1. Introduction, 1^{st} and 2^{nd} variation
2. Existence Theory
3. Curvature Estimates and Compactness
4. Spacetime positive mass theorems

Minimal surfaces model soap films in curved spaces - structures of minimal surfaces vs. ambient curvature

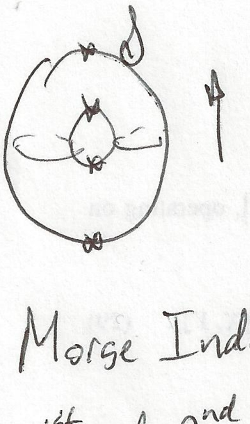
parallel to geodesics: say how ambient curvature and geodesics are related

set up (M^n, g) Riem. m'fold $\Sigma^k \subset M^n$ has an induced metric $g|_\Sigma$

Σ has intrinsic and extrinsic geometry

$g|_\Sigma \to |\Sigma| = Vol(\Sigma)$ look at a class of submanifolds called $\mathcal{S}$.

then get $|\cdot| : \mathcal{S} \to \mathbb{R}_+$ want to understand critical points



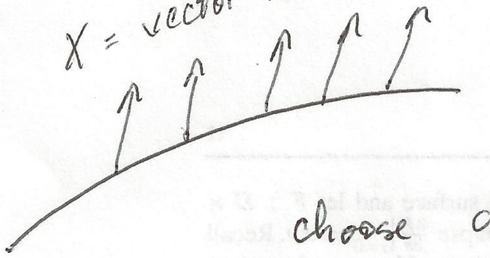
h = height function set $\nabla^{\mathcal{S}} h = 0$ ← 1^{st} variation

at these points set $Hess(h) = 0$ ← 2^{nd} variation

<u>Defn.</u>: Morse Index = # neg. eigenvalues of $Hess(h)$

What are 1^{st} and 2^{nd} variation of volume.

X = vector field



$\Sigma^k \subset M$

choose diffeo's $F_t : M \to M$, $F_0 = Id.$
(think flow of X)

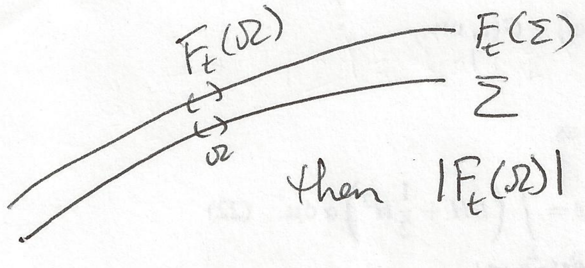
$$\frac{d}{dt}\Big|_{t=0} F_t = X \qquad \text{let } \Sigma_t = F_t(\Sigma)$$

think of these vector fieds as $T_\Sigma \mathcal{S}$

$$1^{st} \text{ variation: } \quad \delta_X \Sigma = \frac{d}{dt}\Big|_{t=0} |\Sigma_t| = \int_\Sigma \operatorname{div}_\Sigma(X)\, d\sigma$$

if $\{e_1, \dots, e_k\}$ = ONB for $T_p\Sigma$ then

$$\operatorname{div}_\Sigma(X) = \sum_{i=1}^k \langle e_i, \nabla_{e_i} X\rangle = \underbrace{\Theta(X) = \text{expansion in } X \text{ direction}}_{\text{in relativity}}$$



then $|F_t(\Omega)| \simeq \underbrace{J_t}_{\text{Jacobian}} |\Omega|$ $\qquad \Theta = \log$ derivative of J_t

Thm: Σ is a critical pnt for volume $\iff$ the mean curv. $H_\Sigma = 0$

Proof: write $X = X^T + X^\perp \Rightarrow \int_\Sigma \operatorname{div}_\Sigma(X)\, d\sigma = \int_\Sigma \operatorname{div}_\Sigma(X^T) + \operatorname{div}(X^\perp)\, d\sigma$
reparameterizations

$$= \int_\Sigma \operatorname{div}_\Sigma(X^\perp)\, d\sigma$$

but $\operatorname{div}_\Sigma(X^\perp) = \sum \langle e_i, \nabla_{e_i} X^\perp\rangle = -\sum \langle (\nabla_{e_i} e_i)^\perp, X^\perp\rangle$

$$= -\langle H, X^\perp\rangle$$

so $\int_\Sigma \langle H, X^\perp\rangle\, d\sigma = 0$ for every X w/ compact support. $\square$

CMC: $\Sigma^{n-1} \subset M^n$, $H = H_0$ = constant.

CMC $\Leftrightarrow$ Σ stationary for volume preserving deformations

volume preserving: $\int_\Sigma \langle X, N \rangle \, d\sigma$, N = normal

MOTS: $M^n \subset N^{n+1} \leftarrow$ lorentz metric
spacelike slice

$\Sigma^{n-1} \subset M \subset N^{n+1}$

k = 2^{nd} fund. form of M in N if Σ bounds a region in M

ν = normal to Σ in M

e_{n+1} = normal to M in N

now take $X = \nu + e_{n+1}$, $\theta_+ = \theta(X)$ = expansion

Defn: if $\theta_+ < 0$ then Σ outer trapped.

if $\theta_+ \equiv 0$ then Σ is MOTS (marginally outer trapped surface)

we have $\theta_+ = \theta(X) = -(H_\Sigma - tr_\Sigma(k)) = 0$

so MOTS eqn says $H_\Sigma = tr_\Sigma(k)$

2^{nd} variation: $\delta_X^2 \Sigma = \frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t|$

again, tangential part doesn't matter. assume $\Sigma^{n-1} \subset M$

ν = normal to Σ

$X = \varphi \cdot \nu$

then $\delta_X^2 \Sigma = \int_\Sigma |\nabla\varphi|^2 - (Rc^M(\nu,\nu) + |A|^2 \varphi^2) \, d\sigma$

. if φ oscillates a lot, then $\delta_X^2 \Sigma \nearrow$

. if $Rc > 0$ then $\delta_X^2 \Sigma \searrow$. if $|A|^2$ large then $\delta_X^2 \Sigma \searrow$, a

Schoenketwel
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write $\int_X^2 \mathcal{I} = -\int_\Sigma \varphi \mathcal{L} \varphi \, d\sigma$

where $\mathcal{L}\varphi = \Delta_\Sigma \varphi + (|A|^2 + Rc(\nu,\nu))\varphi$

Jacobi operator.

can rewrite: $\mathcal{L}\varphi = \Delta_\Sigma \varphi + \frac{1}{2}(R^M - R^\Sigma + |A|^2)\varphi$

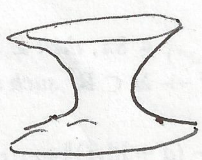
if $\nu = e_n$ then $R^M = \sum_{a,b=1}^{n} Rm_{abab}$

$$= \sum_{i,j=1}^{n-1} Rm_{ijij} + \underbrace{\sum_{a=1}^{n-1} Rm_{anan} + \sum_{b=1}^{n-1} Rm_{nbnb}}_{2Ric(\nu,\nu)}$$

$$= \sum_{i,j=1}^{n-1} \left[Rm^\Sigma_{ijij} - h_{ij}h_{ij} - (h_{ij})^2 \right] + 2Rc(\nu,\nu)$$

Examples: ① in $\mathbb{R}^n$, Σ = hyperplane

② in $\mathbb{R}^3$, catenoid. = minimal surfaces of revolution.



③ in $\mathbb{R}^3$: helicoid

Existence

$\mathcal{S}$ = collection of k-dim'l submanifolds of a fixed Riem. m'fold M
$= \{\Sigma^k ; \Sigma^k \subset (M^n, g)\}$

need: $|\Sigma|$ to be defined
"weak" topology on $\mathcal{S}$

(H1) lower semi-continuity i.e. if $\Sigma_i \to \Sigma$ in the top. of $\mathcal{S}$
then $|\Sigma| \leq \liminf_{i\to\infty} |\Sigma_i|$

(H2) $\forall\, c>0$ the set $\mathcal{S}_c = \{\Sigma \in \mathcal{S} : |\Sigma| \leq c\}$ is compact

Lemma: $\exists\, \Sigma_0 \in \mathcal{S}$ with $|\Sigma_0| = \min_{\Sigma\in\mathcal{S}} |\Sigma|$

Q: what exactly is $\mathcal{S}$ and what is the topology on it?

Rmk: want $\mathcal{S}$ to be a large enough class of submanifolds to get decent results.

small $\mathcal{S}$ => existence easier but regularity much harder

regularity comes from being able to take arbitrary local deformations

Four cases where we can do this.

① $\Gamma \subset M^n$, Γ = smooth emb. curve, Γ bounds a contractible disc.
i.e. $\exists\, f : \mathbb{D} \to M$, $f|_{\partial \mathbb{D}}$ param. Γ
then $\Sigma = f(\mathbb{D})$

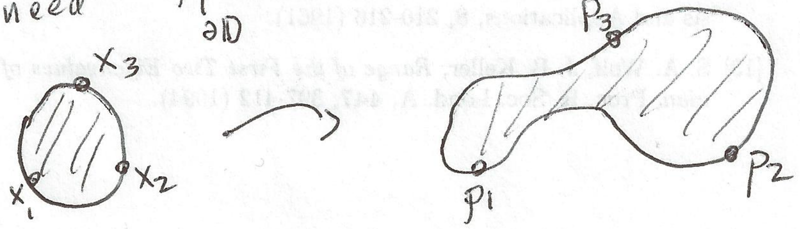
let $E(f) = \frac{1}{2}\int_{\mathbb{D}} |\nabla f|^2\, dx\,dy = \frac{1}{2}\int_{\mathbb{D}} f_x^2 + f_y^2\, dx\,dy$ (assume this is finite)

notice $A(f) = \int_{\mathbb{D}} \| f_x \wedge f_y \|\, dx\,dy \leq E(f)$

equality <=> $\|f_x\| = \|f_y\|$ and $\langle f_x, f_y\rangle = 0$

in this case: $\mathcal{S} = \{ \Sigma = f(D) : f \in W^{1,2}(D, M) \cap C^0(\partial D, M),$
$f|_{\partial D}$ param. Γ in a monotone fashion $\}$

also need $f|_{\partial D}$ satisfies the three point condition.



want $f(x_i) = p_i$

topology: weak $W^{1,2}$ top.

Theorem: under these conditions $\mathcal{S}$ satisfies (H1) and (H2)

Hard part: uniform convergence on the boundary
(follows from monotone property on ∂D)

• in fact, the minimizer is a smooth branched immersed surface.

(2) S^2 = any smooth compact surface, $f : S \to M$

$\hat{\mathcal{S}} = \{ \Sigma = f(S) : f \in W^{1,2}(S, M) \}$

$|\Sigma| = \inf \{ \|f\|^2_{W^{1,2}(\sigma)} : \sigma$ is a conformal structure on $S \}$

place the weak top. on $\hat{\mathcal{S}}$

$\mathcal{S}$ = a connected component of $\hat{\mathcal{S}}$ (sort of like homotopy classes of maps $S \to M$)

require that $f_* : \pi_1(S, *) \to \pi_1(M, *)$ is injective
for some $f \in \mathcal{S}$

then (H1) and (H2) hold (Schoen and Yau)

incompressibility $\Rightarrow$ control over conformal structures on S

regularity comes from regularity for harmonic maps, and minimizers are conformal.

more difficult in higher dimension

(3) $\Gamma^{k-1} \subset M^n$, $\partial\Gamma = \phi$ and Γ bounds some k-dim'l subm'fold Σ

$\mathcal{S} = \{\Sigma$ integral currents : $\partial\Sigma = \Gamma\}$ ← Federer and Flemming

↑ completion of smooth subm'folds in appropriate topology

flat norm:

$$d(\Sigma_1, \Sigma_2) = \inf\{|R|, \dim(R) = k+1,\ \partial R = \Sigma_2 - \Sigma_1\}$$

these are objects in Geometric Measure Theory

these imply (H1) and (H2) hold

(4) $\mathcal{S} = \{k$-dim'l integral currents Σ^k ; $\Sigma^k \in \alpha$ homology class$\}$

then (H1) and (H2) hold

<u>Regularity for codim 1</u>: so $k = n-1$

if $n < 8$ then Σ is smooth and embedded for both (3) and (4)

if $n \geq 8$ then $\exists$ sing. set S, $\dim_{Haus} S \leq n-8$

2nd Method: PDE based, weaker results but works for nonvariational problems.

look at $(M \times \mathbb{R},\ g + dt^2)$ if $f : M \to \mathbb{R}$ Σ_f = graph of f

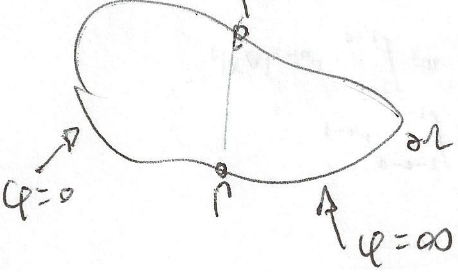
then $H_{\Sigma_f} = 0 = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$ (*)

try to solve Dirichlet BVP: if $\Omega \subset M$

then want (*) w/ $f|_{\partial\Omega} = \varphi$

necessary condition: $\partial\Omega$ is mean convex i.e. $\theta > 0$

if $\Gamma^{n-2} \subset \partial\Omega$, Γ divides $\partial\Omega$ into two pieces



soln → cylinder
(like Ginsburg-Landau)
need curvature estimates
to make this work.

Analogous method for MOTS

have a Riem. m'fold M, $\Omega \subset M$, want $\partial\Omega$ strictly untrapped.
(i.e. $\partial\Omega$ is mean convex)

let $\Gamma^{n-2} \subset \partial\Omega$ which divides $\partial\Omega$ into two parts
call them $(\partial\Omega)_-$ and $(\partial\Omega)_+$

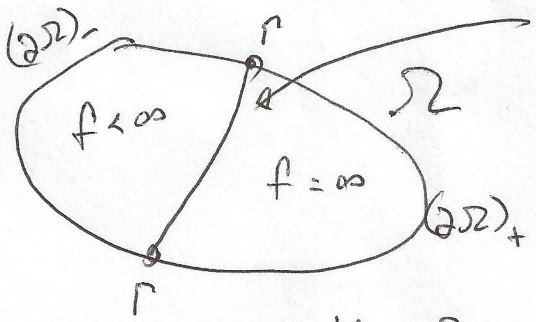
solve the minimal surface eqn:

$$0 = H_{\Sigma_f} = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \qquad f\big|_{\partial\Omega} = \varphi_R = \begin{cases} 0 & \text{on } (\partial\Omega)_- \\ R & \text{on } (\partial\Omega)_+ \end{cases}$$

smooth this to make it C^2

- $\exists$ unique soln for any finite $R \to \Sigma_{f_R}$ = minimal graph in $M \times \mathbb{R}$

as $R \to \infty$, have $\Sigma_{f_R} \to \Sigma_f \subset M \times \mathbb{R}$ a minimal surface



transition is a soln to Plateau problem in M w/ boundary Γ.

Notice: $\Sigma \times \mathbb{R}$ is minimal in $M \times \mathbb{R}$

Generalize this to MOTS: have (M, g, k)

MOTS eqn: $H_\Sigma = \operatorname{tr}_\Sigma(k)$

look at $M \times \mathbb{R}$, extend g to $\hat{g}$, k to $\hat{k}$

$$\hat{g} = g + dt^2, \quad \hat{k} = \pi^{-1}(k), \text{ where } \pi: M\times\mathbb{R} \to M \text{ is projection}$$

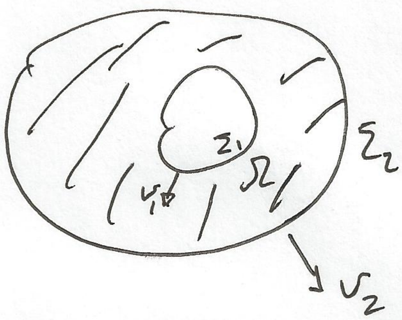
if $\Sigma^{n-1} \subset M$ is a MOTS then $\Sigma \times \mathbb{R} \subset M \times \mathbb{R}$ is a MOTS.

Jang: $f \in C^\infty(M)$, $\Sigma_f \subset M \times \mathbb{R}$

$$H_{\Sigma_f} = tr_{\Sigma_f}(\hat{k}) \to div\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \sum \bar{g}^{ij} k_{ij}$$

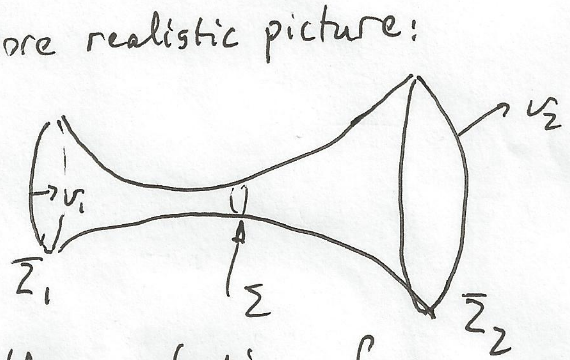
$\bar{g}_{ij} = g_{ij} + f_i f_j$ = induced metric on Σ

1st case: $\Omega \subset M^n$, $\partial\Omega = \Sigma_1 \cup \Sigma_2$



require Σ_1 strictly trapped
Σ_2 strictly untrapped.

More realistic picture:



take Σ homologous to Σ_1 and to Σ_2

this produces a smooth solution for $n \leq 7$

2nd case: $\Gamma^{n-2} \subset \partial\Omega$, $\Omega \subset M$ want $\partial\Omega \setminus \Gamma = (\partial\Omega)_+ \cup (\partial\Omega)_-$

want $(\partial\Omega)_+$ is strictly untrapped for ν = outward normal

$(\partial\Omega)_-$ is strictly trapped for $-\nu$

these together imply $-\underset{\uparrow \text{ expansion}}{H} \geq |tr_{\partial\Omega}(k)|$

then $\exists$ MOTS $\Sigma \subset \Omega$ with $\partial\Sigma = \Gamma$

in the mean curvature case: need to
- solve BVP for any boundary data
- take limits

- solving BVP: try a continuity argument, start w/ bndy val. $\equiv 0$

  i.e. bnd. val. $t \cdot \varphi \quad 0 \le t \le 1$

  can solve for t small.

  if soln blows up, it must blow up at a bndry pnt by max. princ.

  rule this out w/ Hopf boundary pnt. lemma.

  key: $\partial\Omega$ is a barrier $\Rightarrow$ control of $|\nabla f|$ on $\partial\Omega$

  $\Rightarrow$ control of $|\nabla f|$ on Ω

  $\Rightarrow$ control of $\|f\|_{C^2}$ on Ω

- taking limits: use stability

  in the case $H=0$, the stability operator is

$$\mathcal{L}\varphi = \Delta_\Sigma \varphi + \frac{1}{2}(R_M - R_\Sigma + \|A\|^2)\varphi$$

$$\delta^2_{\varphi\nu}\Sigma = -\int_\Sigma \varphi \mathcal{L}\varphi \, dV_\Sigma$$

$-\mathcal{L}$ has a spectrum $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$

Defn: the Morse index of Σ is the # neg. eigenvalues of $-\mathcal{L}$.

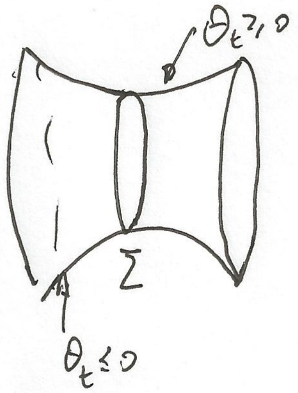
So Σ is stable $\Leftrightarrow \lambda_1 \ge 0 \Leftrightarrow$ Morse index $= 0$.

Prop: Σ is stable $\Leftrightarrow \exists$ pos. function u on Σ w/ $\mathcal{L}(u) \le 0$

if $\exists$ loc. foliation Σ_t, $\Sigma = \Sigma_0$ with $\begin{cases} \theta_t \le 0 \text{ for } t < 0 \\ \theta_t \ge 0 \text{ for } t > 0 \end{cases}$

then Σ is stable.

Picture:



write Σ_t = normal graph over Σ
then feed the graphing function into $\mathcal{L}$.

Sketch of Proof: stable $\Rightarrow \lambda_1 \geq 0$: let u_1 = eigenfunction, choose $u_1 > 0$

then $-\mathcal{L}(u_1) = \lambda_1 u_1 \geq 0$ $\square$

if $u > 0$, $\mathcal{L}(u) \leq 0$, let $w = \log u$

$$\Delta w = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

$$\mathcal{L}u = \Delta u + qu \leq 0 \Rightarrow \Delta u \leq -qu$$

so $\Delta w = -q - |\nabla w|^2$

$$\Rightarrow \int_\Sigma (q + |\nabla w|^2)\varphi^2 \leq 2\int \varphi \langle \nabla\varphi, \nabla w\rangle \leq \int \varphi^2 |\nabla w|^2 + |\nabla \varphi|^2$$

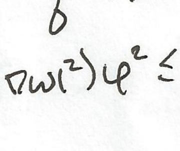
$$\Rightarrow \int_\Sigma q\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2 \Rightarrow \text{stable.}$$

Defn: say an MOTS Σ is stable if $\exists\, u > 0$ on Σ w/ $\mathcal{L}u \leq 0$

here $\mathcal{L}$ = linearized MOTS eqn (NB, this is not self-adjoint)

= Δ + 1st order term + 0th order term.

unstable MOTS:

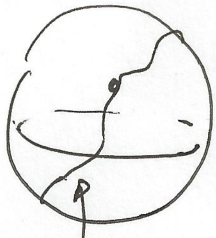


Schoen lecture 3 pages

- under reasonable conditions you can bound the volumes
- also need curvature bounds ⇝ curvature estimates.

not true in general for rescalings of a catenoid!
not stable.

so only expect curvature estimates for <u>stable</u> minimal surfaces



B^n = unit ball in $\mathbb{R}^n$

$\partial\Sigma \cap B = \emptyset$

$\mathcal{C}$ = some collection of minimal surfaces under these hypotheses.

when is it true that $\exists\ C_0 > 0$ so that $\|A_\Sigma\|(0) \le C_0 \quad \forall\, \Sigma \in \mathcal{C}$

True for ① $\mathcal{C} = \{\Sigma \text{ stable, minimal}, |\Sigma| \le V \ \cancel{\text{for some } V}\}$ for some V, and for $n \le 7$

(similar for MOTS)

<u>Rmk:</u> this is connected to Bernstein's theorem.
if $f: \mathbb{R}^n \to \mathbb{R}$ so that Graph(f) is minimal then f is linear

for $n=2$ there are many proofs.
this is true for $n \le 8$ (ie. $\exists$ a minimal ~~nonplanar~~ nonflat graph)

- However, whenever a curvature estimate holds for $\mathcal{C}$ then a Bernstein theorem holds for $\mathcal{C}$

let $\Sigma^{n-1} \subset \mathbb{R}^n$, Σ stable, $|\Sigma \cap B_R| \le V R^{n-1}$ (true for volume minimizers)

$\Rightarrow \Sigma$ is a plane.