&\Gamma^{\beta}_{,2} \\
&= g^{\beta\lambda}_{,2} g^{12} \left(g_{\lambda 1,2} + g_{\lambda 2,1}\right) \\
&+ g^{\beta\lambda}\left[-\left(g^{12}\right)^2 g_{12,2}\left(g_{\lambda 1,2} + g_{\lambda 2,1}\right) + g^{12}\left(g_{\lambda 1,22} + g_{\lambda 2,12}\right)\right] \qquad (3.36) \\
&+ g^{\beta\lambda}_{,22} g^{12} g_{1\lambda} + g^{\beta\lambda}_{,2}\left[-\left(g^{12}\right)^2 g_{12,2} g_{1\lambda} + g^{12} g_{1\lambda,2}\right] \\
&+ \tfrac{1}{2}\left[-2 g^{\beta\lambda} g^{12} g_{12,\lambda} + g^{\beta\lambda} g^{\mu\theta}\left(2 g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right]_{,2}.
\end{aligned}$$

Proof of Proposition 3.5

Then

$$\begin{aligned} g_{\alpha\beta}\Gamma^{\beta}_{,2} &= g_{\alpha\beta}g^{\beta\lambda}_{,2}g^{12}\left(g_{\lambda 1,2}+g_{\lambda 2,1}\right) \\ &+ g_{\alpha\beta}g^{\beta\lambda}\left[-\left(g^{12}\right)^2 g_{12,2}\left(g_{\lambda 1,2}+g_{\lambda 2,1}\right)+g^{12}\left(g_{\lambda 1,22}+g_{\lambda 2,12}\right)\right] \\ &+ g_{\alpha\beta}g^{\beta\lambda}_{,22}g^{12}g_{1\lambda}+g_{\alpha\beta}g^{\beta\lambda}_{,2}\left[-\left(g^{12}\right)^2 g_{12,2}g_{1\lambda}+g^{12}g_{1\lambda,2}\right] \\ &+ \frac{1}{2}g_{\alpha\beta}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda}+g^{\beta\lambda}g^{\mu\theta}\left(2g_{\lambda\mu,\theta}-g_{\mu\theta,\lambda}\right)\right]_{,2}. \end{aligned} \tag{3.37}$$

Proof of Proposition 3.5

Using (3.28) and (3.37), we finally gain

$$
\begin{aligned}
& R_{2\alpha} + \tfrac{1}{2} g_{\alpha\beta} \Gamma^{\beta}_{,2} \\
& = g^{12} g_{1\alpha,22} - \left(g^{12}\right)^2 g_{12,2} g_{2\alpha,1} - \tfrac{1}{2} g_{\alpha\beta,2} g^{\beta\lambda} g^{12} \left(3 g_{1\lambda,2} + g_{2\lambda,1}\right) \\
& + \tfrac{1}{2} g_{\alpha\beta} \left(g^{12} g^{\beta\lambda}_{,2}\right)_{,2} g_{1\lambda} - \tfrac{1}{2} g^{12} g^{\lambda\beta}_{,2} g_{\alpha\beta,2} g_{1\lambda} - \tfrac{1}{2} \left(g^{12}\right)^2 g_{12,2} g_{\alpha\beta,2} g^{\beta\lambda} g_{1\lambda} \\
& - \tfrac{1}{2} g^{12} g^{\beta\lambda} g_{\alpha\beta,22} g_{1\lambda} + \tfrac{1}{2} \left(g^{\lambda\beta} g_{\alpha\beta,2}\right)_{,\lambda} - 3 \left(g^{12} g_{12,2}\right)_{,\alpha} - \tfrac{1}{2} \left(g^{12}\right)^2 g_{12,2} g_{12,\alpha} \\
& + \tfrac{1}{2} g^{12} \left(g_{22,1\alpha} + g_{12,2\alpha}\right) + \tfrac{1}{2} g^{\beta\lambda}_{,2} \left(g_{\lambda\beta,\alpha} + g_{\lambda\alpha,\beta}\right) \\
& + \tfrac{1}{2} g_{\alpha\beta} \left[-2 g^{\beta\lambda} g^{12} g_{12,\lambda} + g^{\beta\lambda} g^{\mu\theta} \left(2 g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right]_{,2} .
\end{aligned}
\tag{3.38}
$$

Proof of Proposition 3.5

Now (3.34) implies that

$$g^{\beta\lambda}g^{12}g_{2\lambda,1} = \Gamma^\beta - g^{\beta\lambda}g^{12}g_{1\lambda,2} - g^{\beta\lambda}_{,2}g^{12}g_{1\lambda} - \frac{1}{2}\left[-2g^{\beta\lambda}g^{12}g_{12,\lambda} + g^{\beta\lambda}g^{\mu\theta}\left(2g_{\lambda\mu,\theta} - g_{\mu\theta,\lambda}\right)\right], \quad (3.39)$$

and

$$g^{12}g_{2\alpha,1} = g_{\alpha\beta}\Gamma^\beta - g^{12}g_{1\alpha,2} - g_{\alpha\beta}g^{\beta\lambda}_{,2}g^{12}g_{1\lambda} - \frac{1}{2}\left[-2g^{12}g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right]. \quad (3.40)$$

The insertion of (3.39) and (3.40) in (3.38) yields the expected relation (3.20).

Proof of Proposition 3.5

Proof of (3.21). From (2.17) we have

$$\begin{aligned} LA_2 &= g^{ik}A_{2,ik} + g^{ki}_{,2}A_{k,i} + g^{ik}\left[A_k, A_2\right]_{,i} \\ &+ g_{j2}\left[\left(g^{ik}g^{jl}\right)_{,i}\left[A_{l,k} - A_{k,l} + \left[A_k, A_l\right]\right] + \Gamma^i_{im}F^{mj} + \Gamma^j_{im}F^{im} + \left[A_i, F^{ij}\right]\right]. \end{aligned} \tag{3.41}$$

Calculation of $g^{ik}A_{2,ik}$, $g^{ki}_{,2}A_{k,i}$ **and** $g^{ik}\left[A_k, A_2\right]_{,i}$. It holds that

$$g^{ik}A_{2,ik} = 2g^{12}A_{2,12} + g^{22}A_{2,22} + 2g^{2\alpha}A_{2,2\alpha} + g^{\alpha\beta}A_{2,\alpha\beta}, \quad \alpha, \beta \in \{3,4\}. \tag{3.42}$$

The expression of g^{22} in terms of $g_{1\lambda}$ and g_{11} via the equalities $g^{2i}g_{1i} = 0$ and $g^{2\alpha} = -g^{12}g^{\alpha\lambda}g_{1\lambda}$ gives

$$g^{22} = -\left(g^{12}\right)^2 g_{11} + \left(g^{12}\right)^2 g^{\alpha\lambda}g_{1\alpha}g_{1\lambda}. \tag{3.43}$$

Using the assumption $A_2 = 0$ on G^1, (3.42) yields

$$g^{ik}A_{2,ik} = 2g^{12}A_{2,12}. \tag{3.44}$$

Proof of Proposition 3.5

The calculation of $g^{ki}_{,2} A_{k,i}$ gives

$$g^{ki}_{,2} A_{k,i} = g^{12}_{,2} \left(A_{1,2} + A_{2,1}\right) + g^{2\alpha}_{,2} \left(A_{\alpha,2} + A_{2,\alpha}\right). \tag{3.45}$$

We calculate $g^{2\alpha}_{,2}$ by using the relation $g^{2\alpha} = -g^{12} g^{\alpha\lambda} g_{1\lambda}$ satisfied on G^1 to have

$$g^{2\alpha}_{,2} = \left(\left(g^{12}\right)^2 g_{12,2} g^{\alpha\lambda} - g^{12} g^{\alpha\lambda}_{,2}\right) g_{1\lambda} - g^{12} g^{\alpha\lambda} g_{1\lambda,2}. \tag{3.46}$$

Since $g^{12}_{,2} = -\left(g^{12}\right)^2 g_{12,2}$ (see (3.7)), we deduce from (3.45) and (3.46) that

$$\begin{aligned} g^{ki}_{,2} A_{k,i} = & -\left(g^{12}\right)^2 g_{12,2} \left(A_{1,2} + A_{2,1}\right) \\ & + A_{\alpha,2} \left[\left(\left(g^{12}\right)^2 g_{12,2} g^{\alpha\lambda} - g^{12} g^{\alpha\lambda}_{,2}\right) g_{1\lambda} - g^{12} g^{\alpha\lambda} g_{1\lambda,2}\right]. \end{aligned} \tag{3.47}$$

Proof of Proposition 3.5

As $g^{11} = g^{1\alpha} = 0$ and $A_2 = 0$ on G^1, the calculation of $g^{ik}\left[A_k, A_2\right]_{,i}$ on G^1 easily gives

$$g^{ik}\left[A_k, A_2\right]_{,i} = 0. \tag{3.48}$$

$(3.44 - 3.48)$ imply

$$\begin{aligned} g^{ik}A_{2,ik} + g^{ki}_{,2}A_{k,i} + g^{ik}\left[A_k, A_2\right]_{,i} &= 2g^{12}A_{2,12} - \left(g^{12}\right)^2 g_{12,2}\left(A_{1,2} + A_{2,1}\right) \\ &+A_{\alpha,2}\left[\left(\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda} - g^{12}g^{\alpha\lambda}_{,2}\right)g_{1\lambda} - g^{12}g^{\alpha\lambda}g_{1\lambda,2}\right]. \end{aligned} \tag{3.49}$$

Proof of Proposition 3.5

Calculation of
$g_{j2}\left[\left(g^{ik}g^{jl}\right)_{,i}[A_{l,k} - A_{k,l} + [A_k, A_l]] + \Gamma^i_{im}F^{mj} + \Gamma^j_{im}F^{im} + \left[A_i, F^{ij}\right]\right]$.
Using the assumption $g_{22} = g_{23} = g_{24} = 0$ on G^1, we get

$$g_{j2}\left[\left(g^{ik}g^{jl}\right)_{,i}[A_{l,k} - A_{k,l} + [A_k, A_l]] + \Gamma^i_{im}F^{mj} + \Gamma^j_{im}F^{im} + \left[A_i, F^{ij}\right]\right]$$
$$= g_{12}\left[\left(g^{ik}g^{1l}\right)_{,i}(A_{l,k} - A_{k,l} + [A_k, A_l]) + \Gamma^i_{im}F^{m1} + \Gamma^1_{im}F^{im} + \left[A_i, F^{i1}\right]\right]. \tag{3.50}$$

Proof of Proposition 3.5

From the assumption $g^{11} = g^{13} = g^{14} = 0$, $A_2 = 0$ on G^1, a simple calculation shows that

$$\begin{aligned} & g^{ik}_{,i} g^{1l} \left(A_{l,k} - A_{k,l} + [A_k, A_l]\right) \\ &= g^{12} \left(g^{1k}_{,1} + g^{2k}_{,2} + g^{\beta k}_{,\beta}\right) \left(A_{2,k} - A_{k,2} + [A_k, A_2]\right) \\ &= g^{12} \left\{ \left(g^{11}_{,1} + g^{12}_{,2}\right) \left(A_{2,1} - A_{1,2}\right) - \left(g^{1\alpha}_{,1} + g^{2\alpha}_{,2} + g^{\alpha\beta}_{,\beta}\right) A_{\alpha,2} \right\}, \end{aligned} \tag{3.51}$$

with

$$g^{1k} g^{1l}_{,1} \left(A_{l,k} - A_{k,l} + [A_k, A_l]\right) = g^{12} \left\{ g^{11}_{,1} \left(A_{1,2} - A_{2,1}\right) + g^{1\alpha}_{,1} A_{\alpha,2} \right\}, \tag{3.52}$$

$$g^{2k} g^{1l}_{,2} \left(A_{l,k} - A_{k,l} + [A_k, A_l]\right) = g^{12}_{,2} \left\{ g^{12} \left(A_{2,1} - A_{1,2}\right) - g^{2\alpha} A_{\alpha,2} \right\}, \tag{3.53}$$

$$g^{\beta k} g^{1l}_{,\beta} \left(A_{l,k} - A_{k,l} + [A_k, A_l]\right) = -g^{\alpha\beta} g^{12}_{,\beta} A_{\alpha,2}. \tag{3.54}$$

Proof of Proposition 3.5

(3.51 − 3.54) yield

$$\left(g^{ik}g^{1l}\right)_{,i}\left(A_{l,k}-A_{k,l}+[A_k,A_l]\right)$$
$$= g^{12}\left\{2g^{12}_{,2}\left(A_{2,1}-A_{1,2}\right)-\left(g^{2\alpha}_{,2}+g^{\alpha\beta}_{,\beta}\right)A_{\alpha,2}\right\}-g^{12}_{,2}g^{2\alpha}A_{\alpha,2}-g^{\alpha\beta}g^{12}_{,\beta}A_{\alpha,2}. \tag{3.55}$$

Inserting the relations

$$g^{2\alpha}=-g^{12}g^{\alpha\lambda}g_{1\lambda},\quad g^{12}_{,2}=-\left(g^{12}\right)^2g_{12,2},\quad g^{12}_{,\beta}=-\left(g^{12}\right)^2g_{12,\beta},$$
$$g^{2\alpha}_{,2}=\left(\left(g^{12}\right)^2g_{12,2}g^{\alpha\lambda}-g^{12}g^{\alpha\lambda}_{,2}\right)g_{1\lambda}-g^{12}g^{\alpha\lambda}g_{1\lambda,2}, \tag{3.56}$$

in (3.55), we get

$$g_{12}\left(g^{ik}g^{1l}\right)_{,i}\left(A_{l,k}-A_{k,l}+[A_k,A_l]\right)$$
$$=-2\left(g^{12}\right)^2g_{12,2}\left(A_{2,1}-A_{1,2}\right)-\left(2\left(g^{12}\right)^2g_{12,2}g^{\alpha\lambda}-g^{12}g^{\alpha\lambda}_{,2}\right)A_{\alpha,2}g_{1\lambda}$$
$$+g^{12}g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda,2}-g^{\alpha\beta}_{,\beta}A_{\alpha,2}+g^{\alpha\beta}g^{12}g_{12,\beta}A_{\alpha,2}. \tag{3.57}$$

Proof of Proposition 3.5

We now handle $g_{j2}\left[\Gamma^i_{im}F^{mj} + \Gamma^j_{im}F^{im} + \left[A_i, F^{ij}\right]\right]$. Using the assumption $g_{22} = g_{23} = g_{24} = 0$ on G^1, we get

$$g_{j2}\Gamma^i_{im}F^{mj} = g_{12}\left(\Gamma^i_{i2}F^{21} + \Gamma^i_{i\beta}F^{\beta 1}\right). \tag{3.58}$$

Simple calculations on G^1 give

$$\Gamma^i_{i2} = 3g^{12}g_{12,2}, \quad 2\Gamma^i_{i\beta} = 2g^{12}g_{12,\beta} + g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right), \tag{3.59}$$

and

$$F^{21} = \left(g^{12}\right)^2\left(A_{2,1} - A_{1,2}\right) + \left(g^{12}\right)^2 g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda},, \quad F^{\beta 1} = -g^{12}g^{\beta\alpha}A_{\alpha,2}. \tag{3.60}$$

Proof of Proposition 3.5

(3.59) and (3.60) give

$$\Gamma^i_{i2}F^{21} = 3\left(g^{12}\right)^3 g_{12,2}\left[\left(A_{2,1} - A_{1,2}\right) + g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda}\right],$$
$$\Gamma^i_{i\beta}F^{\beta 1} = -g^{12}g^{\beta\alpha}A_{\alpha,2}\left[g^{12}g_{12,\beta} + \frac{1}{2}g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right)\right], \tag{3.61}$$

and then

$$g_{12}\Gamma^i_{i2}F^{21} = 3\left(g^{12}\right)^2 g_{12,2}\left[\left(A_{2,1} - A_{1,2}\right) + g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda}\right],$$
$$g_{12}\Gamma^i_{i\beta}F^{\beta 1} = -g^{\beta\alpha}A_{\alpha,2}\left[g^{12}g_{12,\beta} + \frac{1}{2}g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right)\right]. \tag{3.62a}$$

Proof of Proposition 3.5

For the term $g_{j2}\Gamma^j_{im}F^{im}$, since $\Gamma^j_{im} = \Gamma^j_{mi}$ and $F^{im} = -F^{mi}$, a simple computation gives

$$g_{j2}\Gamma^j_{im}F^{im} = 0. \tag{3.62b}$$

The calculation of $g_{j2}\left[A_i, F^{ij}\right]$ gives

$$g_{j2}\left[A_i, F^{ij}\right] = -g^{\beta\alpha}\left[A_\beta, A_{\alpha,2}\right]. \tag{3.63}$$

Finally, from (3.41), (3.49), (3.57), $(3.62a - 3.62b)$, and (3.63), we gain

$$\begin{aligned} LA_2 = {} & 2g^{12}A_{2,12} - g^{\beta\alpha}\left[A_\beta, A_{\alpha,2}\right] + g^{12}g_{12,\beta}g^{\alpha\beta} \\ & - 2\left(g^{12}\right)^2 g_{12,2}A_{1,2} + 2\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda} \\ & + \left[g^{\alpha\beta}\left(-\left[g^{12}g_{12,\beta} + \frac{1}{2}g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right)\right]\right) - g^{\alpha\beta}_{,\beta}\right]A_{\alpha,2}. \end{aligned} \tag{3.64}$$

Proof of Proposition 3.5

Calculation of $\nabla_2 \nabla^k A_k$. By definition it holds that

$$\nabla^k A_k = g^{ik} \nabla_i A_k = g^{ik} \left(A_{k,i} - \Gamma^l_{ik} A_l \right). \tag{3.65}$$

Using the assumption $g^{11} = g^{1\alpha} = 0$ and $A_2 = 0$ on G^1 and the equality $g^{2\alpha} = -g^{12} g^{\alpha\lambda} g_{1\lambda}$ on G^1, we have

$$g^{ki} A_{k,i} = g^{12} \left(A_{1,2} + A_{2,1} \right) - g^{12} g^{\alpha\lambda} \left(A_{\alpha,2} + A_{2,\alpha} \right) g_{1\lambda}. \tag{3.66}$$

Using the notation $\Gamma^l = g^{ik} \Gamma^l_{ik}$, the calculation of $g^{ik} \Gamma^l_{ik} A_l$ gives $g^{ik} \Gamma^l_{ik} A_l = \Gamma^l A_l$. Since $\Gamma^1 = 0$ on G^1 at this step of the construction process, we deduce that

$$\nabla^k A_k = g^{12} \left(A_{1,2} + A_{2,1} \right) - g^{12} g^{\alpha\lambda} A_{\alpha,2} g_{1\lambda} + \Gamma^\alpha A_\alpha \text{ on } G^1. \tag{3.67}$$

Proof of Proposition 3.5

From (3.34) we deduce that on G^1 the following equality holds

$$\begin{aligned}&\Gamma^\alpha_{,2}\\&= \left[g^{12}g^{\beta\lambda}_{,2}\right]_{,2} g_{1\lambda} + g^{12}g^{\beta\lambda}_{,2} g_{1\lambda,2} + g^{12}g^{\alpha\lambda}\left(g_{2\lambda,12} + g_{1\lambda,22}\right)\\&+ \left(g^{12}g^{\alpha\lambda}\right)_{,2}\left(g_{2\lambda,1} + g_{1\lambda,2}\right)\\&+ \left(-g^{12}g^{\alpha\lambda}g_{12,\lambda} + \tfrac{1}{2}\left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right]\right)_{,2}.\end{aligned} \tag{3.68}$$

(3.67) and (3.34) yield

$$\begin{aligned}&\nabla^k A_k\\&= g^{12}\left(A_{1,2} + A_{2,1}\right) + \left[g^{12}g^{\beta\lambda}_{,2}A_\alpha - g^{12}g^{\alpha\lambda}A_{\alpha,2}\right] g_{1\lambda}\\&+ g^{12}g^{\alpha\lambda}A_\alpha\left(g_{2\lambda,1} + g_{1\lambda,2}\right) - g^{12}g^{\alpha\lambda}g_{12,\lambda}A_\alpha\\&+ \tfrac{1}{2}\left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right] A_\alpha.\end{aligned} \tag{3.69}$$

Proof of Proposition 3.5

Differentiation of (3.69) w.r.t. x^2 gives

$$\begin{aligned}
&\nabla_2\left(\nabla^k A_k\right) \\
&= g^{12}\left(A_{1,22}+A_{2,12}\right)+g^{12}_{,2}\left(A_{1,2}+A_{2,1}\right) \\
&+\left[g^{12}g^{\beta\lambda}_{,2}A_\alpha-g^{12}g^{\alpha\lambda}A_{\alpha,2}\right]_{,2}g_{1\lambda}+\left[g^{12}g^{\beta\lambda}_{,2}A_\alpha-g^{12}g^{\alpha\lambda}A_{\alpha,2}\right]g_{1\lambda,2} \\
&+\left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2}\left(g_{2\lambda,1}+g_{1\lambda,2}\right)+g^{12}g^{\alpha\lambda}A_\alpha\left(g_{2\lambda,12}+g_{1\lambda,22}\right) \\
&+\left(-g^{12}g^{\alpha\lambda}g_{12,\lambda}A_\alpha+\frac{1}{2}\left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta}+g_{\delta\beta,\mu}-g_{\mu\beta,\delta}\right)\right]A_\alpha\right)_{,2}.
\end{aligned} \tag{3.70}$$

Proof of Proposition 3.5

As $g^{12}_{,2} = -\left(g^{12}\right)^2 g_{12,2}$ (see (3.7)) and $\nabla^k A_k = \Delta$, we gain

$$\begin{aligned}\Delta_{,2} &= g^{12}\left(A_{1,22} + A_{2,12}\right) - \left(g^{12}\right)^2 g_{12,2}\left(A_{1,2} + A_{2,1}\right)\\ &+ \left[g^{12} g^{\beta\lambda}_{,2} A_\alpha - g^{12} g^{\alpha\lambda} A_{\alpha,2}\right]_{,2} g_{1\lambda} + \left[g^{12} g^{\beta\lambda}_{,2} A_\alpha - g^{12} g^{\alpha\lambda} A_{\alpha,2}\right] g_{1\lambda,2}\\ &+ \left(g^{12} g^{\alpha\lambda} A_\alpha\right)_{,2}\left(g_{2\lambda,1} + g_{1\lambda,2}\right) + g^{12} g^{\alpha\lambda} A_\alpha\left(g_{2\lambda,12} + g_{1\lambda,22}\right)\\ &+ \left(-g^{12} g^{\alpha\lambda} g_{12,\lambda} A_\alpha + \frac{1}{2}\left[g^{\alpha\delta} g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right] A_\alpha\right)_{,2}.\end{aligned} \tag{3.71}$$

Proof of Proposition 3.5

(3.64) and (3.71) yield

$$
\begin{aligned}
&LA_2 - 2\Delta_{,2} = -2g^{12}A_{1,22} + 2\left(g^{12}\right)^2 g_{12,2}A_{2,1} + 2\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda} \\
&+ \left[4\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_\alpha + g^{12}g^{\alpha\lambda}g^{\beta\mu}g_{\mu\beta,2}A_\alpha + 2g^{12}g^{\alpha\lambda}A_{\alpha,2}\right]_{,2} g_{1\lambda} \\
&+ \left[4\left(g^{12}\right)^2 g_{12,2}g^{\alpha\lambda}A_\alpha + g^{12}g^{\alpha\lambda}g^{\beta\mu}g_{\mu\beta,2}A_\alpha + 2g^{12}g^{\alpha\lambda}A_{\alpha,2}\right] g_{1\lambda,2} \\
&-2\left(g^{12}g^{\alpha\lambda}A_\alpha\right)_{,2}\left(g_{2\lambda,1} + g_{1\lambda,2}\right) - 2g^{12}g^{\alpha\lambda}A_\alpha\left(g_{2\lambda,12} + g_{1\lambda,22}\right) \\
&+ \left[g^{\alpha\beta}\left(g^{12}g_{12,\beta} - \left[g^{12}g_{12,\beta} + \tfrac{1}{2}g^{\lambda\mu}\left(g_{\mu\beta,\lambda} + g_{\lambda\mu,\beta} - g_{\beta\lambda,\mu}\right)\right]\right) - g^{\alpha\beta}_{,\beta}\right] A_{\alpha,2} \\
&-g^{\beta\alpha}\left[A_\beta, A_{\alpha,2}\right] + \left(2g^{12}g^{\alpha\lambda}g_{12,\lambda}A_\alpha - \left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right] A_\alpha\right)_{,2}
\end{aligned}
\qquad (3.72)
$$

Proof of Proposition 3.5

From (3.40) we have

$$(g_{2\lambda,1} + g_{1\lambda,2}) = g_{12}g_{\nu\lambda}\Gamma^\nu + g^{\beta\mu}g_{\mu\beta,2}g_{1\lambda} + g_{12,\lambda} - \frac{1}{2}g_{12}\left[g^{\beta\mu}\left(g_{\mu\lambda,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}\right)\right]. \qquad (3.73)$$

Thus

$$\begin{aligned}&(g_{2\lambda,12} + g_{1\lambda,22})\\ &= (g_{12}g_{\nu\lambda})_{,2}\Gamma^\nu + g_{12}g_{\nu\lambda}\Gamma^\nu_{,2} + \left[\tfrac{1}{2}g^{\beta\mu}g_{\mu\beta,2}\right]_{,2}g_{1\lambda}\\ &+g^{\beta\mu}g_{\mu\beta,2}g_{1\lambda,2} + g_{12,2\lambda} - \tfrac{1}{2}\left(g_{12}\left[g^{\beta\mu}\left(g_{\mu\lambda,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}\right)\right]\right)_{,2}. \qquad (3.74)\end{aligned}$$

Proof of Proposition 3.5

One proceeds in the same way to calculate $A_{2,1}$ from (3.69) and (3.73) to have

$$\begin{aligned} A_{2,1} &= g_{12}\Delta - A_\alpha g_{12}\Gamma^\alpha - A_{1,2} + g^{\alpha\lambda}A_{\alpha,2}g_{1\lambda} - A_\alpha g^{\alpha\lambda}g_{12,\lambda} \\ &\quad + \frac{1}{2}g_{12}g^{\alpha\lambda}A_\alpha\left[g^{\beta\mu}\left(g_{\mu\lambda,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}\right)\right] + g^{\alpha\lambda}g_{12,\lambda}A_\alpha \\ &\quad - \frac{1}{2}g_{12}\left[g^{\alpha\delta}g^{\beta\mu}\left(g_{\mu\delta,\beta} + g_{\delta\beta,\mu} - g_{\mu\beta,\delta}\right)\right]A_\alpha. \end{aligned} \tag{3.75}$$

Inserting (3.73), (3.74) and (3.75) in (3.72) and using the equality $g^{\beta\mu}g_{\mu\beta,2} = 4g^{12}g_{12,2}$ on G^1, we gain the desired relation (3.21).

Proof of Proposition 3.5

Proof of item (*ii*). One uses the expression of $A_{2,1}$ given above in (3.75) to obtain, from (3.33), the following expression of $\tau_{2\alpha}$

$$\tau_{2\alpha} = -2g^{12}A_{\alpha,2}.A_{1,2} + A_{\alpha,2}.\Delta - A_\nu.A_{\alpha,2}\Gamma^\nu + 2g^{\nu\lambda}g^{12}A_{\alpha,2}.A_{\nu,2}g_{1\lambda} - g^{\beta\lambda}\left(A_{\alpha,\lambda} - A_{\lambda,\alpha} + [A_\lambda, A_\alpha]\right).A_{\beta,2} + \left(\Phi_{,2}\right).\left(\Phi_{,\alpha} + [A_\alpha, \Phi]\right). \tag{3.76}$$

The insertion of (3.76) in (3.22) gives the desired system (3.23) with the appropriate known coefficients given by (3.24). This completes the proof of Proposition 3.5.

Construction of (g_{13}, g_{14}, A_1) on $G^1_{T_1}$

The proof of the following Proposition 3.6, that provides the construction of (g_{13}, g_{14}, A_1) on $G^1_{T_1}$, is a direct consequence of Proposition 3.5.

Proposition 3.6. Let $a_0,\ a_1,\ b_0,\ b_1,\ c_0,\ c_1 \in C^\infty(\Gamma)$. Then system (3.23) has a unique solution (g_{13}, g_{14}, A_1) in $C^\infty\left(G^1_{T_1}\right)$ satisfying

$$(g_{13},\ g_{14},\ A_1) = (a_0,\ b_0,\ c_0) \ \text{ on } \Gamma,$$

and

$$(g_{13,2},\ g_{14,2},\ A_{1,2}) = (a_1,\ b_1,\ c_1) \ \text{ on } \Gamma.$$

The relations $\Gamma^\beta = 0$ and $\Delta = 0$ on $G^1_{T_1}$

Now the relations $\Gamma^\beta = 0$ and $\Delta = 0$ on $G^1_{T_1}$ are arranged in the following Proposition 3.7.

Proposition 3.7. (i) On $G^1_{T_1}$, the reduced system

$$\widetilde{R}_{2\alpha} = \tau_{2\alpha},$$
$$LA_2 = J_2,$$

is equivalent to

$$\begin{aligned} g_{3\beta}\Gamma^\beta_{,2} + \left(g^{12}g_{12,2}g_{3\beta} + \tfrac{1}{2}g_{3\beta,2} - A_\beta.A_{3,2}\right)\Gamma^\beta + A_{3,2}.\Delta = 0,\\ g_{4\beta}\Gamma^\beta_{,2} + \left(g^{12}g_{12,2}g_{4\beta} + \tfrac{1}{2}g_{4\beta,2} - A_\beta.A_{4,2}\right)\Gamma^\beta + A_{4,2}.\Delta = 0,\\ 2A_\beta\Gamma^\beta_{,2} - 2\Delta_{,2} - 2g^{12}g_{12,2}\Delta + 2\left(g^{12}g_{12,2}A_\beta + A_{\beta,2}\right)\Gamma^\beta = 0. \end{aligned} \quad (3.77)$$

(ii) if $\Gamma^\beta = 0$ and $\Delta = 0$ on Γ then $\Gamma^\beta = 0$ and $\Delta = 0$ on $G^1_{T_1}$.

Proof of Proposition 3.7

Proof of item (i). By definition of $\widetilde{R}_{ij}$ (see (2.17)) and since $\Gamma^1 = 0$ on G^1 at this step of the construction process, the reduced system

$$\widetilde{R}_{23} = \tau_{23}, \quad \widetilde{R}_{24} = \tau_{24}, \quad LA_2 = J_2,$$

is equivalent to

$$\begin{aligned} &R_{23} - \tfrac{1}{2} g_{3\beta}\Gamma^{\beta}_{,2} = \tau_{23}, \\ &R_{24} - \tfrac{1}{2} g_{4\beta}\Gamma^{\beta}_{,2} = \tau_{24}, \\ &LA_2 = J_2. \end{aligned}$$

In view of (3.22), this is equivalent to (3.77).
Proof of item (ii). (3.77) is a linear homogeneous system of first order ODEs on $G^1_{T_1}$, with unknown $\left(\Gamma^3, \Gamma^4, \Delta\right)$, with variable x^2, with C^∞ coefficients depending smoothly on parameters x^3 and x^4. This yields $\Gamma^\alpha = 0$ and $\Delta = 0$ in $G^1_{T_1}$, if $\Gamma^\alpha = 0$ and $\Delta = 0$ on Γ. (The conditions $\Gamma^\alpha = 0$ and $\Delta = 0$ on Γ can be obtained upon a judicious choice of the data a_0, a_1, b_0, b_1, c_0, c_1 on Γ (see Proposition 3.6).

Last step

The last step of the hierarchical construction process is now described. Consider the reduced equations $\widetilde{R}_{\alpha\beta} = \tau_{\alpha\beta}$ which are equivalent to $R_{\alpha\beta} = \tau_{\alpha\beta}$, since

$$\widetilde{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\left(g_{k\alpha}\Gamma^k_{,\beta} + g_{k\beta}\Gamma^k_{,\alpha}\right) \text{ and } \Gamma^1 = \Gamma^3 = \Gamma^4 = 0 \text{ on } G^1_{T_1}.$$

As usual, the problem is to find a good combination of $g^{\alpha\beta}R_{\alpha\beta}$ and Γ^2 that will provide an ODE with unknown g_{11}. Analogously to Proposition 3.5, the following proposition is very important.

An important proposition : Proposition 3.8

Proposition 3.8. (i) On $G^1_{T_1}$, the following combinations hold

$$\begin{aligned} & g^{\alpha\beta}R_{\alpha\beta} - 2\Gamma^2_{,2} - 2g^{12}g_{12,2}\Gamma^2 \\ & = -2\left(g^{12}\right)^2 g_{11,22} + 4\left(g^{12}\right)^3 g_{12,2}g_{11,2} \\ & \quad + \left\{4\left(g^{12}\right)^4\left(g_{12,2}\right)^2 + \frac{1}{2}\left(g^{12}\right)^2\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}\right\}g_{11} \\ & \quad + \frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right) - 2W - 2g^{12}g_{12,2}S, \\ & g^{\alpha\beta}\tau_{\alpha\beta} = K. \end{aligned} \tag{3.78}$$

An important proposition : Proposition 3.8

Here, at this level of the construction process, $N_{\alpha\beta}$, $M_{\alpha\beta}$, W, S, K are known on $G^1_{T_1}$ and given by

$$
\begin{aligned}
&N_{\alpha\beta} \\
&= -g_{\alpha\beta,2}\left[\left(g^{12}\right)^2 g_{22,1}g^{2\mu}g_{1\mu} - g^{12}g^{2\mu}g_{2\mu,1}\right] - 2\left(g^{12}\right)^2 g_{12,2}\left(g_{1\beta,\alpha} + g_{1\alpha,\beta}\right) \\
&-g^{12}\left(2g^{2\mu}g_{12,2} + g^{\mu\lambda}g_{2\lambda,1}\right)\left(g_{\beta\mu,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \\
&+g^{12}\left(g_{2\beta,1\alpha} + g_{2\alpha,1\beta}\right) + g_{\alpha\beta,2}\left(g^{12}g^{2\mu}g_{1\mu}\right)_{,2} + g^{12}g^{2\mu}g_{1\mu}g_{\alpha\beta,22} \\
&+g^{12}_{,2}\left(g_{1\beta,\alpha} + g_{1\alpha,\beta}\right) + g^{2\mu}_{,2}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) + g^{12}\left(g_{1\beta,2\alpha} + g_{1\alpha,2\beta}\right) \\
&+g^{2\mu}\left(g_{\beta\mu,2\alpha} + g_{\mu\alpha,2\beta} - g_{\alpha\beta,2\mu}\right) - g^{2\lambda}_{,\lambda}g_{\alpha\beta,2} + g^{\lambda\mu}_{,\lambda}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \\
&-g^{2\lambda}g_{\alpha\beta,2\lambda} + g^{\lambda\mu}\left(g_{\mu\beta,\lambda\alpha} + g_{\mu\alpha,\lambda\beta} - g_{\alpha\beta,\lambda\mu}\right) \\
&-\left[2g^{12}g_{12,\alpha} + g^{\lambda\mu}\left(g_{\mu\lambda,\alpha} + g_{\mu\alpha,\lambda} - g_{\alpha\lambda,\mu}\right)\right]_{,\beta},
\end{aligned}
\tag{3.79}
$$

An important proposition : Proposition 3.8

$$
\begin{aligned}
& M_{\alpha\beta} \\
& = -g^{12} g_{\alpha\beta,2} \left[-g^{12} g_{22,1} g^{2\mu} g_{1\mu} + 2 g^{2\lambda} g_{2\lambda,1} + g^{\lambda\mu} \left(g_{1\mu,\lambda} - g_{1\lambda,\mu} \right) \right] \\
& + \left[g^{12} g^{2\mu} g_{1\mu} g_{\alpha\beta,2} + g^{12} \left(g_{1\beta,\alpha} + g_{1\alpha,\beta} \right) + g^{2\mu} \left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu} \right) \right] \\
& \times \left(3 g^{12} g_{22,1} + g^{\lambda\mu} g_{\lambda\mu,2} \right) \\
& + \left[2 g^{12} g_{12,\lambda} + g^{\mu\theta} \left(g_{\mu\theta,\lambda} + g_{\theta\lambda,\mu} - g_{\mu\lambda,\theta} \right) \right] \\
& \times \left[-g^{2\lambda} g_{\alpha\beta,2} + g^{\mu\lambda} \left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu} \right) \right] \\
& - \left(g^{12} \right)^2 \left(g_{12,\beta} + g_{2\beta,1} - g_{1\beta,2} \right) \left(g_{12,\alpha} + g_{2\alpha,1} - g_{1\alpha,2} \right) \\
& + g^{12} g_{\mu\alpha,2} \left[g^{2\mu} \left(g_{2\beta,1} + g_{12,\beta} - g_{1\beta,2} \right) + g^{\lambda\mu} \left(g_{1\lambda,\beta} - g_{1\beta,\lambda} \right) \right] \\
& - 2 g^{12} g^{2\lambda} g_{1\lambda} g^{\theta\mu} g_{\theta\beta,2} g_{\alpha\mu,2} - g^{12} g^{\lambda\mu} g_{\lambda\beta,2} \left(g_{1\mu,\alpha} + g_{1\alpha,\mu} \right) \\
& - g^{\theta\mu} g_{\theta\beta,2} g^{2\lambda} \left(g_{\lambda\mu,\alpha} + g_{\lambda\alpha,\mu} - g_{\alpha\mu,\lambda} \right) \\
& - \left[g^{12} \left(g_{12,\beta} + g_{1\beta,2} - g_{2\beta,1} \right) + g^{2\mu} g_{\mu\beta,2} \right] \\
& \times \left[g^{12} \left(g_{12,\alpha} + g_{1\alpha,2} - g_{2\alpha,1} \right) + g^{2\lambda} g_{\lambda\alpha,2} \right] \\
& - g^{12} g^{\lambda\theta} g_{\theta\alpha,2} \left(g_{1\beta,\lambda} + g_{1\lambda,\beta} \right) - g^{2\mu} g^{\lambda\theta} g_{\theta\alpha,2} \left(g_{\mu\beta,\lambda} + g_{\mu\lambda,\beta} - g_{\lambda\beta,\mu} \right) + m_{\alpha\beta},
\end{aligned}
\tag{3.80}
$$

An important proposition : Proposition3.8

with

$$\begin{aligned}
&m_{\alpha\beta}\\
&= g^{12}g_{\lambda\beta,2}\left[g^{2\lambda}\left(g_{2\alpha,1}+g_{12,\alpha}-g_{1\alpha,2}\right)+g^{\lambda\mu}\left(g_{1\mu,\alpha}-g_{1\alpha,\mu}\right)\right]\\
&-\left[-g^{2\mu}g_{\lambda\beta,2}+g^{\theta\mu}\left(g_{\theta\lambda,\beta}+g_{\theta\beta,\lambda}-g_{\lambda\beta,\theta}\right)\right]\\
&\times\left[-g^{2\lambda}g_{\alpha\mu,2}+g^{\delta\lambda}\left(g_{\delta\mu,\alpha}+g_{\delta\alpha,\mu}-g_{\alpha\mu,\delta}\right)\right],
\end{aligned}$$

$$\begin{aligned}
S &= 2g^{12}g^{2\lambda}g_{1\lambda,2}+\tfrac{1}{2}g^{12}g^{\lambda\mu}\left(2g_{1\lambda,\mu}\right)\\
&\quad+\tfrac{1}{2}g^{2\mu}\left[g^{\lambda2}\left(2g_{\mu\lambda,2}-g_{\lambda2,\mu}\right)+g^{\lambda\theta}\left(2g_{\mu\lambda,\theta}-g_{\lambda\theta,\mu}\right)\right],\\
W &= \left[2g^{12}g^{2\lambda}g_{1\lambda,2}+g^{12}g^{\lambda\mu}\left(g_{1\lambda,\mu}\right)\right]_{,2}\\
&\quad+\tfrac{1}{2}\left\{g^{2\mu}\left[g^{\lambda2}\left(2g_{\mu\lambda,2}-g_{\lambda2,\mu}\right)+g^{\lambda\theta}\left(2g_{\mu\lambda,\theta}-g_{\lambda\theta,\mu}\right)\right]\right\}_{,2},\\
K &= 2g^{\alpha\beta}g^{2\lambda}F_{2\alpha}.F_{\lambda\beta}+g^{\alpha\beta}g^{\mu\lambda}F_{\mu\alpha}.F_{\lambda\beta}-F_{12}.F^{12}-F_{34}.F^{34}\\
&\quad-F_{2\lambda}.\left[g^{21}g^{\lambda2}F_{12}+g^{23}g^{\lambda2}F_{32}+g^{23}g^{\lambda4}F_{34}+g^{24}g^{\lambda2}F_{42}+g^{24}g^{\lambda3}F_{43}\right]\\
&\quad+g^{\alpha\beta}\left(\Phi_{,\alpha}+[A_\alpha,\Phi]\right).\left(\Phi_{,\beta}+[A_\beta,\Phi]\right)+V\left(\Phi^2\right).
\end{aligned} \tag{3.81}$$

An important proposition : Proposition3.8

(ii) The equation

$$g^{\alpha\beta}R_{\alpha\beta} - 2\Gamma^2_{,2} - 2g^{12}g_{12,2}\Gamma^2 = g^{\alpha\beta}\tau_{\alpha\beta}, \tag{3.82}$$

is equivalent to the following second order ODE on $G^1_{T_1}$ with unknown g_{11},

$$-2\left(g^{12}\right)^2 g_{11,22} + 4\left(g^{12}\right)^3 g_{12,2}g_{11,2} + \chi g_{11} + \psi = 0, \tag{3.83}$$

where

$$\begin{aligned} \chi &= 4\left(g^{12}\right)^4\left(g_{12,2}\right)^2 + \tfrac{1}{2}\left(g^{12}\right)^2\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}, \\ \psi &= \tfrac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right) - 2W - 2g^{12}g_{12,2}S - K. \end{aligned} \tag{3.84}$$

Proof of Proposition 3.8

Proof of item (i). By definition of the Ricci curvature tensor, we have

$$R_{\alpha\beta} = \Gamma^k_{\alpha\beta,k} - \Gamma^k_{\alpha k,\beta} + \Gamma^k_{lk}\Gamma^l_{\alpha\beta} - \Gamma^k_{l\beta}\Gamma^l_{\alpha k}. \tag{3.85}$$

As in the previous steps, each term in the r.h.s of (3.85) is calculated meticulously on G^1. Here one uses the equality

$$g^{12}_{,1} = \left(g^{12}\right)^3 g_{22,1}g_{11} + \left(g^{12}\right)^2 g_{22,1}g^{2\mu}g_{1\mu} - g^{12}\left(g^{12}g_{12,1} + g^{2\mu}g_{2\mu,1}\right),$$

which follows from the equalities $\left(g^{2i}g_{2i}\right)_{,1} = 0, \quad g^{2i}g_{1i} = 0$ on G^1, to gain

Proof of Proposition 3.8

$$
\begin{aligned}
2\Gamma^k_{\alpha\beta,k} = &-\left(g^{11}_{,1} + g^{12}_{,2}\right) g_{\alpha\beta,1} - 2g^{12} g_{\alpha\beta,12} + \left(g^{12}\right)^2 g_{\alpha\beta,2} g_{12,1} \\
&+ \left(g^{12}\right)^2 g_{\alpha\beta,2} g_{11,2} + \left\{\left[\left(g^{12}\right)^2 g_{\alpha\beta,2}\right]_{,2} - \left(g^{12}\right)^3 g_{22,1} g_{\alpha\beta,2}\right\} g_{11} \\
&- g_{\alpha\beta,2}\left[\left(g^{12}\right)^2 g_{22,1} g^{2\mu} g_{1\mu} - g^{12} g^{2\mu} g_{2\mu,1}\right] \\
&+ g^{11}_{,1}\left(g_{1\beta,\alpha} + g_{1\alpha,\beta}\right) + g^{1\mu}_{,1}\left(g_{\beta\mu,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \\
&+ g^{12}\left(g_{2\beta,1\alpha} + g_{2\alpha,1\beta}\right) + g_{\alpha\beta,2}\left(g^{12} g^{2\mu} g_{1\mu}\right)_{,2} \\
&+ g^{12} g^{2\mu} g_{1\mu} g_{\alpha\beta,22} + g^{12}_{,2}\left(g_{1\beta,\alpha} + g_{1\alpha,\beta}\right) + g^{2\mu}_{,2}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \\
&+ g^{12}\left(g_{1\beta,2\alpha} + g_{1\alpha,2\beta}\right) + g^{2\mu}\left(g_{\beta\mu,2\alpha} + g_{\mu\alpha,2\beta} - g_{\alpha\beta,2\mu}\right) \\
&- g^{2\lambda}_{,\lambda} g_{\alpha\beta,2} + g^{\lambda\mu}_{,\lambda}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right) \\
&- g^{2\lambda} g_{\alpha\beta,2\lambda} + g^{\lambda\mu}\left(g_{\mu\beta,\lambda\alpha} + g_{\mu\alpha,\lambda\beta} - g_{\alpha\beta,\lambda\mu}\right). \qquad (3.86a)
\end{aligned}
$$

Proof of Proposition 3.8

Similarly, we have

$$2\Gamma^k_{\alpha k,\beta} = \left[2g^{12}g_{12,\alpha} + g^{\lambda\mu}\left(g_{\mu\lambda,\alpha} + g_{\mu\alpha,\lambda} - g_{\alpha\lambda,\mu}\right)\right]_{,\beta}. \qquad (3.86b)$$

(3.86a) and (3.86b) yield

$$\begin{aligned} &2\left(\Gamma^k_{\alpha\beta,k} - \Gamma^k_{\alpha k,\beta}\right) \\ &= -\left(g^{11}_{,1} + g^{12}_{,2}\right)g_{\alpha\beta,1} - 2g^{12}g_{\alpha\beta,12} + \left(g^{12}\right)^2 g_{\alpha\beta,2}g_{12,1} + \left(g^{12}\right)^2 g_{\alpha\beta,2}g_{11,2} \\ &+ \left\{\left[\left(g^{12}\right)^2 g_{\alpha\beta,2}\right]_{,2} - \left(g^{12}\right)^3 g_{22,1}g_{\alpha\beta,2}\right\}g_{11} + N_{\alpha\beta}, \end{aligned} \qquad (3.86c)$$

where $N_{\alpha\beta}$ is known and given on G^1 by (3.79).

Proof of Proposition 3.8

The calculation of $\Gamma^k_{lk}\Gamma^l_{\alpha\beta}$ and $\Gamma^k_{l\beta}\Gamma^l_{\alpha k}$ gives

$$
\begin{aligned}
&4\Gamma^k_{lk}\Gamma^l_{\alpha\beta}\\
&= -2\left(g^{12}\right)^2 g_{\alpha\beta,2}g_{12,1} - g^{12}\left(3g^{12}g_{22,1} + g^{\lambda\mu}g_{\lambda\mu,2}\right)g_{\alpha\beta,1}\\
&+ \left(g^{12}\right)^2 g_{\alpha\beta,2}\left(4g^{12}g_{22,1} + g^{\lambda\mu}g_{\lambda\mu,2}\right)g_{11}\\
&-g^{12}g_{\alpha\beta,2}\left[-g^{12}g_{22,1}g^{2\mu}g_{1\mu} + 2g^{2\lambda}g_{2\lambda,1} + g^{\lambda\mu}\left(g_{1\mu,\lambda} + g_{\lambda\mu,1} - g_{1\lambda,\mu}\right)\right]\\
&+ \left[g^{12}g^{2\mu}g_{1\mu}g_{\alpha\beta,2} + g^{12}\left(g_{1\beta,\alpha} + g_{1\alpha,\beta}\right) + g^{2\mu}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right)\right]\\
&\times\left(3g^{12}g_{22,1} + g^{\lambda\mu}g_{\lambda\mu,2}\right)\\
&+ \left[2g^{12}g_{12,\lambda} + g^{\mu\theta}\left(g_{\mu\theta,\lambda} + g_{\theta\lambda,\mu} - g_{\mu\lambda,\theta}\right)\right]\\
&\times\left[-g^{2\lambda}g_{\alpha\beta,2} + g^{\mu\lambda}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right)\right],
\end{aligned}
\tag{3.87a}
$$

and

Proof of Proposition 3.8

$$
\begin{aligned}
4\Gamma^k_{l\beta}\Gamma^l_{\alpha k} &= 2\left(g^{12}\right)^2 g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,2} g_{11} - 2g^{12} g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,1} \\
&+ \left(g^{12}\right)^2 \left(g_{12,\beta} + g_{2\beta,1} - g_{1\beta,2}\right)\left(g_{12,\alpha} + g_{2\alpha,1} - g_{1\alpha,2}\right) \\
&- g^{12} g_{\mu\alpha,2}\left[g^{2\mu}\left(g_{2\beta,1} + g_{12,\beta} - g_{1\beta,2}\right) + g^{\lambda\mu}\left(g_{\lambda\beta,1} + g_{1\lambda,\beta} - g_{1\beta,\lambda}\right)\right] \\
&+ 2g^{12} g^{2\lambda} g_{1\lambda} g^{\theta\mu} g_{\theta\beta,2} g_{\alpha\mu,2} \\
&+ g^{12} g^{\lambda\mu} g_{\lambda\beta,2}\left(g_{1\mu,\alpha} + g_{1\alpha,\mu}\right) + g^{\theta\mu} g_{\theta\beta,2} g^{2\lambda}\left(g_{\lambda\mu,\alpha} + g_{\lambda\alpha,\mu} - g_{\alpha\mu,\lambda}\right) \\
&+ \left[g^{12}\left(g_{12,\beta} + g_{1\beta,2} - g_{2\beta,1}\right) + g^{2\mu} g_{\mu\beta,2}\right] \\
&\times \left[g^{12}\left(g_{12,\alpha} + g_{1\alpha,2} - g_{2\alpha,1}\right) + g^{2\lambda} g_{\lambda\alpha,2}\right] \\
&+ g^{12} g^{\lambda\theta} g_{\theta\alpha,2}\left(g_{1\beta,\lambda} + g_{1\lambda,\beta}\right) + g^{2\mu} g^{\lambda\theta} g_{\theta\alpha,2}\left(g_{\mu\beta,\lambda} + g_{\mu\lambda,\beta} - g_{\lambda\beta,\mu}\right) \\
&- g^{12} g_{\lambda\beta,2}\left[g^{2\lambda}\left(g_{2\alpha,1} + g_{12,\alpha} - g_{1\alpha,2}\right) + g^{\lambda\mu}\left(g_{\mu\alpha,1} + g_{1\mu,\alpha} - g_{1\alpha,\mu}\right)\right] \\
&+ \left[-g^{2\mu} g_{\lambda\beta,2} + g^{\theta\mu}\left(g_{\theta\lambda,\beta} + g_{\theta\beta,\lambda} - g_{\lambda\beta,\theta}\right)\right] \\
&\times \left[-g^{2\lambda} g_{\alpha\mu,2} + g^{\delta\lambda}\left(g_{\delta\mu,\alpha} + g_{\delta\alpha,\mu} - g_{\alpha\mu,\delta}\right)\right].
\end{aligned}
\tag{3.87b}
$$

Proof of Proposition 3.8

From (3.87a) and (3.87b) it follows that

$$4\left(\Gamma^k_{lk}\Gamma^l_{\alpha\beta} - \Gamma^k_{l\beta}\Gamma^l_{\alpha k}\right)$$
$$= -2\left(g^{12}\right)^2 g_{\alpha\beta,2}g_{12,1} - g^{12}\left(3g^{12}g_{22,1} + g^{\lambda\mu}g_{\lambda\mu,2}\right)g_{\alpha\beta,1}$$
$$+\left(g^{12}\right)^2 g_{\alpha\beta,2}\left(4g^{12}g_{22,1} + g^{\lambda\mu}g_{\lambda\mu,2}\right)g_{11} - 2\left(g^{12}\right)^2 g^{\lambda\mu}g_{\lambda\beta,2}g_{\alpha\mu,2}g_{11}$$
$$+2g^{12}g^{\lambda\mu}g_{\lambda\beta,2}g_{\alpha\mu,1} + 2g^{12}g_{\mu\alpha,2}g^{\lambda\mu}g_{\lambda\beta,1} + M_{\alpha\beta}, \tag{3.87c}$$

where $M_{\alpha\beta}$ is known and given on G^1 by (3.80).

(3.86c) and (3.87c) yield

$$\begin{aligned}
R_{\alpha\beta} = & -\frac{1}{2}\left(g^{11}_{,1} + g^{12}_{,2}\right) g_{\alpha\beta,1} - g^{12} g_{\alpha\beta,12} + \frac{1}{2}\left(g^{12}\right)^2 g_{\alpha\beta,2} g_{11,2} \\
& + \frac{1}{2}\left\{\left[\left(g^{12}\right)^2 g_{\alpha\beta,2}\right]_{,2} - \left(g^{12}\right)^3 g_{22,1} g_{\alpha\beta,2}\right\} g_{11} \\
& - \frac{1}{4} g^{12}\left(3 g^{12} g_{22,1} + g^{\lambda\mu} g_{\lambda\mu,2}\right) g_{\alpha\beta,1} \\
& + \frac{1}{4}\left(g^{12}\right)^2 g_{\alpha\beta,2}\left(4 g^{12} g_{22,1} + g^{\lambda\mu} g_{\lambda\mu,2}\right) g_{11} \\
& - \frac{1}{2}\left(g^{12}\right)^2 g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,2} g_{11} + \frac{1}{2} g^{12} g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,1} + \frac{1}{4}\left(N_{\alpha\beta} + M_{\alpha\beta}\right).
\end{aligned} \tag{3.88}$$

Proof of Proposition 3.8

Now using the following relations (see (3.6), (3.7), and proof of Proposition 3.4)

$$\begin{aligned} g^{11}_{,1} &= -\left(g^{12}\right)^2 g_{22,1} = -2\left(g^{12}\right)^2 g_{12,2}, \\ g^{12}_{,2} &= -\left(g^{12}\right)^2 g_{12,2}, \quad g^{\lambda\mu} g_{\lambda\mu,2} = 4 g^{12} g_{12,2}, \end{aligned}$$

together with (3.88), we gain

$$\begin{aligned} R_{\alpha\beta} = & \tfrac{1}{2} g^{12} g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,1} + \tfrac{1}{4} g^{12} g_{\alpha\beta,2} g^{\lambda\mu} g_{\lambda\mu,1} \\ & + \tfrac{1}{2} g^{12} g_{\mu\alpha,2} g^{\lambda\mu} g_{\lambda\beta,1} - \left(g^{12}\right)^2 g_{12,2} g_{\alpha\beta,1} \\ & - g^{12} g_{\alpha\beta,12} + \tfrac{1}{2}\left(g^{12}\right)^2 g_{\alpha\beta,2} g_{11,2} \\ & + \tfrac{1}{2}\left\{\left[\left(g^{12}\right)^2 g_{\alpha\beta,2}\right]_{,2} + 4\left(g^{12}\right)^3 g_{12,2} g_{\alpha\beta,2} - \left(g^{12}\right)^2 g^{\lambda\mu} g_{\lambda\beta,2} g_{\alpha\mu,2}\right\} g_{11} \\ & + \tfrac{1}{4}\left(N_{\alpha\beta} + M_{\alpha\beta}\right). \end{aligned} \tag{3.89}$$

Proof of Proposition 3.8

From (3.89), after some supplementary calculation, we obtain

$$
\begin{aligned}
& g^{\alpha\beta}R_{\alpha\beta} \\
&= -g^{12}g^{\alpha\beta}_{,2}g_{\alpha\beta,1} - g^{12}g^{\alpha\beta}g_{\alpha\beta,12} + 2\left(g^{12}\right)^3 g_{12,2}g_{11,2} \\
&+ \left[4\left(g^{12}\right)^4\left(g_{12,2}\right)^2 + \frac{1}{2}\left(g^{12}\right)^2\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}\right]g_{11} + \frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right).
\end{aligned}
\tag{3.90}
$$

We now expand the equality $2\Gamma^2 = g^{ij}g^{2m}\left(2g_{mi,j} - g_{ij,m}\right)$ by using the equalities

$$g^{11} = g^{13} = g^{14} = 0, \quad g_{22} = g_{23} = g_{24} = 0,$$

assumed on G^1 to gain

$$\Gamma^2 = \left(g^{12}\right)^2 g_{11,2} - \frac{1}{2}g^{12}g^{\lambda\mu}g_{\lambda\mu,1} + S, \tag{3.91}$$

where S is known and given on G^1 by (3.81).

(3.91) gives

$$\Gamma^2_{,2} = \left(g^{12}\right)^2 g_{11,22} - 2\left(g^{12}\right)^3 g_{12,2}g_{11,2} + \frac{1}{2}\left(g^{12}\right)^2 g_{12,2}g^{\lambda\mu}g_{\lambda\mu,1}$$
$$- \frac{1}{2}g^{12}g^{\lambda\mu}_{,2}g_{\lambda\mu,1} - \frac{1}{2}g^{12}g^{\lambda\mu}g_{\lambda\mu,12} + W, \qquad (3.92)$$

where W is known and given on G^1 by (3.81). (3.90), (3.91) and (3.92) yield the first equality of (3.78).

Proof of Proposition 3.8

We now prove the second equality of (3.78). From (2.17) we have

$$g^{\alpha\beta}\tau_{\alpha\beta} = g^{\alpha\beta}F_{\alpha k}.F_{\beta i}g^{ki} - \frac{1}{2}F_{kl}.F^{kl} + g^{\alpha\beta}\widehat{\nabla}_\alpha\Phi.\widehat{\nabla}_\beta\Phi + V\left(\Phi^2\right). \quad (3.93)$$

It is worth noting at this step of the construction process that, apart from g_{11} and $F_{1\alpha}$, all the g_{ij} and F_{ij} are known on G^1. One deduces that, apart from g^{22} and $F^{2\alpha}$, all the g^{ik} and F^{ik} are known on G^1. More precisely, the following equalities hold on G^1.

$$
\begin{aligned}
g^{22} &= -\left(g^{12}\right)^2 g_{11} - g^{12} g^{2\lambda} g_{1\lambda}, \\
F^{12} &= \left(g^{12}\right)^2 F_{21} + g^{12} g^{2\lambda} F_{2\lambda}, \\
F^{1\alpha} &= g^{12} g^{\alpha\lambda} F_{2\lambda}, \\
F^{2\alpha} &= g^{21} g^{\alpha 2} F_{12} + g^{21} g^{\alpha\lambda} F_{1\lambda} + g^{22} g^{\alpha\lambda} F_{2\lambda} + g^{23} g^{\alpha 2} F_{32} + g^{23} g^{\alpha 4} F_{34} \\
&\quad + g^{24} g^{\alpha 2} F_{42} + g^{24} g^{\alpha 3} F_{43}, \\
F^{34} &= g^{32} g^{4\lambda} F_{2\lambda} + g^{33} g^{42} F_{32} + g^{33} g^{44} F_{34} + g^{34} g^{42} F_{42} + g^{34} g^{43} F_{43}.
\end{aligned} \tag{3.94}
$$

Proof of Proposition 3.8

We will then examine all the terms of the r.h.s of (3.93) in order to highlight the unknown functions. The following equalities hold on G^1 via direct calculations

$$\begin{aligned} & g^{\alpha\beta}g^{ki}F_{\alpha k}.F_{\beta i} \\ & = 2g^{12}g^{\alpha\beta}F_{1\alpha}.F_{2\beta} + g^{22}g^{\alpha\beta}F_{2\alpha}.F_{2\beta} \\ & +2g^{\alpha\beta}g^{2\lambda}F_{2\alpha}.F_{\lambda\beta} + g^{\alpha\beta}g^{\mu\lambda}F_{\mu\alpha}.F_{\lambda\beta}. \end{aligned} \tag{3.95}$$

As the tensor $\left(F^{ij}\right)$ is antisymmetric, from the above computations, we obtain

$$\begin{aligned} & \tfrac{1}{2}F_{kl}.F^{kl} \\ & = 2g^{12}g^{\alpha\lambda}F_{1\alpha}.F_{2\lambda} + g^{22}g^{\lambda\beta}F_{2\beta}.F_{2\lambda} + F_{12}.F^{12} + F_{34}.F^{34} \\ & +F_{2\lambda}.\left[g^{21}g^{\lambda 2}F_{12} + g^{23}g^{\lambda 2}F_{32} + g^{23}g^{\lambda 4}F_{34} + g^{24}g^{\lambda 2}F_{42} + g^{24}g^{\lambda 3}F_{43}\right]. \end{aligned} \tag{3.96}$$

(3.95) and (3.96) yield

$$g^{\alpha\beta}g^{ki}F_{\alpha k}.F_{\beta i} - \frac{1}{2}F_{kl}.F^{kl}$$
$$= 2g^{\alpha\beta}g^{2\lambda}F_{2\alpha}.F_{\lambda\beta} + g^{\alpha\beta}g^{\mu\lambda}F_{\mu\alpha}.F_{\lambda\beta} - F_{12}.F^{12} - F_{34}.F^{34}$$
$$-F_{2\lambda}.\left[g^{21}g^{\lambda 2}F_{12} + g^{23}g^{\lambda 2}F_{32} + g^{23}g^{\lambda 4}F_{34} + g^{24}g^{\lambda 2}F_{42} + g^{24}g^{\lambda 3}F_{43}\right]. \tag{3.97}$$

The second equality of (3.78) follows straightforwardly from (3.93) and (3.97).

Proof of Proposition 3.8

Proof of item (*ii*)**.** In view of (3.78), the equation

$$g^{\alpha\beta}R_{\alpha\beta} - 2\Gamma^2_{,2} - 2g^{12}g_{12,2}\Gamma^2 = g^{\alpha\beta}\tau_{\alpha\beta},$$

is equivalent to

$$\begin{aligned} K = & -2\left(g^{12}\right)^2 g_{11,22} + 4\left(g^{12}\right)^3 g_{12,2}g_{11,2} \\ & + \left\{4\left(g^{12}\right)^4\left(g_{12,2}\right)^2 + \frac{1}{2}\left(g^{12}\right)^2\left(g^{\alpha\beta}g_{\alpha\beta,2}\right)_{,2}\right\} g_{11} \\ & + \frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right) - 2W - 2g^{12}g_{12,2}S. \end{aligned} \tag{3.98}$$

(3.98) is arranged under the simplified form (3.83) with χ and ψ given by (3.84).

Construction of g_{11} on $G^1_{T_1}$

The proof of the following Proposition 3.9, that provides the construction of g_{11} on $G^1_{T_1}$, is straightforward.

Proposition 3.9. Let $d_0, d_1 \in C^\infty(\Gamma)$. Then (3.83) has a unique solution $g_{11} \in C^\infty\left(G^1_{T_1}\right)$ satisfying $g_{11} = d_0$ and $g_{11,2} = d_1$ on Γ.

The relation $\Gamma^2 = 0$ on $G^1_{T_1}$

The relation $\Gamma^2 = 0$ on $G^1_{T_1}$ is arranged in the following Proposition 3.10.

Proposition 3.10. (i) On $G^1_{T_1}$ the reduced system

$$\widetilde{R}_{\alpha\beta} = \tau_{\alpha\beta}, \tag{3.99}$$

implies the following homogenous ODE on $G^1_{T_1}$ with unknown Γ^2

$$\Gamma^2_{,2} + g^{12} g_{12,2} \Gamma^2 = 0. \tag{3.100}$$

(ii) Assume $\Gamma^2 = 0$ on Γ. Then $\Gamma^2 = 0$ on $G^1_{T_1}$.

Proof of Proposition 3.10

Proof Since $g_{23} = g_{24} = 0$, $\Gamma^1 = \Gamma^3 = \Gamma^4 = 0$ on G^1 at this final step of the construction process, it follows from the definition of $\widetilde{R}_{ij}$ (see (2.17)) that the reduced equation (3.99) reads $R_{\alpha\beta} = \tau_{\alpha\beta}$. Thus, in view of (3.82), equation $g^{\alpha\beta}R_{\alpha\beta} = g^{\alpha\beta}\tau_{\alpha\beta}$ implies (3.100). (3.100) is a linear homogenous first order ODE on $G^1_{T_1}$, with unknown function Γ^2, of the real variable x^2, with C^∞ coefficients depending smoothly on real parameters x^3 and x^4. Thus, assuming $\Gamma^2 = 0$ on Γ implies $\Gamma^2 = 0$ on $G^1_{T_1}$.

Compatibility condition

- We have successfully adapted Rendall method through which, given a positive real number $0 < T \leq T_0$, appropriate free data $\underset{\omega}{h}_{\alpha\beta}$, $\underset{\omega}{A}_{\alpha}$ and $\underset{\omega}{\Phi}$ in $C^\infty(G^\omega_T)$ and some adequate conditions, initial data for the reduced Einstein-Yang-Mills-Higgs system are constructed on G^ω_T, $\omega = 1, 2$ (see $(2.1 - 2.3)$ for the definition of G^ω_T and T_0).
- We have established that the solution of the evolution problem with those initial data satisfies the relations $\Gamma^i = 0$ and $\Delta = 0$ on $G^\omega_{T_1}$ for some $T_1 \in (0, T]$.
- In fact, setting $g_{\alpha\beta} = \Omega h_{\alpha\beta}$ on $G^1_T \cup G^2_T$, where $h_{\alpha\beta} = \underset{\omega}{h}_{\alpha\beta}$ on G^ω_T, $\omega = 1, 2$, $\left(\underset{\omega}{h}_{\alpha\beta}\right)_{\alpha,\beta=3,4}$ a symmetric positive definite matrix function with determinant 1 at each point of G^ω_T, $\omega = 1, 2$, and Ω an unknown positive function, we have constructed C^∞ initial data as follows :

Compatibility condition

- (i) Construction of $g_{\alpha\beta}$, g_{12} on $G^1_{T_1}$ such that $\Gamma^1 = 0$ and $g_{22,1} = 2g_{12,2}$ on $G^1_{T_1}$ under the following conditions : $g_{22} = g_{23} = g_{24} = 0$ on G^1_T, $A_2 = 0$ on G^1_T ; g_{12}, Ω and $\Omega_{,2}$ are given C^∞ functions on Γ, such that $\Gamma^1 = 0$ and $g_{22,1} = 2g_{12,2}$ on Γ.
- (ii) Construction of $g_{\alpha\beta}$, g_{12} on $G^2_{T_1}$ such that $\Gamma^2 = 0$ and $g_{11,2} = 2g_{12,1}$ on $G^2_{T_1}$ under the following supplementary conditions (in addition to conditions in (i)) : $g_{11} = g_{13} = g_{14} = 0$ on G^2_T, $A_1 = 0$ on G^2_T ; $\Omega_{,1}$ is a given C^∞ function on Γ, such that $\Gamma^2 = 0$ and $g_{11,2} = 2g_{12,1}$ on Γ.
- (iii) Construction of $g_{1\alpha}$, A_1 on $G^1_{T_1}$ such that $\Gamma^\alpha = 0$ and $\Delta = 0$ on $G^1_{T_1}$ under the following supplementary condition (in addition to conditions in (i)) : $g_{1\alpha,2}$ and $A_{1,2}$ are given C^∞ functions on Γ, such that $\Gamma^\alpha = 0$ and $\Delta = 0$ on Γ.

Compatibility condition

- (iv) Construction of $g_{2\alpha}$, A_2 on $G^2_{T_1}$ such that $\Gamma^\alpha = 0$ and $\Delta = 0$ on $G^2_{T_1}$ under the following supplementary condition (in addition to conditions in (ii)) : $g_{2\alpha,1}$ and $A_{2,1}$ are given C^∞ functions on Γ, such that $\Gamma^\alpha = 0$ and $\Delta = 0$ on Γ.
- $(5i)$ Construction of g_{11} on $G^1_{T_1}$ such that $\Gamma^2 = 0$ on $G^1_{T_1}$ under the following supplementary condition (in addition to conditions in (i) and (iii)) : $g_{11,2}$ is a given C^∞ function on Γ, such that $\Gamma^2 = 0$ on Γ.
- $(6i)$ Construction of g_{22} on $G^2_{T_1}$ such that $\Gamma^1 = 0$ on $G^2_{T_1}$ under the following supplementary condition (in addition to conditions in (ii) and (iv)) : $g_{22,1}$ is a given C^∞ function on Γ, such that $\Gamma^1 = 0$ on Γ.

Compatibility condition

We now show how the above adequate conditions in (i), (ii), (iii), (iv), $(5i)$ and $(6i)$ are arranged.

- Firstly, take $g_{12} = -1$ on Γ (this is a non-restrictive property that can naturally be imposed to any metric in standard coordinates, c.f. Rendall 1990, p. 232). Then choose $\left(\underset{\omega\alpha\beta}{h}\right)_{\alpha,\beta=3,4}$, a C^∞ symmetric positive definite matrix function on G^ω_T with determinant 1 at each point and set $g_{\alpha\beta} = \Omega h_{\alpha\beta}$, where $h_{\alpha\beta} = \underset{\omega\alpha\beta}{h}$ on G^ω_T, $\omega = 1, 2$. Let $\Omega = v_0$ on Γ, where v_0 is a given C^∞ function on Γ. Take also

$$\begin{aligned} g_{22} = g_{23} = g_{24} = 0, &\quad A_2 = 0 \text{ on } G^1_T, \\ g_{11} = g_{13} = g_{14} = 0, &\quad A_1 = 0 \text{ on } G^2_T. \end{aligned}$$

  Then all the components g_{ij} of the metric are determined on Γ, since $g_{\alpha\beta} = \Omega h_{\alpha\beta}$, $g_{11} = g_{1\alpha} = 0$, $g_{22} = g_{2\alpha} = 0$ on Γ, Ω and $h_{\alpha\beta}$ are known on Γ.

Compatibility condition

- Next choose $\underset{\omega}{\Phi}$, $\underset{\omega\,3}{A}$ and $\underset{\omega\,4}{A}$, which are given C^∞ functions on G^ω_T such that

$$\underset{1}{\Phi} = \underset{2}{\Phi} \text{ on } \Gamma, \quad \underset{1\,3}{A} = \underset{2\,3}{A} \text{ on } \Gamma, \quad \underset{1\,4}{A} = \underset{2\,4}{A} \text{ on } \Gamma.$$

Let $\Omega_{,1} = v_1$ and $\Omega_{,2} = v_2$ on Γ, where v_1 and v_2 are two given C^∞ functions on Γ. Then Eqs. (3.15) and (3.15a) on G^1_T as well as their following counterparts on G^2_T

$$g_{12,1} = \frac{1}{2} g_{12} \frac{\Omega_{,1}}{\Omega} \text{ on } G^2_T,$$

and

$$\frac{1}{4} g^{\alpha\beta}_{,1} g_{\alpha\beta,1} - \frac{1}{2}\left(g^{\alpha\beta} g_{\alpha\beta,1}\right)_{,1} = \tau_{11} \text{ on } G^2_T,$$

are satisfied and it holds that $g_{11,2} = g_{22,1} = \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta,1}$ on Γ. This insures $\Gamma^1 = \Gamma^2 = 0$ on $\Gamma \equiv G^1_T \cap G^2_T$.

Compatibility condition

- Finally, let $g_{13,2} = \underset{23}{\widetilde{b}}$, $g_{14,2} = \underset{24}{\widetilde{b}}$, $A_{1,2} = \underset{21}{\widetilde{A}}$ on Γ, where $\underset{23}{\widetilde{b}}$, $\underset{24}{\widetilde{b}}$, $\underset{21}{\widetilde{A}}$ are given C^∞ functions on Γ. Then there is only one way to choose $g_{2\beta,1}$ on Γ such that $\Gamma^3 = \Gamma^4 = 0$ on Γ. In fact, by the definition of Γ^β (see (2.13)), on G^1_T it holds that

$$\begin{aligned} g^{12} g_{2\alpha,1} = {} & g_{\alpha\beta}\Gamma^\beta - g^{12} g_{1\alpha,2} - g_{\alpha\beta} g^{\beta\lambda}_{,2} g^{12} g_{1\lambda} \\ & - \tfrac{1}{2}\left[-2g^{12} g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right], \end{aligned} \tag{4.1}$$

and on G^2_T it holds that

$$\begin{aligned} g^{12} g_{1\alpha,2} = {} & g_{\alpha\beta}\Gamma^\beta - g^{12} g_{2\alpha,1} - g_{\alpha\beta} g^{\beta\lambda}_{,1} g^{12} g_{2\lambda} \\ & - \tfrac{1}{2}\left[-2g^{12} g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right]. \end{aligned} \tag{4.2}$$

Compatibility condition

Since $g_{1\lambda} = 0 = g_{2\lambda}$ on Γ, it follows from (4.1) and (4.2) that

$$\begin{aligned} g^{12} g_{2\alpha,1} &= g_{\alpha\beta}\Gamma^{\beta} - g^{12} g_{1\alpha,2} \\ &\quad - \tfrac{1}{2}\left[-2g^{12} g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right] \text{ on } \Gamma. \end{aligned} \tag{4.3}$$

In view of (4.3), on Γ, $\Gamma^{\beta} = 0$ is equivalent to

$$g^{12} g_{2\alpha,1} = -g^{12} g_{1\alpha,2} - \frac{1}{2}\left[-2g^{12} g_{12,\alpha} + g^{\mu\theta}\left(2g_{\alpha\mu,\theta} - g_{\mu\theta,\alpha}\right)\right].$$

Compatibility condition

We now proceed to arrange the condition $\Delta = 0$ on Γ. Since $A_2 = 0$ on G^1_T, $A_1 = 0$ on G^2_T, $\underset{\omega\,\alpha}{A}$ are given C^∞ functions on G^ω_T, there is only one way to choose $A_{2,1}$ on Γ such that $\Delta = 0$ on Γ. In fact, from the definitions of Δ and Γ^k (see (2.15) and (2.13)), on G^1_T it holds that

$$\begin{aligned} & g^{12} A_{2,1} \\ & = \Delta - g^{12} A_{1,2} + \left(2 g^{12} \Gamma^1_{12} + g^{\alpha\beta} \Gamma^1_{\alpha\beta} \right) A_1 + g^{12} g^{\alpha\lambda} A_\alpha \left(g_{1\lambda,2} + g_{2\lambda,1} \right) \\ & - \left[2 g^{12} \Gamma^\beta_{2\alpha} A_\beta g^{\alpha\lambda} + 2 \left(g^{12} \right)^2 g_{12,2} g^{\alpha\lambda} A_\alpha - g^{12} g^{\alpha\lambda} A_{\alpha,2} \right] g_{1\lambda} \\ & - g^{12} g^{\alpha\lambda} A_\alpha g_{12,\lambda} - g^{\alpha\beta} \left(A_{\beta,\alpha} - \Gamma^\lambda_{\alpha\beta} A_\lambda \right). \end{aligned} \tag{4.4}$$

Since $g_{1\lambda} = 0$ and $A_1 = 0$ on Γ, (4.4) implies that on Γ the following equality holds

$$g^{12}A_{2,1} = \Delta - g^{12}A_{1,2} + g^{12}g^{\alpha\lambda}A_\alpha\left(g_{1\lambda,2} + g_{2\lambda,1}\right) \\ - g^{12}g^{\alpha\lambda}A_\alpha g_{12,\lambda} - g^{\alpha\beta}\left(A_{\beta,\alpha} - \Gamma^\lambda_{\alpha\beta}A_\lambda\right). \tag{4.5}$$

In view of (4.5), on Γ, $\Delta = 0$ is equivalent to

$$g^{12}A_{2,1} = -g^{12}A_{1,2} + g^{12}g^{\alpha\lambda}A_\alpha\left(g_{1\lambda,2} + g_{2\lambda,1}\right) \\ - g^{12}g^{\alpha\lambda}A_\alpha g_{12,\lambda} - g^{\alpha\beta}\left(A_{\beta,\alpha} - \Gamma^\lambda_{\alpha\beta}A_\lambda\right).$$

It follows from the above discussion that all necessary data are given on Γ and all necessary assumptions fulfilled.

Conclusion

We can now sum up the C^∞ resolution of the Goursat problem for the EYMH system in the following theorem.

Theorem

Let $T \in (0, T_0]$ be a real number and $\omega \in \{1, 2\}$. Let $\underset{\omega}{h}_{33}, \underset{\omega}{h}_{34}, \underset{\omega}{h}_{44}$ be C^∞ scalar functions on G^ω_T such that $\left(\underset{\omega}{h}_{\alpha\beta}\right)$ is a symmetric positive definite matrix with determinant 1 at each point of G^ω_T and $\left(\underset{1}{h}_{33}, \underset{1}{h}_{34}, \underset{1}{h}_{44}\right) = \left(\underset{2}{h}_{33}, \underset{2}{h}_{34}, \underset{2}{h}_{44}\right)$ on Γ. Let $\underset{\omega}{\widetilde{\Phi}}, \underset{\omega}{\widetilde{A}}_3, \underset{\omega}{\widetilde{A}}_4$ be C^∞ functions on G^ω_T such that $\left(\underset{1}{\widetilde{\Phi}}, \underset{1}{\widetilde{A}}_3, \underset{1}{\widetilde{A}}_4\right) = \left(\underset{2}{\widetilde{\Phi}}, \underset{2}{\widetilde{A}}_3, \underset{2}{\widetilde{A}}_4\right)$ on Γ. Let C^∞ functions $\widetilde{\Omega}, \widetilde{\Omega}_1, \widetilde{\Omega}_2, \underset{2}{\widetilde{b}}_3, \underset{2}{\widetilde{b}}_4, \underset{2}{\widetilde{A}}_1$ be given on Γ. Then there exists $T_1 \in (0, T]$, a unique C^∞ scalar function Ω on $G^1_{T_1} \cup G^2_{T_1}$, a unique C^∞ Lorentz metric g_{ij} on L_{T_1}, a unique C^∞ Yang-Mills potential A_k on L_{T_1} and a unique C^∞ Higgs function Φ on L_{T_1} such that :

Theorem

(1) $g_{\alpha\beta} = \Omega h_{\alpha\beta}$ *on* $G^1_{T_1} \cup G^2_{T_1}$, *where* $h_{\alpha\beta} = \underset{\omega\alpha\beta}{h}$ *on* G^ω_T.
(2) $u = (g_{ij}, A_k, \Phi)$ *satisfies the Einstein-Yang-Mills-Higgs equations on* L_{T_1},
(3) *the given coordinates on* $\mathbb{R}^4$ *are standard coordinates for* g_{ij} *and the Lorentz gauge condition* $\nabla_k A^k = 0$ *is satisfied on* L_{T_1}, *with* $A_2 = 0$ *on* $G^1_{T_1}$ *and* $A_1 = 0$ *on* $G^2_{T_1}$,
(4) $u = (g_{ij}, A_k, \Phi)$ *induce the given data on* $G^1_{T_1} \cup G^2_{T_1}$,
(5) *on* Γ *it holds that* : $\Omega = \widetilde{\Omega}$, $\Omega_{,1} = \widetilde{\Omega}_1$, $\Omega_{,2} = \widetilde{\Omega}_2$, $g_{13,2} = \underset{2\,3}{\widetilde{b}}$, $g_{14,2} = \underset{2\,4}{\widetilde{b}}$, *and* $A_{1,2} = \underset{2\,1}{\widetilde{A}}$.

Future challenges

- The handling of the local Goursat problem for the EYMH system is quite satisfactory.
- The global resolution of the evolution problem associated to the EYMH system with the same gauge conditions and the same initial hypersurfaces is still open. Hope that it works for small initial data (By using Lindblad-Rodnianski or Caciotta-Nicolo methods).
- Great opportunities for numerical analysis : The constraints equations are much more simpler (they reduce to ODEs) than the elliptic PDEs that arise in the classical Cauchy problem in GR.
- Solve the constraints and the evolution problems for other physically interesting models such as Einstein-Vlasov, Einstein-Euler, etc.

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