A crash course in Riemannian geometry

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Basic definitions

Let M be an n-dimensional smooth manifold.

Definition

A Riemannian metric g on M is a smoothly varying, rank two, positive definite, symmetric, covariant tensor field on M.

This means that, as a function, g evaluated at $p \in M$ takes in two vectors $X, Y \in T_pM$ and produces a number. The further properties mean the following:

- That g is a tensor means $g(f_1X_1 + f_2X_2, Y) = f_1g(X_1, Y) + f_2g(X_2, Y)$ for all functions f_1, f_2 and vectorfields X_1, X_2, Y, and that a similar linearity holds over the second variable as well.
- That g is symmetric means $g(X, Y) = g(Y, X)$ for all vectors $X, Y \in T_pM$.
- That g is positive definite means $g(X, X) > 0$ for all nonzero vectors $X \in T_pM$.

Recall that, near any $p \in M$, we can find smooth coordinates for a neighborhood $U \subset M$, by finding a diffeomorphism $U \mapsto \mathbf{B} \subset \mathbf{R}^n$, where $\mathbf{B}$ is the unit ball, which sends p to 0. In these coordinates, we can identify g on U with a smoothly-varying, $n \times n$, positive definite, symmetric matrix. Because g is like a usual inner product from linear algebra, we'll often write $g(X, Y) = \langle X, Y \rangle$ and $g(X, X) = \|X\|^2$.

Having chosen coordinates $\{x^1, \ldots, x^n\}$ we write the entries in the matrix associated to g as g_{ij}. We also write the inverse of this matrix as g^{ij}, and use g to raise and lower indices of tensors. For instance, if f is a smooth function, it has a natural differential

$$df = \partial_1 f dx^1 + \partial_2 f dx^2 + \cdots + \partial_n f dx^n = \sum_{i=1}^{n} \partial_i f dx^i.$$

With the Riemannian metric g, we can "raise the index" on df to get a vector field, which we call the gradient of f; it has the form

$$\text{grad}(f) = g^{ij} \partial_j f \partial_i.$$

(Notice the implicit sum over i and j.)

It is also possible to write down the divergence of a vector field and the Laplacian of a function:

$$\mathrm{div}(X) = \sum_{i,j} \frac{1}{\sqrt{\det g}} \partial_i \sqrt{\det g} X^j$$

and

$$\Delta u = \mathrm{div}(\mathrm{grad}(u)) = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j u \right).$$

We can extend the notion of a Riemannian metric to that of a **pseudo-Riemannian** metric, which drops the condition that g is positive definite. In particular, we say g is a **Lorentzian metric** if, for each $p \in M$ the metric g has 1 negative eigenvalue and $n - 1$ positive eigenvalues. In this case, we say a vector $X \in T_pM$ is time-like if $\langle X, X \rangle < 0$ and space-like if $\langle X, X \rangle > 0$. It turns out that most of what we do below holds in the Lorentz case as well, with a few additional minus signs.

The Riemannian metric g allows us to measure the lengths of parameterized curves $\gamma : (a, b) \to M$, by the formula

$$\text{length}(\gamma) = \int_a^b \sqrt{\langle \gamma', \gamma' \rangle} dt.$$

This in turn gives us a distance function on M (by minimizing lengths of curves), and also a way to measure volumes, areas, etc.

To further analyze the metric (and other) properties the Riemannian metric g gives the underlying manifold M, we will develope some tools. First we need to be able to differentiate vectorfields, and tensor fields. We do this using the **Levi-Civita connection** ∇, defined by

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k, \qquad \Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{1}$$

Any connection, and in particular the Levi-Civita connection, will satisfy the following rules:

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = f\nabla_X Y + (X(f))Y.$$

Thus we see that, if $X = X^i\partial_i$ and $Y = Y^j\partial_j$ then

$$\nabla_X Y = [X^i Y^j \Gamma^k_{ij} + X(Y^k)]\partial_k.$$

There is a very natural way to extend ∇ to all tensor fields, and in particular $\nabla_X f = X(f)$ is the usual directional derivative for all smooth functions f. For instance, if T is a rank two covariant tensor, then ∇T is the rank three covariant tensor given by

$$\nabla T(X,Y,Z) = X(T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z). \quad (2)$$

The Levi-Civita connection is the unique compatible, torsion-free connection for g. This means

$$\nabla g = 0 \Leftrightarrow \nabla_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle Y, \nabla_X Z\rangle$$

and

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

The analouge of straight lines are the geodesic curves. These are curves which have a parameterization $\gamma : (a, b) \to M$ which satisfy

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \Leftrightarrow \ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0. \tag{3}$$

It follows from ODE theory that, given any basepoint $p \in M$ and initial tangent vector $V \in T_pM$, there is a unique geodesic γ which satisfies

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = V.$$

A standard computation shows that geodesics are critical curves for the length functional, and then a bit more work shows that geodesics are in fact locally length minimizing. It is also often useful to choose coordinates based on geodesics. That is, we choose a base point p, and then an orthonormal (with respect to g) basis $\{e_1, \ldots, e_n\}$ for T_pM, and then let x^i be the geodesic with initial point p, initial veolocity e_i and length $|x^i|$. These are called **normal coordinates** centered at p, and in these coordinates we have

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2) \Rightarrow \Gamma_{ij}^k(p) = 0. \tag{4}$$

Effectively, this says that the any Riemannian metric looks like the usual Euclidean metric, to first order, in normal coordinates.

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Intrinsic curvature tensors

Next we define curvature operator

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[x,y]} Z \qquad (5)$$

and the Riemann curvature tensor

$$\mathrm{Rm}(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle. \qquad (6)$$

It serves to be careful of the signs here, because some authors will define Rm with the opposite sign convention.

We also define the sectional curvature K of a two-plane spanned by X and Y:

$$K(X, Y) = \frac{\mathrm{Rm}(X, Y, Y, X)}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}. \tag{7}$$

One can check that this definition of K only depends on the two-dimensional plane spanned by X and Y.

In local coordinates,

$$\mathrm{Rm}_{ijkl} = [\partial_j \Gamma_{ik}^m - \partial_i \Gamma_{jk}^m + \Gamma_{ik}^p \Gamma_{jp}^m - \Gamma_{jk}^p \Gamma_{ip}^m] g_{lm}. \tag{8}$$

We have the algebraic Bianchi identities:

$$\mathrm{Rm}_{ijkl} = \mathrm{Rm}_{klij}, \quad \mathrm{Rm}_{ijkl} = -\mathrm{Rm}_{jikl}, \tag{9}$$

and

$$\mathrm{Rm}_{ijkl} + \mathrm{Rm}_{jkil} + \mathrm{Rm}_{kijl} = 0. \tag{10}$$

We also have the differenital Bianchi identity:

$$\nabla_i \mathrm{Rm}_{jklm} + \nabla_j \mathrm{Rm}_{kilm} + \nabla_k \mathrm{Rm}_{ijlm} = 0. \tag{11}$$

A Riemannian metric g is locally isometric to the Euclidean metric if and only if Rm vanishes identically. Also, the tensor Rm is uniquely determined by all its sectional curvatures K.

In many cases, Rm is too unwieldy to work with. So we form various traces. The first is the Ricci tensor:

$$\mathrm{Rc}_{ij} = g^{kl}\,\mathrm{Rm}_{ikjl}, \tag{12}$$

which is another rank two, covariant, symmetric tensor field. The second is the scalar curvature

$$S = \mathrm{tr}(\mathrm{Rc}) = g^{ij}\,\mathrm{Rc}_{ij}\,. \tag{13}$$

In local coordinates these have the expressions

$$\mathrm{Rc}_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^p_{ij}\Gamma^k_{kp} - \Gamma^p_{kj}\Gamma^k_{ip} \tag{14}$$

and

$$S = g^{ij}[\partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^p_{ij}\Gamma^k_{kp} - \Gamma^p_{kj}\Gamma^k_{ip}]. \tag{15}$$

If we take the trace of the differential Bianchi identity (11) over j and l, we get

$$2(\operatorname{div}\mathrm{Rc})_k = \nabla_k S \Leftrightarrow \operatorname{div}\mathrm{Rc} = \frac{1}{2}\nabla S, \tag{16}$$

which is the contracted Bianchi identity.
Now suppose $\mathrm{Rc} = f(x)g$ for some function f, which implies $S = nf$ and $\nabla S = n\nabla f$. However, $\operatorname{div}\mathrm{Rc} = \nabla f$, and so the contracted Bianchi identity now reads

$$\nabla f = \operatorname{div}\mathrm{Rc} = \frac{1}{2}\nabla S = \frac{n}{2}\nabla f \Rightarrow (n-2)\nabla f = 0.$$

We immediately conclude the following theorem.

Theorem

Let $n > 2$ and suppose $\mathrm{Rc} = f(x)g$ for some smooth function f. Then f must be constant on connected components of M.

Geometric interpretation of curvature

To see the geometric effect of curvature, we begin by recalling that geodesics are locally length-minimizing curves. Thus, if $\gamma(t)$ is a geodesic and X is a vector field along γ which vanishes at its endpoints, we have

$$0 = \frac{d}{ds}\Big|_{s=0} \text{length}(\Phi_s(\gamma)),$$

where Φ_s is the one-parameter family of diffeomorphisms generated by X. In other words, geodesics are the critical points of the length functional, subject to the constraint of fixing the endpoints of the curve.

We see curvature when we examine the stability of the length functional:

$$\frac{d^2}{ds^2}\bigg|_{s=0} \text{length}(\Phi_s(\gamma)) = \int_a^b \|\nabla_{\dot{\gamma}}(X^\perp)\|^2 - \text{Rm}(X^\perp, \dot{\gamma}, \dot{\gamma}, X^\perp)dt,$$

where $X^\perp$ is the component of X perpendicular to $\dot{\gamma}$. In particular, if X is the (normal) velocity field of a one-parameter family of geodesics, we have the **Jacobi equation**:

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X + R(X, \dot{\gamma})\dot{\gamma} = 0. \tag{17}$$

Thus we see that geodesics tend to spread apart in the presence of negative curvature, and they focus together in positive curvature.

If we let $\{x^1, x^2, \ldots, x^n\}$ be normal coordinates centered at $p \in M$, and write our Rm in these coordinates, the (8) becomes

$$\mathrm{Rm} = -\frac{1}{2}(\partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il} + \partial_j \partial_l g_{ik}).$$

However, it will be more informative to write out an expansion for g_{ij} using the Jacobi equation (17).

Fix two vectors $V, W \in T_pM$ and let $\gamma_s(t)$ be the geodesic with inital position p and initial velocity $V + sW$; this is a one-parameter family of geodesics, parameterized by s. Thus we have

$$X = \frac{\partial}{\partial s}\gamma_s(t), \qquad \nabla_{\dot{\gamma}_s}\nabla_{\dot{\gamma}_s}X + R(\dot{\gamma}_s, X)\dot{\gamma}_s = 0.$$

Now let $f(t) = \|X(\gamma_0(t))\|^2$, and expand f in Taylor series centered at $t = 0$ to see

$$\begin{aligned} f(t) &= \langle W, W\rangle t^2 - \frac{8}{4!}\,\mathrm{Rm}(V, W, V, W) \\ &\quad + \frac{20}{5!}\langle \nabla_V(R(V, W)V), W\rangle + \cdots . \end{aligned}$$

In turn, this implies

$$\begin{aligned} g_{ij} &= \delta_{ij} + \frac{1}{3}\,\mathrm{Rm}_{iklj}\, x^k x^l + \frac{1}{6}\nabla_m\,\mathrm{Rm}_{iklj}\, x^k x^l x^m + \mathcal{O}(|x|^4) \\ &= \delta_{ij} - \frac{1}{3}\,\mathrm{Rm}_{ikjl}\, x^k x^l - \frac{1}{6}\nabla_n\,\mathrm{Rm}_{ikjl}\, x^k x^l x^m + \mathcal{O}(|x|^4) \end{aligned} \tag{.18}$$

Notice here that Rm is only evaluated at p, the center of the coordinate system.
We have already seen that, in normal coordinates, any Riemannian metric is locally Euclidean to first order. Now we see that the second order correction term is given by curvature.

We can develope a similar expansion for volume. Use (18) to see

$$\det(g) = 1 - \frac{1}{3} \operatorname{Rc}_{ij} x^i x^j - \frac{1}{6} (\nabla_k \operatorname{Rc}_{ij}) x^i x^j x^k + \mathcal{O}(|x|^4). \quad (19)$$

(At the heart of this computation is the fact that the derivative of a determinant is a trace.)
We related this to volume by recalling that the local expression for the Riemannian volume element is

$$dV_g = \sqrt{\det(g)} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

We can simplify our expression for dV_g a little bit, by writing

$$dV_g = \mu(\theta, r)dr \wedge d\sigma = r^{n-1}\left(1 - \frac{r^2}{6}\,\mathrm{Rc}(\theta, \theta) + \mathcal{O}(r^3)\right) dr \wedge d\sigma,$$

where $\theta \in \mathbf{S}^{n-1}$ is a unit vector in $\mathbf{R}^n$ and $d\sigma$ is the usual volume element on the unit sphere $\mathbf{S}^{n-1}$. As above, we only evaluate Rc at p, the center of the coordinate system. Integrating this last expression from 0 to r, we have

$$V_g(\mathbf{B}_r) = \omega_n r^n \left(1 - \frac{1}{6(n+2)} S(p) r^2 + \mathcal{O}(r^3)\right) \tag{20}$$

where ω_n is the volume of the n-dimensional Euclidean unit ball. Thus we see that the volume of a small geodesic ball is the same as in Euclidean space to first order, and the scalar curvature gives us the second order correction term.

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Submanifolds and the second fundamental form

We consider a submanifold Σ immersed in M which means there is a a full-rank differentiable map from Σ to M.
The immersion $\Sigma \mapsto M$ induces a metric on Σ by pulling back the Riemannian metric on M, and and we write the associated Levi-Civita connection on Σ as ∇^Σ. For vector fields X, Y on Σ we have

$$\nabla^\Sigma_X Y = (\nabla_X Y)^T,$$

where Z^T is the component of Z tangent to Σ.

We have identified the tangential component of $\nabla_X Y$, and we call the normal component

$$(\nabla_X Y)^\perp = A(X, Y)$$

the second fundamental form. One can show that A is a symmetric, rank-two covariant tensor on Σ (even though neither connections ∇ or ∇^Σ are tensorial!) .
It is important to notice that A depends on the way Σ is immersed in M. In particular it is possible to immerse Σ in M in several ways such that the induced metrics are the same, but the second fundamental forms are very different.

The simplest example of this phenomenon is the fact that a flat plane in $\mathbf{R}^3$ is locally isometric to a right circular cylinder. However, the second fundamental form for a flat plane is identically zero, whereas it is never zero for a cylinder.

We refer to A, and quantities related to it, as **extrinsic curvature**, because they depend on how Σ sits in M, rather than the intrinsic properties of its induced metric.

Let $\Sigma \mapsto M$ be an immersed submanifold, and let $p, q \in \Sigma$. Notice that we now have two ways to measure the distance between p and q: in Σ and M. Because of the inclusion, we have $\mathrm{dist}_M(p, q) \leq \mathrm{dist}_\Sigma(p, q)$, and this inequality is usually strict. If we always have $\mathrm{dist}_\Sigma(p, q) = \mathrm{dist}_M(p, q)$ then the geodesics in Σ must also be geodesics in M. In this case, it turns out that $A(X, Y) = 0$ for all vector fields X, Y on Σ, and we call Σ **totally geodesic**.

For instance, flat planes are totally geodesic in $\mathbf{R}^3$, and great spheres (the intersection of any three-dimensional linear space through 0 with $\mathbf{S}^3$) are totally geodesic in $\mathbf{S}^3$.

An example from Lorenz geometry

Everything we have done so far refers to Riemannian metrics, but in the Lorentzian case there is another interpretation of the scalar curvature function. To see this, we need to recall the Einstein field equations for a Lorentz manifold with n spacial dimensions:

$$\mathrm{Rc} - \frac{1}{2} S g = n(n-1)\omega_n T, \tag{21}$$

where T is the stress-energy tensor of the physical system and (as above) ω_n is the volume of the Euclidean unit ball.

Now let Σ be an n-dimensional Riemannian manifold, embedded as a totally geodesic, spacelike slice in the $(n+1)$-dimensional Lorentz manifold M, oriented by the unit normal vector N. We can think of Σ as a space-like slice of M. Then (21) tells us that the energy seen by an observer moving perpendicular to Σ is

$$T(N, N) = \frac{S}{2n(n-1)\omega_n}.$$

Here we have used the Gauss equation (see below) to combine the Ricci and scalar curvature terms.

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Extrinsic vs. intrinsic curvature

There are several formulas relating the extrinsic and intrinsic curvatures of Σ and the intrinsic curvature of the ambient manifold M. The first of these is we've already seen, and it's called the Gauss formula:

$$\nabla_X Y = \nabla_X^\Sigma Y + A(X, Y). \tag{22}$$

The second one is also named after Gauss. It's called the Gauss equation, and has the form

$$\begin{aligned} \mathrm{Rm}(X, Y, Z, W) &= \mathrm{Rm}^\Sigma(X, Y, Z, W) \\ &\quad -\langle A(X, W), A(Y, Z)\rangle + \langle A(X, Z), A(Y, W)\rangle. \end{aligned} \tag{23}$$

Let X and Y be vector fields tangent to Σ and let N be a vector field normal to Σ. Now if we differentiate $\langle N, Y\rangle$ along X we see the Weingarten equation:

$$\langle \nabla_X N, Y\rangle = -\langle N, A(X, Y)\rangle. \tag{24}$$

Finally we have the Codazzi equation

$$\begin{aligned} \mathrm{Rm}(X, Y, Z, W) &= -\langle \nabla A(X, Y, Z), W\rangle \\ &\quad +\langle \nabla A(Y, X, Z), W\rangle. \end{aligned} \tag{25}$$

At this point it will be convenient to specialize to hypersurfaces, so we take $\dim(\Sigma) = n - 1 = \dim(M) - 1$. In this case there the normal space $(T_p\Sigma)^\perp$ is 1-dimensional, and we can (locally) choose a unit normal vector N. In this case $A(X, Y)$ must be a scalar multiple of N, and we write

$$A(X, Y) = -B(X, Y)N.$$

Notice that if we reverse the sign of N we also reverse the sign of B. By the Weingarten equation (24) we have

$$B(X, Y) = \langle \nabla_X N, Y \rangle,$$

and so B is essentially the gradient map of the normal vector N.

The scalar-valued two-tensor h is now a symmetric bilinear form on each tangent space $T_p\Sigma$. As such, it has real eigenvalues

$$\kappa_1(p), \kappa_2(p), \ldots, \kappa_{n-1}(p)$$

at each point. We call each κ_i a **principle curvature** of Σ, and say $p \in \Sigma$ is an **umbilic point** if $\kappa_i(p) = \kappa_j(p)$.

We can form various functions of the principle curvatures, in particular the **mean curvature**

$$H = \frac{1}{n-1}\operatorname{tr}(B) = \frac{\kappa_1 + \cdots + \kappa_{n-1}}{n-1}$$

and the Gauss curvature

$$\mathcal{K} = \det(B) = \kappa_1 \cdot \kappa_2 \cdots \kappa_{n-1}.$$

The mean curvature depends on the way Σ is immersed in M, but (remarkably!) $\mathcal{K}$ only depends on the Riemannian metric induced on Σ. This result is the famous "Theorem Egregium" of Gauss.

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There are some examples relating extrinsic and intrinsic curvature which you know quite well. For instance, we've already mentioned that flat planes in $\mathbf{R}^3$ are totally geodesic, and so they have no extrinsic curvature. A sphere $\mathbf{S}^2_r$ or radius r in $\mathbf{R}^3$ does have extrinsic curvature. Orient $\mathbf{S}^2_r$ with the inward unit normal, and then for all $X, Y \in T_p\mathbf{S}^2_r$ we have

$$B(X, Y) = \frac{1}{r}\langle X, Y\rangle, \quad H = \frac{1}{r}, \quad \mathcal{K} = \frac{1}{r^2}.$$

Notice that every point on a sphere is umbilic.

We consider a cylinder $C \mapsto \mathbf{R}^3$, embedded by $(t,\theta) \mapsto (r\cos\theta, r\sin\theta, t)$ oriented by the normal $N = (-\cos\theta, -\sin\theta, 0)$. Then we have

$$B(\partial_t, \partial_t) = 0, \quad B(\partial_\theta, \partial_\theta) = \frac{1}{r}, \quad B(\partial_t, \partial_\theta) = 0,$$

and

$$H = \frac{2}{r}, \qquad \mathcal{K} = 0.$$

We have just seen that spheres in Euclidean space have constant mean curvature, and also that Riemannian metrics are locally Euclidean to first order, so it makes sense that the mean curvature of a geodesic sphere is close to a constant. Let $\Sigma_r \subset M$ be a small geodesic sphere centered at p, and parameterize Σ_r by $\theta \in \mathbf{S}^{n-1}$. Then we have

$$H_{\Sigma_r}(\theta) = \frac{1}{r} - \frac{r}{3(n-1)} \operatorname{Rc}(\theta, \theta) + \mathcal{O}(r^2),$$

where Rc is evaluate only at the center p.

For this last example, we take the ambient manifold M to be Minkowski space $\mathbf{R}^{n,1}$, *i.e.* $\mathbf{R}^{n+1}$ with the metric

$$g = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2.$$

Recall that an immersed hypersurface $\Sigma \subset \mathbf{R}^{n,1}$ is space-like if its induced metric is positive definite, which, in this case, implies

$$\Sigma = \{(t, x) : t = u(x), \|\nabla u\| < 1\}.$$

We can choose a time-forward oriented normal N, so that

$$\langle N, N \rangle = -1, \quad N(p) \perp T_p\Sigma, \quad N^t > 0.$$

One particular example of such a surface is the usual hyperbolic space $\mathbf{H}^n$, which sits inside $\mathbf{R}^{n,1}$ as the locus

$$\mathbf{H}^n = \{(t, x) : t^2 - \|x\|^2 = 1, t > 0\}.$$

We can rewrite the defining equation as $t = \sqrt{1 + \|x\|^2} = u(x)$, and so the normal vector is

$$N = \frac{1}{\sqrt{1 - \|\nabla u\|^2}}(1, \nabla u) = (\sqrt{1 + \|x\|^2}, x).$$

Differentiating, we see

$$\nabla_j N = \partial_j N = \left(\frac{x_j}{\sqrt{1 + \|x\|^2}}, e_j \right), \quad \nabla_t N = \partial_t N = 0,$$

and so

$$B_{ij} = \delta_{ij}$$

and all the principle curvatures are 1. From here, the Gauss equation implies all the sectional curvatures of $\mathbf{H}^n$ are -1.
In general, a hypersurface $\Sigma \subset \mathbf{R}^{n,1}$, all of whose principle curvatures are positive, will have negative sectional curvature.
On the other hand, a hypersurface $\Sigma \subset \mathbf{R}^{n+1}$, all of whose principle curvatures are positive, will have all positive sectional curvatures.