Lecture 4: The spacetime positive mass theorem

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Workshop on Geometric Analysis

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AIMS, Cape Town

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Plan of Lecture

The lecture will have four parts:

Part 1: ADM energy and linear momentum.

Part 2: The Riemannian positive energy theorem.

Part 3: Asymptotic conditions and a density theorem.

Part 4: The spacetime positive mass theorem.

Initial Data

Suppose $\mathcal{S}$ is a spacetime satisfying the Einstein equations

$$Ric(g) - \frac{1}{2}R\ g = T.$$

We have seen that the spacetime metric evolves from initial data (M, g, k) where g is a Riemannian metric and k a symmetric $(0, 2)$ tensor.

The Constraint Equations

Using the Einstein equations together with the Gauss and Codazzi equations, the constraint equations may be written

$$\mu = \frac{1}{2}(R_M + Tr_g(k)^2 - \|k\|^2)$$
$$J_i = \sum_{j=1}^{n} \nabla^j \pi_{ij}$$

for $i = 1, \ldots, n$ where $\pi_{ij} = k_{ij} - Tr_g(k) g_{ij}$.

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In case there is no matter present, the vacuum constraint equations become

$$R_M + Tr_g(k)^2 - \|k\|^2 = 0$$
$$\sum_{j=1}^{n} \nabla^j \pi_{ij} = 0$$

for $i = 1, \ldots, n$ where R_M is the scalar curvature of M.

Energy Conditions

For spacetimes with matter, the stress-energy tensor is normally required to satisfy the **dominant energy condition** which says that the energy-momentum density vector of the matter fields is non-spacelike for any observer. For an initial data set this is the inequality $\mu \geq \|J\|$.

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In the time symmetric case ($k = 0$) the dominant energy condition is equivalent to the inequality $R_M \geq 0$. In case the maximal case $Tr_g(k) = 0$ the dominant energy condition implies $R_M \geq 0$

Asymptotic Flatness

The most natural boundary condition for the Einstein equations is the condition of asymptotic flatness. This boundary condition describes isolated systems which are the analogues of finite mass distributions in Newtonian gravity. The requirement is that the initial manifold M outside a compact set be diffeomorphic to the exterior of a ball in R^3 and that there be coordinates x in which g and k have appropriate falloff

$$g_{ij} = \delta_{ij} + O_2(|x|^{2-n}),\ k_{ij} = O_1(|x|^{1-n}).$$

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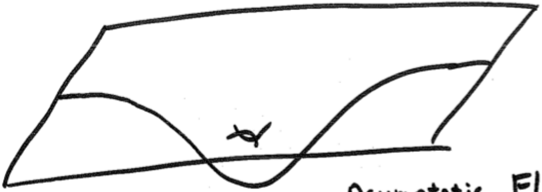

Asymptotic Flatness

Minkowski and Schwarzschild Solutions

The following are two basic examples of asymptotically flat spacetimes:

1) The Minkowski spacetime is R^{n+1} with the flat metric $g = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$. It is the spacetime of special relativity.

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2) The Schwarzschild spacetime is determined by initial data with $k = 0$ and

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij}$$

for $|x| > 0$. It is a vacuum solution describing a static black hole with mass E. It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

ADM Energy and Linear Momentum

For general asymptotically flat initial data sets there is a notion of total energy-momentum which was defined by Arnowitt, Deser, and Misner. There is no energy density for the gravitational field so these quantities are computed in terms of the asymptotic behavior of g and k. For these definitions we fix asymptotically flat coordinates x.

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\nu_0^j \, d\sigma_0$$

$$P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{j=1}^{n} \pi_{ij}\nu_0^j \, d\sigma_0, \qquad i = 1, 2, \ldots, n$$

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These limits exist under quite general asymptotic decay conditions. For the constant time slices in the Schwarzschild metric we have $E = m$. Generally (E, P) can be thought of as a vector in the asymptotic Minkowski space, and for a more general slice in these spacetimes we have $m = \sqrt{E^2 - |P|^2}$.

Statement of Theorem

In this talk we will describe the proof of the following theorem due to (EHLS) M. Eichmair, L. Huang, D. Lee, and the speaker (arXiv:1110.2087).

Theorem (Spacetime positive mass theorem)

Let $3 \leq n < 8$, and let (M, g, k) be an n-dimensional asymptotically flat initial data set satisfying the dominant energy condition. Then

$$E \geq |P|,$$

where (E, P) is the ADM energy-momentum vector of (M, g, k).

Previous Results

Our theorem is an improvement of earlier results.

• $R \geq 0$ implies $E \geq 0$ by S & Yau for $3 \leq n < 8$. This includes the maximal (and Riemannian) case.

• Dominant energy condition implies $E \geq 0$. Done by S & Yau for n=3, and same method extended recently by Eichmair for $3 < n < 8$.

• For spin manifolds of any dimension $E \geq |P|$ follows from argument of E. Witten.

Part 2: The Riemannian Positive Energy Theorem

The mean curvature proof of the positive energy theorem in the Riemannian case is based on the study of stable minimal hypersurfaces. Recall that the stability condition for a minimal hypersurface may be written

$$\int_\Sigma [\|\nabla\varphi\|^2 - \frac{1}{2}(R^M - R^\Sigma + \|A\|^2)\varphi^2]\, dv \geq 0.$$

for all smooth φ with compact support on Σ.

Stability and Scalar Curvature I

For $n = 3$, if Σ is compact we may choose $\varphi = 1$ and we obtain

$$\int_\Sigma R^\Sigma \, dv \geq \frac{1}{2} \int_\Sigma (R^M + \|A\|^2) \, dv.$$

By the Gauss-Bonnet theorem this implies that the Euler characteristic of Σ is positive if Σ is orientable and $R^M > 0$.

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In higher dimensions the stability argument implies that a stable hypersurface in a manifold of positive scalar curvature admits a metric of positive scalar curvature (conformal to the induced metric). To see this we let $u > 0$ be a first eigenfunction of the Jacobi operator $\mathcal{L}$. It follows that

$$\mathcal{L}(u) = \Delta u + \frac{1}{2}(R^M - R^\Sigma + \|A\|^2)u \leq 0$$

Stability and Scalar Curvature II

Thus if $R^M > 0$ we have

$$\Delta u - \frac{1}{2} R^\Sigma u < 0.$$

It follows that $v = u^\alpha$ with $\alpha = \frac{n-3}{2(n-2)}$ satisfies

$$\Delta v - \frac{n-3}{4(n-2)} R^\Sigma v < 0.$$

The operator on the left is the conformal Laplacian on Σ and it follows from the conformal transformation of scalar curvature that the metric $v^{4/(n-3)} g$ has positive scalar curvature.

Simplifying the asymptotics

Assume that M is asymptotically flat with $R \geq 0$. In order to show that $E \geq 0$, we first prove a density theorem which shows that, given any $\epsilon > 0$, there is a scalar flat metric $\bar{g}$ which has conformally flat asymptotics meaning that near infinity

$$\bar{g}_{ij} = u^{\frac{4}{n-2}} \delta_{ij}$$

and such that $\bar{E} \geq E - \epsilon$.

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This means that the metric is equal to the standard slice of the Schwarzschild metric to leading order at infinity. This is because u is a harmonic function asymptotic to 1 so we have

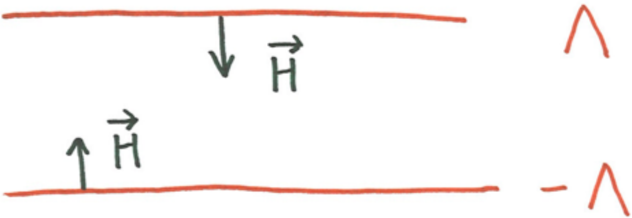
$$u(x) = 1 + \frac{E}{2|x|^{n-2}} + O(|x|^{1-n}).$$

Barrier Construction

Therefore in suitable coordinates near infinity we have

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij} + O(|x|^{1-n})$$

where E is the energy. If $E < 0$ we get the following picture



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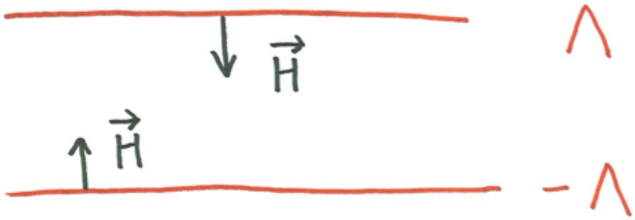

We may use this condition to construct a stable minimal hypersurface asymptotic to a plane.

Dimension three

In case $n = 3$ the proof is completed by observing that the constant function 1 is a limit of functions of compact support in the Dirichlet seminorm, and so we can justify it as a variation, and so we have from stability

$$\int_\Sigma R_\Sigma \, da > 0.$$

On the other hand, because of the asymptotically planar condition the Gauss-Bonnet theorem implies

$$\int_\Sigma R_\Sigma \, da = 2\pi(\chi(\Sigma) - 1).$$

Since Σ is a noncompact surface we have $\chi(\Sigma) \leq 1$, a contradiction.

Strong Stability

For $n \geq 4$ it is not sufficient to have such a stable hypersurface. Instead it is necessary to choose a special such hypersurface which is **strongly stable** in the sense that the second variation of volume is nonnegative for variations which are translations near infinity (not just of compact support). This is achieved by doing an extra minimization over boundary heights. This can be done because of the barriers and is essential for $n \geq 4$.

For $n = 3$ and Σ asymptotically planar, it follows that stability implies strong stability, but this is not true in higher dimensions.

Completion of proof

Using the strong stability we can find a conformal factor $v > 0$ as in the compact case with

$$\Delta v - \frac{n-3}{4(n-2)} v = 0$$

and with the asymptotic behavior

$$v(x) = 1 + b|x|^{3-n} + O(|x|^{2-n})$$

with $b < 0$. It follows that Σ with the metric $v^{4/(n-3)}g$ is asymptotically flat with zero scalar curvature and with negative mass. This contradicts the positive energy theorem in dimension $n-1$ completing the proof inductively.

Part 3: Asymptotic Conditions and a Density Theorem

We now move to the general case. There is a replacement for the Schwarzschild asymptotics used in the Riemannian case discovered by the speaker earlier. This is called harmonic asymptotics and it means that near infinity

$$g = u^{\frac{4}{n-2}}\delta, \ \pi = u^{\frac{2}{n-2}}(L_X\delta - div(X)\delta)$$

where $u > 0$ and X is a vector field, and $L_X\delta$ the Lie derivation of the euclidean metric with respect to X.

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where $u > 0$ and X is a vector field, and $L_X\delta$ the Lie derivation of the euclidean metric with respect to X.

It was shown by J. Corvino and the speaker that any vacuum initial data set can be approximated by one with harmonic asymptotics in a norm for which E and P are continuous. This density theorem was extended from vacuum to the dominant energy condition in EHLS.

Asymptotics for Energy and Linear Momentum

In harmonic asymptotics we have

$$\begin{aligned} u(x) &= 1 + a|x|^{2-n} + O_{2+\alpha}(|x|^{1-n}) \\ X_i(x) &= b_i|x|^{2-n} + O_{2+\alpha}(|x|^{1-n}) \end{aligned}$$

for some $\alpha > 0$.

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A computation shows $a = \frac{E}{2}$, $b_i = -\frac{n-1}{n-2}P_i$ for $i = 1, \ldots, n$.

A Density Theorem

Let (M^n, g, π) be an asymptotically flat initial data set. Assume that the dominant energy condition $\mu \geq |J|_g$ holds. For every $\epsilon > 0$ there exists asymptotically flat initial data $(M, \bar{g}, \bar{\pi})$ of the same type with *harmonic asymptotics*, where $(\bar{g}, \bar{\pi})$ approximates (g, π) in an appropriate weighted Sobolev space, and such that the strict dominant energy condition

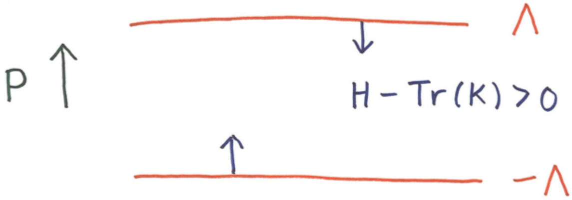
$$\bar{\mu} > |\bar{J}|_{\bar{g}}$$

holds, and

$$|E - \bar{E}| < \epsilon \quad \text{and} \quad |P - \bar{P}| < \epsilon.$$

Part 4: The Spacetime Positive Mass Theorem

We assume we are in harmonic asymptotics and we show that if $E < |P|$ then we have a picture reminiscent of the Riemannian case



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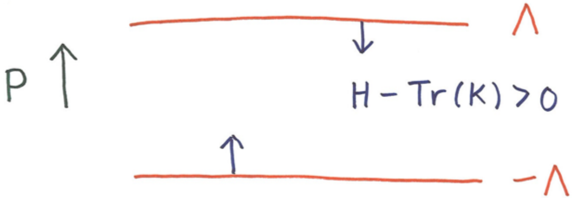

This is based on the calculation in harmonic asymptotics of the expansion of the hypersurfaces $x_n = \Lambda$ where we have chosen coordinates for which P points in the positive x_n direction

$$H - tr_{\Sigma}(k) = (n-1)(|P| - E)\Lambda|x|^{-n} + O(|x|^{-n}).$$

The Three Dimensional Case

From the trapping condition on the slab we are able to construct a hypersurface Σ asymptotic to a plane with $H - Tr(k) = 0$ and an appropriate stability condition which is identical to that for stable hypersurfaces. In case $n = 3$ the proof is completed by observing that the constant function 1 is a limit of functions of compact support in the Dirichlet seminorm, and so we can justify it as a variation, and so we have from stability

$$\int_{\Sigma} R_{\Sigma}\ da > 0.$$

On the other hand, because of the asymptotically planar condition the Gauss-Bonnet theorem implies

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Since Σ is a noncompact surface we have $\chi(\Sigma) \leq 1$, a contradiction.

The Higher Dimensional Case

For $n \geq 4$ an additional difficulty appears since we can no longer justify the function 1 as a variation; that is, we need to construct a strongly stable MOTS. This was accomplished by minimization in the Riemannian case. An interesting and subtle feature of the argument is that we are able to accomplish this even though the equation is not variational.

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Once we find a strongly stable asymptotically planar MOTS Σ the argument proceeds as in the Riemannian case. If we have $E < |P|$, we construct Σ and use the strong stability condition to find an asymptotically flat metric on Σ with $R = 0$ and $E < 0$. This contradicts the Riemannian postive energy theorem in dimension $n - 1$.

Height Picking Heuristics

Let $\Sigma_{\rho,h}$ be a stable MOTS with boundary at height h on the boundary of the cylinder of radius ρ centered on the x_n-axis. We observe that in the Riemannian case the condition

$$\frac{d^2}{d^2h}|\Sigma_{\rho,h}| \geq 0$$

is equivalent to (first variation formula)

$$\frac{d}{dh}F(h) \geq 0 \; where \; F(h) = \int_{\partial\Sigma\rho,h} \langle \partial_n, \eta \rangle \; d\sigma$$

where η is the outer unit conormal vector along the boundary.

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The barrier condition implies that $F(-\Lambda) < 0$ while $F(\Lambda) > 0$, and so it is expected that F has positive derivative at some point $h \in (-\Lambda, \Lambda)$. In the general case we replace the volume minimization by this condition. The difficulty is that the $\Sigma_{\rho,h}$ are not unique and we do not expect F to be continuous.

Height Picking

It turns out that the family of hypersurfaces $\Sigma_{\rho,h}$ (fixed ρ) form a foliation with gaps, and the function F is continuous except at a countable set of jump discontinuities, and that the jumps are always down

$$\lim_{h\uparrow h_0} F(h) \geq F(h_0) \geq \lim_{h\downarrow h_0} F(h)$$

with at least one strict inequality.

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It can then be shown that there is $h_\rho \in (-\Lambda, \Lambda)$ such that Σ_{ρ,h_ρ} has a Jacobi field X (for the expansion) which agrees with ∂_n on the boundary and such that the boundary term F increases when a deformation is made along X. It is then possible to take a limit of $\Sigma_{\rho,\sigma_\rho}$ and to show that the limit is strongly stable. The proof can then be completed as described above.