Invariants of 3-manifolds via generators and relations of the 1-2-3 bordism 2-category

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Workshop on Geometric Analysis, AIMS, 3 Dec 2012

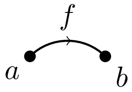
Outline

1. 2-categories
2. Three-dimensional topological quantum field theories
3. Presentation theorem
4. Modular tensor categories
5. Classification theorem
6. Sample calculations

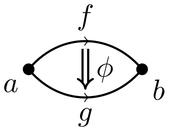
1. 2-categories

A **2-category** consists of *objects*, drawn as

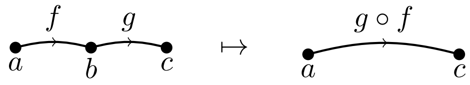
together with *1-morphisms*, drawn as



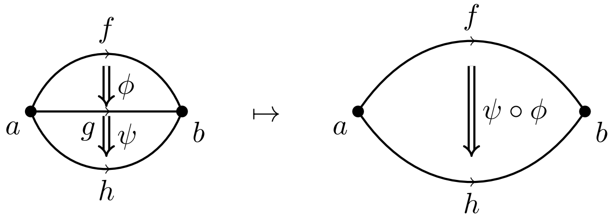
as well as *2-morphisms*, drawn as



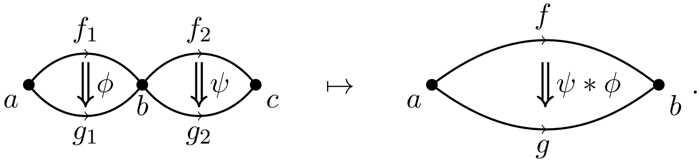
The 1-morphisms can be composed:



The 2-morphisms can be composed both *vertically*,



and *horizontally*,



These composition operations must satisfy various coherence equations.

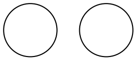
Examples of 2-categories:

1. **Top**: objects are topologial spaces, 1-morphisms are continuous maps, 2-morphisms are homotopies.
2. **Cat**: objects are categories, 1-morphisms are functors, 2-morphisms are natural transformations.
3. **Alg**: objects are algebras, 1-morphisms are bimodules, 2-morphisms are bimodule homomorphisms.
4. **LinCat**: objects are $\mathbb{C}$-linear categories, 1-morphisms are profunctors, 2-morphisms are natural transformations.
5. ... (any structure that has morphisms *between* the morphisms!)

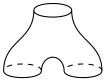
2. Three-dimensional TQFTs

Definition. The oriented 1-2-3 bordism 2-category $\mathbf{Bord}^{\mathrm{or}}_{\mathbf{123}}$ is defined as follows:

- An object is a closed oriented 1-manifold A, for example:

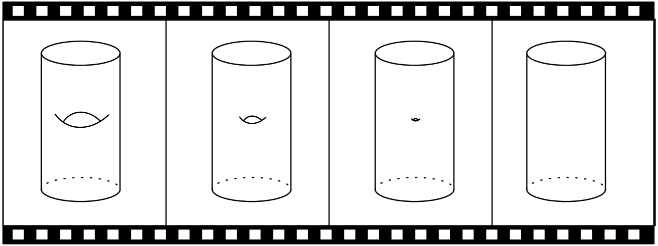


- A 1-morphism $A \xrightarrow{\Sigma} B$ is a *bordism* from A to B (a compact oriented 2-manifold with $\partial\Sigma \cong \overline{A} \coprod B$), for example:



$$\begin{array}{c} B \\ \uparrow \Sigma \\ A \end{array}$$

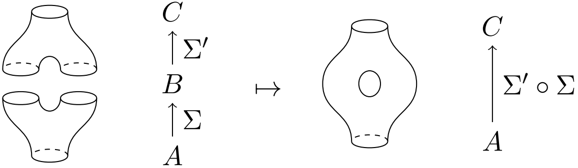
- A 2-morphism $\Sigma \underset{A}{\overset{B}{\overset{M}{\Longrightarrow}}} \Sigma'$ is a 3-dimensional bordism from Σ to Σ', for example:



$\Sigma = f^{-1}(0)$ $f^{-1}(0.3)$ $f^{-1}(0.6)$ $\Sigma' = f^{-1}(1)$

We can visualize M as a 'movie' with the help of a Morse function $f\colon M \to [0,1]$ so that the fibers $f^{-1}(t)$ interpolate from Σ to Σ'.

Composition of 1- and 2-morphisms is given by *gluing* the bordisms together, for example:



There is another composition operation — *disjoint union* of bordisms. This makes $\mathbf{Bord}^{\text{or}}_{\mathbf{123}}$ into a 'symmetric monoidal 2-category'.

Definition. An oriented 123 topological quantum field theory is a symmetric monoidal 2-functor

$$Z\colon \mathbf{Bord}^{or}_{\mathbf{123}} \to \mathbf{LinCat}.$$

3. Presentation theorem

Theorem. [BB, CD, CS-P, JV] The symmetric monoidal 2-category $\mathbf{Bord}^{\mathrm{or}}_{\mathbf{123}}$ has the following presentation:

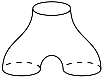
- Generating objects:

 (a circle)

- Generating 1-morphisms:


cap


cup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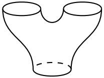
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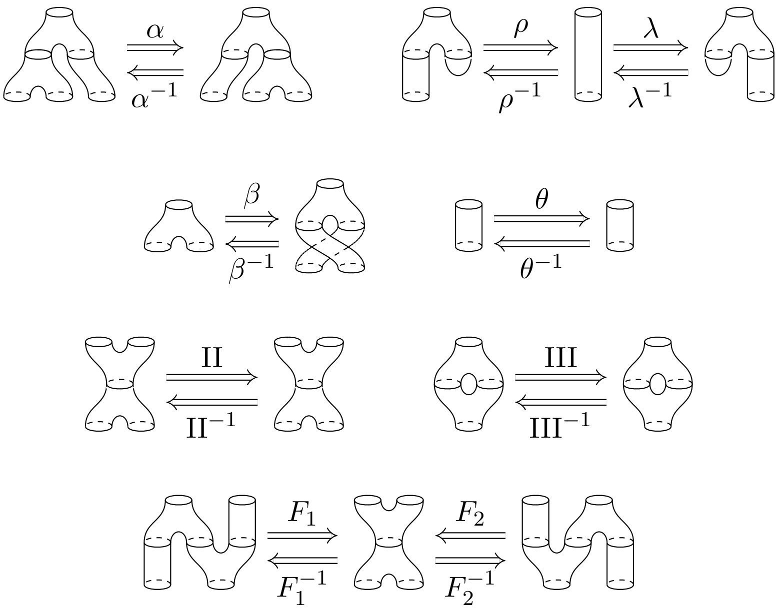
- Generating 2-morphisms:

...P.T.O...

Invertible 2-morphisms:



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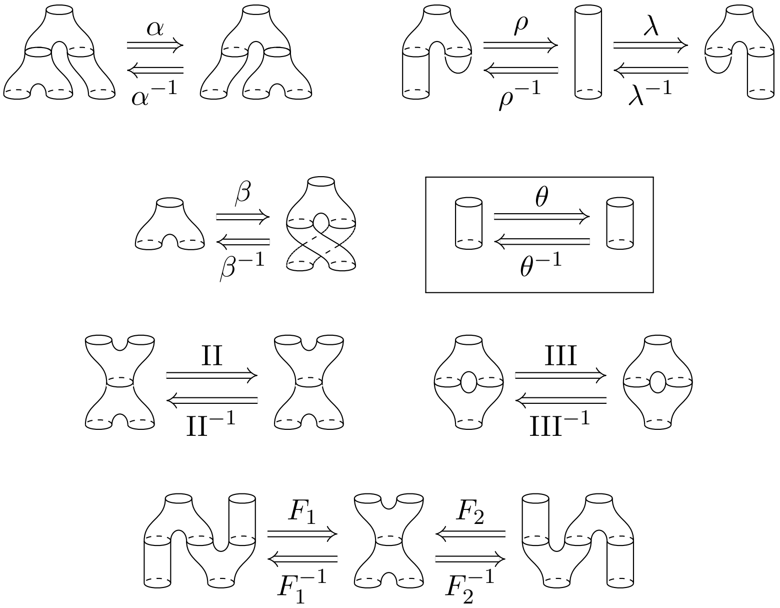

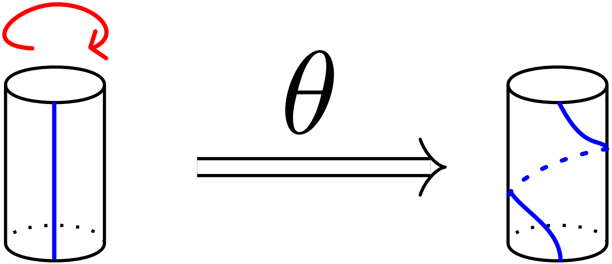
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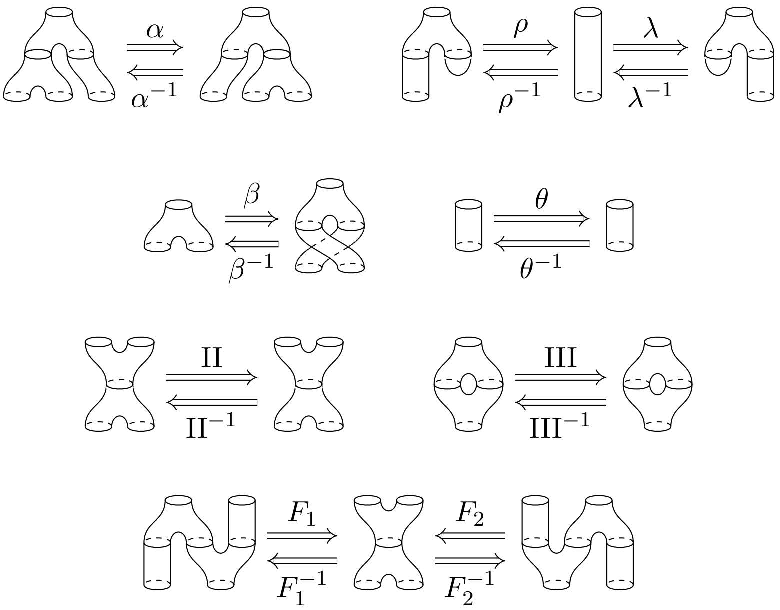

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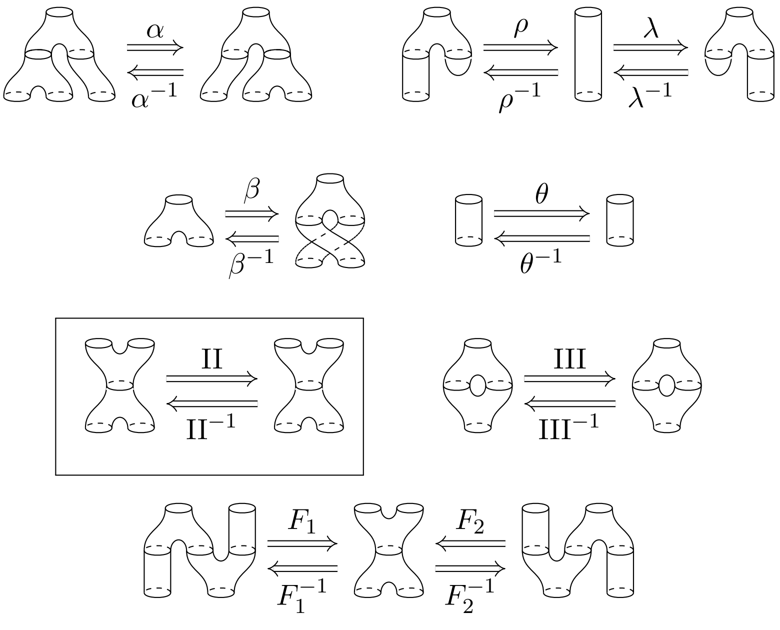
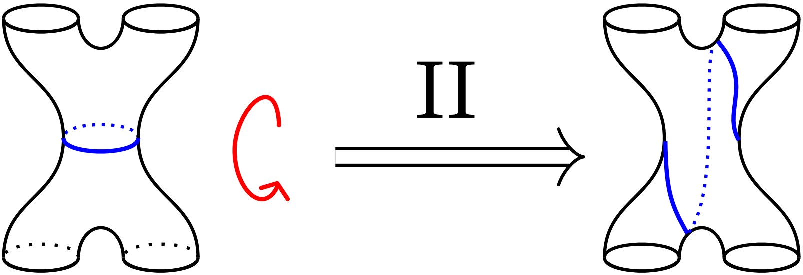
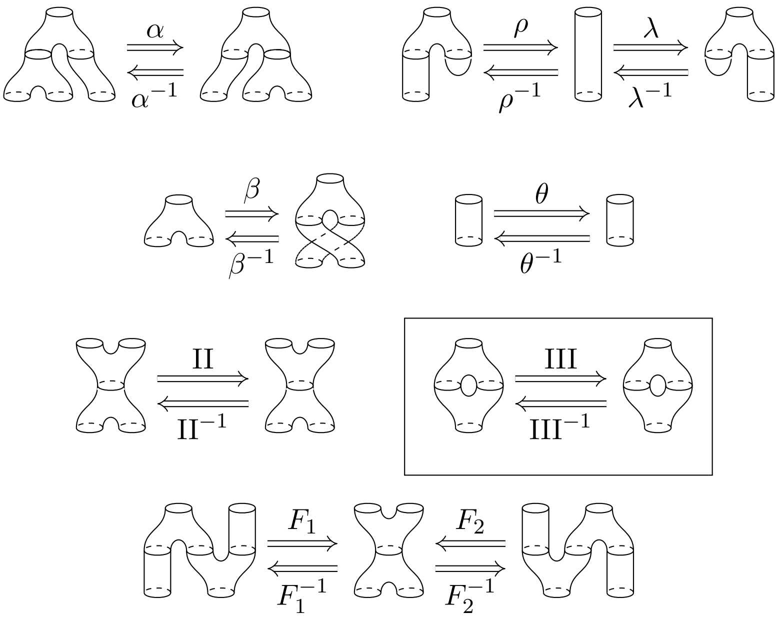
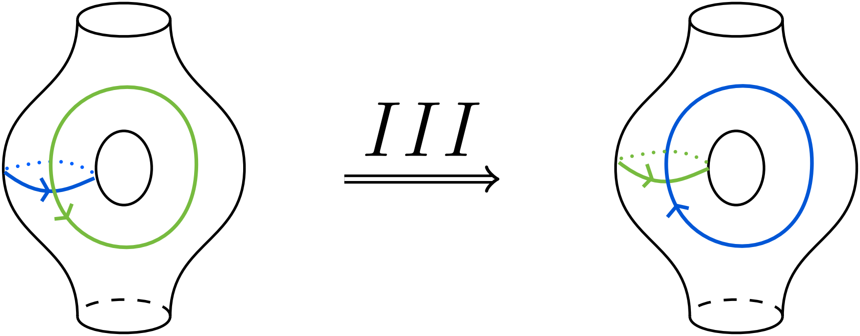


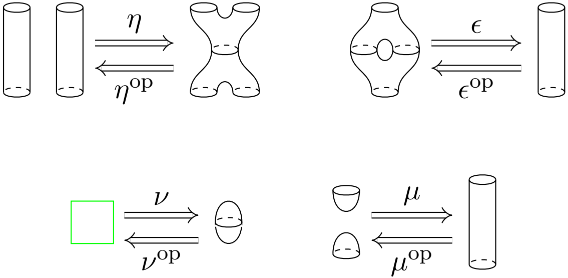

II

Invertible 2-morphisms:

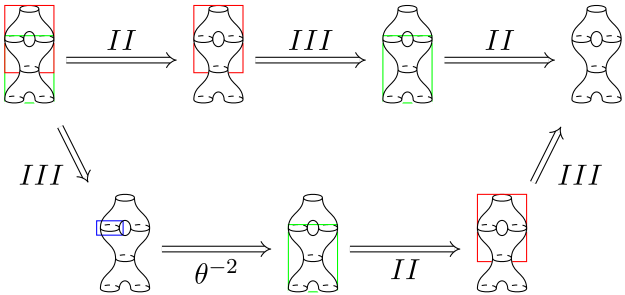



III

Noninvertible 2-morphisms:

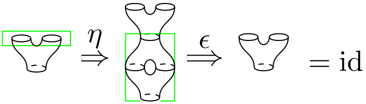


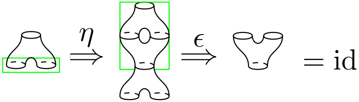
- Relations between the invertible generators, eg.



Compare with $\underbrace{SL_2(\mathbb{Z})}_{\Gamma(\text{torus})} = \langle s, t \mid (st)^3 = s^2,\ s^4 = 1 \rangle$.

- Relations between the non-invertible generators, eg.





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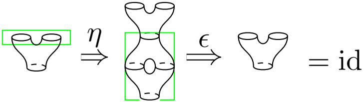

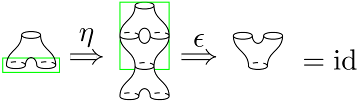

Proof. Uses Cerf theory methods from Chris Schommer-Pries's thesis. ∎

4. Modular tensor categories

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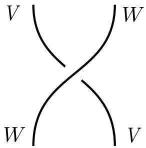
'Ribbon' means that there is a tensor product and nontrivial braiding isomorphisms $V \otimes W \to W \otimes V$, so the objects behave like braided ribbons when drawn in string diagrams:

$$V \otimes W \xrightarrow{\sigma} W \otimes V$$

drawn as



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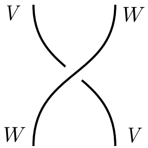
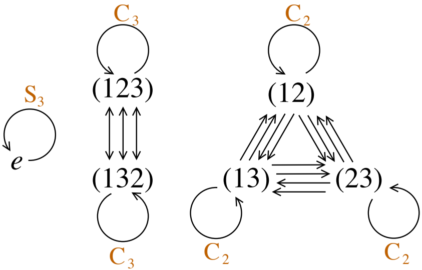


Examples: Representations of loop groups, representations of quantum groups.

Toy example: let G be a finite group. Let ΛG be the *loop groupoid* of G:

- An *object* of ΛG is an element $g \in G$.
- For every $h \in G$, there is an *arrow* $g \xrightarrow{h} hgh^{-1}$.

For example, here is a picture of ΛS_3:



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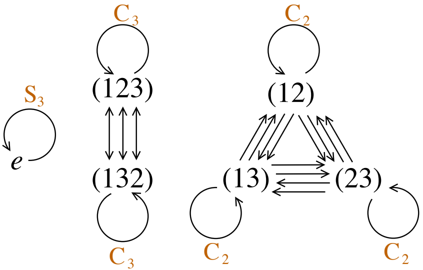


Rep (ΛG) forms a modular category.

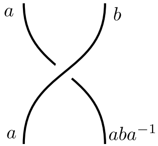
The tensor product on $\operatorname{Rep}(\Lambda G)$ is

$$(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b$$

while the braiding $\sigma : V \otimes W \to W \otimes V$ is given on homogenous elements by

$$v_a \otimes w_b \mapsto w_b \otimes V(a \xrightarrow{b^{-1}})(v_a).$$

which is drawn as



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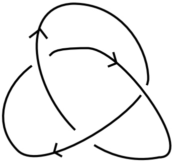
For instance, given a representation V supported on a conjugacy class $\mathcal{A}$ in G, the trefoil

$K =$ 

computes as

$$Z(K) = \sum_{\substack{a,b\in\mathcal{A}\\ aba=bab}} [V^*(b^{-1}\sum_{ijk} \overset{a^{-1}}{\to})]_{kj}[V(a \overset{b}{\to})]_{ki}[V^*(a^{-1} \overset{ba^{-1}b^{-1}}{\to})]_{ji}$$

For instance, for $G = S_3$, and V the sign representation supported on the class of (12), this gives 6.

5. Classification theorem

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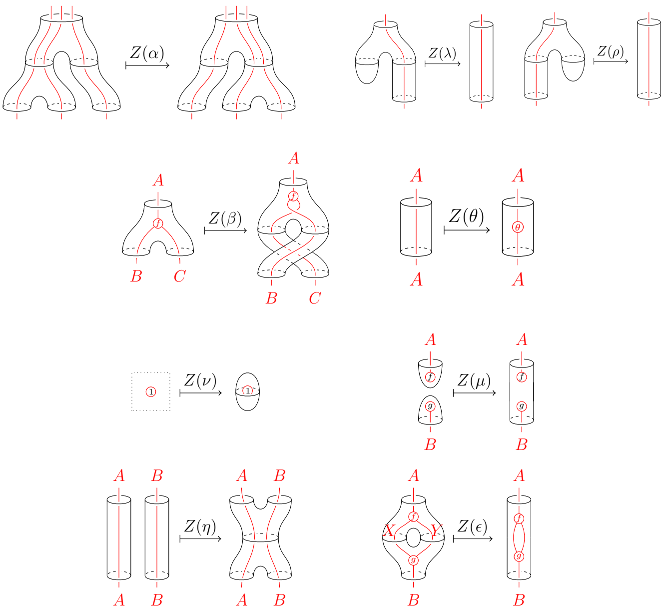
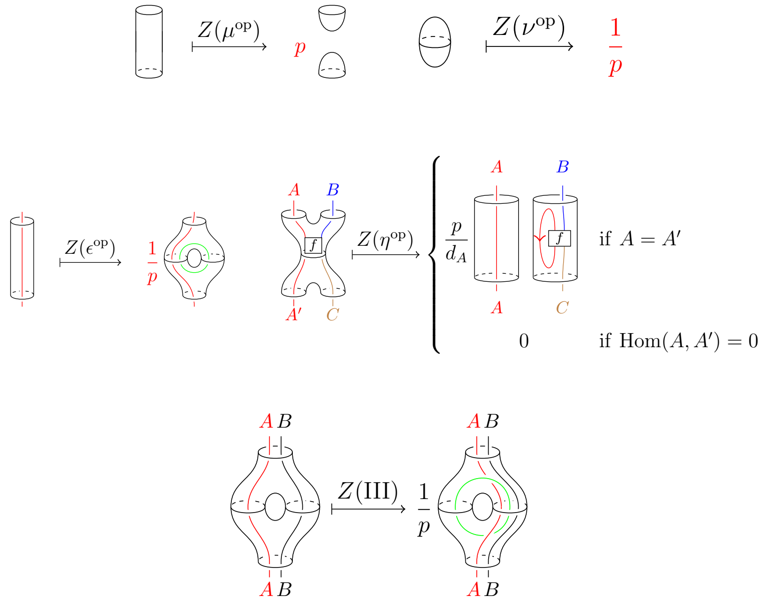
Many ideas here in fact date back to the 'early days' of TQFT in the 1990's. But a key aspect of our approach is a new 'internal string diagrams' calculus to check the relations.

Lemma. In any oriented 123 TQFT $Z : \mathbf{Bord}_{123}^{\mathrm{or}} \to \mathbf{LinCat}$, the generators must act as follows:





Here, $p = \sum_i \theta_i d_i^2$. (In the finite group model, $p = |G|$).

6. Sample calculations

Let's use the internal string diagram calculus to compute some invariants of 3-manifolds!

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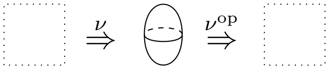
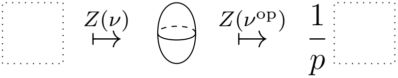
Let's start with S^3. One can build it by creating and then killing a 2-sphere:



Using the internal string diagram calculus, we compute:



So, $Z(S^3) = \frac{1}{p}$!

Now suppose M_K is a 3-manifold obtained by surgery on a framed knot $K \subset S^3$. For instance, K might be the trefoil knot.

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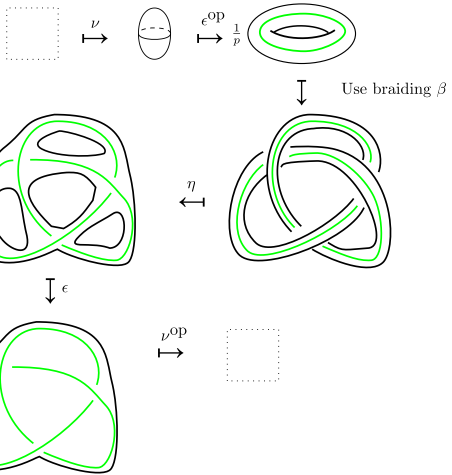
We can build up M_K using the generators as follows:



So, $Z(M_K) = \frac{1}{p^2} \sum_i d_i Z(K, V_i)$. This shows that our approach is equivalent to the traditional surgery approach.

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We have just seen that if G is a finite group, and K is the trefoil knot, then

$$Z(M_K) = \frac{1}{|G|^2} \sum_{\text{reps } V_i \text{ of } \Lambda G} \dim(V_i)\, Z(K, V_i).$$

Recall that earlier we computed $Z(K, V_i)$ as a certain sum over the matrix elements of the representation V_i.

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There is a 'dual' way to determine $Z(M_K)$ in terms of the homotopy group of M_K:

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The fact that these two numbers are equal is rather interesting! For TQFT's based on loop groups and quantum groups, these kinds of equations can be very striking.

Summary

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- Using a new calculus of *internal string diagrams*, we are able to demonstrate the equivalence between oriented 123 TQFTs and anomaly-free modular tensor categories in a clean and efficient way.
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- Using a new calculus of *internal string diagrams*, we are able to demonstrate the equivalence between oriented 123 TQFTs and anomaly-free modular tensor categories in a clean and efficient way.
- In particular, we show that the category assigned to the circle is actually *rigid*, resolving a long-standing issue. This uses the 2-categorical structure (the noninvertible adjunctions) in an important way.
- Once a 3-manifold is presented in terms of the generators, computing its invariant is straightforward using internal string diagrams.