Mean Curvature in Riemannian Geometry and General Relativity

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1 Introduction

In this series of three lectures we describe some connections between the theory of minimal submanifolds and mean curvature with Riemannian Geometry and General Relativity. Our goal will be to put the known results in context, and to outline problems which remain open. We begin with a discussion of the first and second variation formulae for volume. We place particular emphasis on the notion of stability for minimal submanifolds because it is primarily through stability and the Jacobi operator that the ambient curvature influences the behavior of minimal submanifolds. By definition a minimal submanifold is stable if the second variation of volume is nonnegative for all compactly supported deformations. We summarize the known analytic estimates and compactness theorems for stable hypersurfaces, and then proceed to discuss connections with scalar curvature and general relativity.

We describe the constraint equations and asymptotic flatness for initial data sets for Einstein's equations. We then discuss the ADM mass and state the positive mass theorem. We discuss the Schwarzschild spacetime, and formulate the Penrose inequality. We then describe the Huisken/Ilmanen and Bray work on the Riemannian Penrose inequality. We emphasize throughout the close connection of these results with the mean curvature theory; indeed they are really sharp quantitative statements about the area of minimal surfaces in three dimensional manifolds with nonnegative scalar curvature.

In the final part of this paper we discuss issues connected with area minimizing and stable submanifolds in higher codimensions. We describe the theory of calibrations, and place particular emphasis on the holomorphic and special lagrangian calibrations. We introduce the lagrangian Plateau problem and outline its connection with constructions of special lagrangian and minimal lagrangian submanifolds of Kähler-Einstein manifolds.

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2 Background and Notation

Let (M^{n+1}, g) be a Riemannian manifold, and let D be an affine connection. D is called the Levi-Civita connection of g if

(1) $D_X g = 0$, for any vector field X;

(2) D satisfies the torsion free condition: $D_X Y - D_Y X = [X, Y]$, for any vector fields X and Y.

It is a standard result that given a metric g, there exists a unique Levi-Civita connection.

Using the Riemannian metric g, we can define the length of a curve, and the Riemannian distance function for a pair of points x and y

$$d(x, y) := \inf\{L(\gamma) : \gamma \text{ is a rectifiable curve connecting x and y}\}.$$

Geodesics are defined to be the critical curves for length functional. Locally they are length minimizing, but in the large they tend to be unstable critical points in many cases.

The Riemann curvature tensor may be considered a $(0, 4)$ tensor $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle$ where R is the curvature tensor of the Levi-Civita connection D and $\{e_1, \ldots, e_{n+1}\}$ is a basis for $T_x M$. We define the Ricci tensor to be the trace of the curvature tensor, $R_{ij} = \sum_{k,l} g^{kl} R_{ikjl}$, and the scalar curvature R to be the trace of Ricci tensor $R = \sum_{i,j} g^{ij} R_{ij}$.

Now we study hypersurfaces in M^{n+1}. Let Σ^n be a hypersurface in M. We define the induced Riemannian connection ∇ on Σ by

$$\nabla_X Y := (D_X Y)^T.$$

where $(D_X Y)^T$ is the projection of $D_X Y$ onto the tangent space of Σ. It is easy to check that ∇ is an affine connection and is indeed the Levi-Civita connection for the induced metric on Σ.

Let ν be a chosen unit normal vector field for Σ. We define the scalar valued *second fundamental form* h as

$$h(X, Y) := \langle D_X Y, \nu\rangle,$$

for tangent vector fields X and Y on Σ. We can show that h is in fact a symmetric $(0, 2)$ tensor.

The *mean curvature* H is defined to be the trace of second fundamental form, i.e

$$H := \sum_{i,j} g^{ij} h_{ij},$$

where $h_{ij} = h(e_i, e_j)$.

Definition 2.1. *A hypersurface is called minimal if the mean curvature is identically zero, i.e. $H = 0$.*

The following fundamental equations relate the intrinsic geometry (defined by g) and the extrinsic geometry (defined by h) for a hypersurface. Let $\Sigma^n \subset M^{n+1}$ be a hypersurface and $e_1, \ldots, e_n$ be a orthonormal basis on Σ.

Proposition 2.2 (Gauss Equation)**.** *The Riemann curvature tensor of the induced metric on Σ is given by*

$$R^{\Sigma}_{ijkl} = R^{M}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk},$$

for $1 \leq i, j, k, l \leq n$.

Proposition 2.3 (Codazzi Equation)**.** *The following compatibility conditions hold*

$$\nabla_{e_i} h_{jk} - \nabla_{e_j} h_{ik} = R^{M}_{ijk(n+1)},$$

for $1 \leq i, j, k, l \leq n$.

The basic philosophy of this paper is to study the geometric properties of the ambient manifold M using minimal hypersurfaces. In particular, it will be important to understand how curvature properties of M influence the behavior of the minimal submanifolds in M.

Notice that the theory of minimal surfaces in R^{n+1} may be viewed as the "local theory". This is because the complete minimal submanifolds in R^{n+1} arise by rescaling a potential "blow-up" sequence in a manifold M^{n+1}, and then the global behavior of this submanifold reflects local properties of the original submanifolds.

A final remark here is that in applications, the minimal submanifolds we study will typically arise from a minimizing procedure, and hence will have a minimizing or stability property. We proceed to discuss the variational theory in more detail.

3 First and Second Variation

In this section we record the first and second variation formulae. Let $\Sigma^k \subset M^n$ be a submanifold. Let X be a vector field in M, and F_t be the flow generated by X. A standard computation (see [L]) gives the first and second variation formulae:

Lemma 3.1 (First Variation)**.**

$$\delta\Sigma(X) := \frac{d}{dt} Vol(F_t(\Sigma))_{t=0} = \int_{\Sigma} div_{\Sigma}(X) d\mu_{\Sigma}$$

where $div_{\Sigma}(X) := \sum_{i,j} g^{ij}\langle D_{e_i}X, e_j\rangle = -\langle X, H\rangle + div(X^T)$. Moreover if $X = 0$ on $\partial\Sigma$ then

$$\delta\Sigma(X) = -\int_{\Sigma} \langle X, H\rangle d\mu_{\Sigma}.$$

A submanifold with vanishing first variation for all compactly supported vector fields X is said to be *stationary*, and we see from the first variation formula that a smooth submanifold Σ is stationary if and only if it is minimal.

Lemma 3.2 (Second Variation)**.**

$$\begin{aligned}\frac{d^2}{dt^2}Vol(F_t(\Sigma))|_{t=0} &= \int_\Sigma \{\sum_{i=1}^k |D_{e_i}^\perp X|^2 + div_\Sigma(D_X X) \\ &+ \sum_{i=1}^k R^M(e_i, X, e_i, X) \\ &+ \sum_{i=1}^n \langle D_{e_i} X, e_i\rangle^2 - \sum_{i,j=1}^k \langle D_{e_i} X, e_j\rangle\langle D_{e_j} X, e_i\rangle\} d\mu_\Sigma.\end{aligned}$$

If Σ is a stationary hypersurface, then the second variation formula above has a simpler form. Assume the the normal bundle of Σ is trivial and let ν be a unit normal vector field. Define a vector field $X = \varphi(x)\nu$ where $\varphi(x)$ is a smooth function on Σ such that $\varphi = 0$ on $\partial\Sigma$. If we assume that Σ is minimal, then the second variation formula becomes:

Lemma 3.3 (Second variation for hypersurface)**.** *If $X = \varphi\nu$, then*

$$\delta^2\Sigma(X) = -\int_\Sigma \varphi L\varphi d\mu_\Sigma,$$

where $L\varphi := \triangle\varphi + (\|h\|^2 + Ric(\nu,\nu))\varphi$.

A minimal submanifold Σ is called *stable* if the second variation is non-negative, i.e. $\delta^2\Sigma(X) \geq 0$ for any vector field X with compact support on Σ. In the hypersurface case, stability is equivalent to the condition that $\lambda_1(-L,\Omega) \geq 0$ for any compact domain Ω in Σ, where λ_1 is the first Dirichlet eigenvalue.

4 Curvature Estimates and Compactness Theorems

In this section we state(without proof) some results concerning curvature estimate for minimal stable hypersurfaces. Let Σ^n be a minimal stable hypersurface in M^{n+1} and let h be the second fundamental form. The following estimate is proven in [S1].

Theorem 4.1 (Schoen)**.** *If $n = 2$ then $\|h(x)\| \leq c(M, d(x,\partial\Sigma))$. If M is $\mathbb{R}^3$, then $\|h(x)\| \leq c \cdot d(x,\partial\Sigma)^{-2}$ for an absolute constant c.*

In the cases $n = 3,4,5$ the curvature estimate holds with a constant depending also on the volume of Σ (see [SSY]).

Theorem 4.2 (Schoen-Simon-Yau). *If $n = 3, 4, 5$ and Σ is stable immersion, then*

$$\|h(x)\| \leq c(M, |\Sigma|, d(x, \partial\Sigma)),$$

where $|\Sigma|$ is the volume of Σ. If M is $\mathbb{R}^{n+1}$, then $\|h(x)\| \leq c \cdot d(x, \partial\Sigma)^{-2}$ for an absolute constant c. If $n = 6$ and Σ is a proper embedding, then the same estimate holds.

Curvature estimates of this type imply strong compactness theorems on large classes of stable hypersurfaces. The first theorem implies that the limit of immersed stable two dimensional minimal surfaces in a fixed three manifold is a stable minimal lamination (provided we stay a fixed distance from the boundaries). Because singularities are generally present even in area minimizing hypersurfaces for $n \geq 7$, we do not expect the same estimate to hold in general dimensions. The following compactness theorem holds in all dimensions (see [SS]).

Theorem 4.3 (Schoen-Simon). *For arbitrary n, let*

$$\mathcal{G} := \{\Sigma : \Sigma \text{ proper embedded stable minimal hypersurface in } M \text{ with } |\Sigma| \leq C\}.$$

Then any element in the closure of $\mathcal{G}$ has singular set of Hausdorff dimension no greater than $n - 7$.

It is an unsolved question whether Theorem 4.1 holds for $n = 3, 4, 5, 6$. It would be very interesting to have such an estimate. It is also unknown whether there is a version of Theorem 4.3 without the volume bound. Very recently Neshan Wickramasekera [W] in his Stanford PhD thesis has given a partial extension of Theorem 4.3 in the case that the hypersurfaces are immersed rather than embedded.

5 Geometry and Second Variation

The simplest example of the philosophy which is involved in the use of minimal submanifold theory to study curvature is the following theorem.

Theorem 5.1. *If $Ric_M > 0$, then there is no compact stable minimal hypersurface in M.*

Proof. Assume Σ is a compact stable minimal hypersurface in M. Let $\varphi \equiv 1$ in Lemma 3.3. The stability of Σ gives us that

$$\int_\Sigma [\|h\|^2 + Ric(\nu, \nu)] d\mu_\Sigma \leq 0,$$

which contradicts the fact that $Ric_M > 0$. □

If we combine this with the existence results (see [F]) which represent integral homology classes with area minimizing submanifolds, we get as a special case the vanishing of the n-th Betti number, $b_n(M^{n+1}) = 0$. This result is dual to Bochner's theorem on the vanishing of harmonic 1-forms under the positive Ricci curvature assumption.

Now we will study the case where only the scalar curvature is positive. In this setting the minimal hypersurface theory becomes much more powerful. Assume $R_M \geq 0$, and apply the Gauss Equation to express the intrinsic scalar curvature of Σ

$$R^{\Sigma}_{ijij} = R^{M}_{ijij} + h_{ii}h_{jj} - h^2_{ij}.$$

Summing for $1 \leq i, j \leq n$ we get

$$R^{\Sigma} = \sum_{i,j=1}^{n} R^{M}_{ijij} - |h|^2 = R^M - 2Ric(\nu, \nu) - |h|^2.$$

Therefore,

$$|h|^2 + Ric(\nu, \nu) = \frac{1}{2}|h|^2 + \frac{1}{2}R^M - \frac{1}{2}R^{\Sigma}$$

To illustrate the usefulness of this expression, consider the case $n = 2$ and let $\varphi \equiv 1$. The stability assumption then implies

$$\int_{\Sigma} R^{\Sigma} d\mu > 0,$$

which by the Gauss-Bonnet formula gives

$$\chi(\Sigma) > 0.$$

This gives us the following result (see [SY1]).

Theorem 5.2. *Let (M^3, g) be a Riemannian manifold with positive scalar curvature $R^M > 0$. Then any compact stable minimal surface in M is topologically a sphere.*

For arbitrary dimension, the corresponding result is that the induced metric on Σ is conformally equivalent to a metric of positive scalar curvature (see [SY2]). This makes it possible to inductively study the topological structure of manifolds of positive scalar curvature. There has been extensive work in this direction (see [GL1], [GL2], [SY2], [SY3], [St]) which uses both the stable hypersurface approach and the dual approach using harmonic spinors. One of the basic unresolved questions in this theory is whether a compact $K(\pi, 1)$ manifold of dimension four or more can carry a metric of positive scalar curvature. There is a more refined theory which has to do with the Yamabe invariants of compact manifolds (see [S2], [LB], [P] for a general discussion and some results).

There is a rather direct relationship between the scalar curvature theory and problems about the Einstein equations of General Relativity. We now describe this connection.

6 General Relativity

A spacetime in General Relativity is a Lorentz four dimensional manifold $(N^4, \bar{g})$, where $\bar{g}$ has signature type $(-,+,+,+)$. The evolution of the gravitational field $\bar{g}$ is then determined by the Einstein Equations

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor, $\bar{R}$ is the scalar curvature, and $T_{\mu\nu}$ is the stress-energy tensor of any matter fields which are present.

6.1 Dominant Energy Condition

An observer moves tangent to a timelike vector e_0, and if e_0, e_1, e_2, e_3 is a Lorentz frame, then the observed energy-momentum density is represented by the vector $\sum T_{0\mu}e_\mu$. The condition that this vector is forward pointing and timelike for every observer is the *dominant energy condition*:

$$T_{00} \geq \sqrt{\sum_{i=1}^{3} T_{0i}^2}.$$

Since the Einstein equations are an evolution equation of hyperbolic type, a spacetime is determined by initial data given on a three dimensional spacelike hypersurface in N. It is readily observed from the Gauss equations that a totally geodesic hypersurface in a Ricci flat manifold has vanishing scalar curvature; thus we expect such initial data to satisfy scalar curvature conditions. Using the Gauss and Codazzi equations together with the Einstein equations we first rewrite the dominant energy condition. Let M^3 be a space-like hypersurface, g be the induced metric and p be the second fundamental form. We then see that $\mu = T_{00}$ is given by

$$\mu := \frac{1}{16\pi}[R_M - \|p\|^2 + (Tr(p))^2],$$

and $J = \sum_{i=1}^{3} T_{0i}e_i$ is

$$J := \frac{1}{8\pi}[div(p) - \nabla Tr(p)]$$

where the covariant derivatives are taken in the induced geometry on M. Thus the dominant energy condition becomes:

$$\mu \geq \|J\|. \tag{1}$$

An important special case is $p = 0$, and in this case we see that (1) is equivalent to the condition that M has nonnegative scalar curvature $R(g) \geq 0$. In this paper we usually restrict our attention to this case, which we refer to as the Riemannian case.

6.2 Asymptotic Flatness

The condition that a spacetime be an isolated gravitating system with all other matter/gravity at infinite distance is called asymptotic flatness. An initial data set (M^3, g, p) is said to be *asymptotically flat* if there is a compact set $K \subset M$ such that $M^3 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1$(where B_1 is the unit ball in $\mathbb{R}^3$) and such that under this diffeomorphism, the metric and second fundamental form of $M^3 \setminus K$ can be written as

$$g_{ij} = \delta_{ij} + O(|x|^{-1}), \quad p_{ij} = O(|x|^{-2}).$$

One can define the total ADM mass (and linear momentum) for an asymptotically flat initial data set (see for example [HE]). We first consider an important example of an asymptotically flat manifold.

6.3 Schwarzschild Initial Data

The Schwarzschild manifold is $(\mathbb{R}^3 \setminus \{0\}, s)$, where s is a metric given by $s_{ij} = (1 + m/2r)^4 \delta_{ij}$ and m is a positive constant and is by definition the total mass of the manifold. This manifold has zero scalar curvature everywhere and hence defines initial data for the vacuum Einstein equations (see [HE] for a discussion of this spacetime). Physically the Schwarzschild corresponds to the gravitational field exterior to a rotationally symmetric black hole of mass m. For simplicity, in our general discussion of asymptotic flatness, we can think of asymptotically flat manifolds as being asymptotic to a Schwarzschild manifold, i.e. we can write the asymptotic condition on g as

$$g_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij} + O(\frac{1}{r^2}).$$

That this is no loss of genrality in discussing the ADM mass is shown in [?]. We can then simply define the ADM mass of the manifold to be m. (More generally, the ADM mass is defined as a certain boundary integral over large coordinate spheres).

7 Positive Mass Theorem

In this section we give a brief discussion of the Positive Mass Theorem both in the Riemannian case and the general case.

7.1 Positive Mass Theorem(Riemannian Case)

With the terminology we have set up, we may state the Positive Mass Theorem (Riemannian case) which was first proven by Schoen and Yau [SY4] using minimal surface theory, and later by Witten [W], [PT] using the theory of the Dirac operator:

Theorem 7.1 (Positive Mass Theorem(Riemannian Case))**.** *Let (M^3, g) be a complete, smooth asymptotically flat manifold with nonnegative scalar curvature and total mass m. Then*

$$m \geq 0,$$

with equality if and only if (M^3, g) is isometric to $\mathbb{R}^3$ with the standard Euclidean metric.

Now we outline the idea of the minimal surface proof of this theorem:

Step 1. We may normalize the asymptotic data so that g is conformal to the Euclidean metric near infinity. More precisely $g_{ij} = u^4\delta_{ij}$ near ∞ where u is a function such that

$$\triangle u \leq 0, \quad u = 1 + \frac{m}{2|x|} + O(|x|^{-2}).$$

Step 2. Assuming $m < 0$, then $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_3| \leq C\}$ is a mean convex region for large C. This is a direct calculation using the asymptotic assumption on g. It follows that there exists a stable minimal surface Σ asymptotic to a plane $x_3 = a$ for some $a \in (-C, C)$. This follows from solving the Plateau problem with boundary a large circle in the x_1x_2 plane, and using the mean convexity obtained in the previous step to show that we may let the radius of this circle go to infinity and find a limit of these area minimizing surfaces which is a stable minimal surface which is asymptotic to a horizontal plane.

Step 3. We show that this is inconsistent with the stability of Σ and the non-negativity of the scalar curvature; in fact, stability of Σ gives (we have to justify choosing $\varphi = 1$)

$$\int_\Sigma (2^{-1}\|h\|^2 + 2^{-1}R_M)da \leq \int_\Sigma K_\Sigma da, \tag{2}$$

where K_Σ is the Gauss curvature of the surface Σ. Then because of the fact that Σ is asymptotic to a plane, the Gauss-Bonnet formula gives

$$2^{-1}\int_\Sigma K_\Sigma da = 2\pi\chi(\Sigma) - 2\pi \leq 0.$$

On the other hand by a conformal argument we may assume the scalar curvature of M is positive, i.e. $R_M > 0$. Then (2) implies that

$$2^{-1}\int_\Sigma K_\Sigma da > 0,$$

a contradiction.

7.2 Positive Mass Theorem(General Case)

We discuss briefly an unpublished proof due to the author of the general case of the Positive Mass Theorem. This proof follows along the same lines as the

Riemannian case. The original proof [SY5] is more complicated, but the development there has other useful applications particularly to the existence of apparent horizons assuming largeness conditions on the initial data ([SY6],)

Trapped Surfaces and the horizon equation: Let S be a 2 dimensional surface in an initial data set M^3 with induced metric g and second fundamental form p. Let e_0, e_1, e_2 and e_3 be a Lorentz frame. Let ν be the outward unit normal vector field to S in M. Then $\nu \pm e_0$ are orthogonal null vectors. We say that S is *trapped for* $\nu \pm e_0$ if

$$H_S \pm Tr_S(p) < 0.$$

Then the apparent horizon equation for S is the marginally trapped condition

$$H_S \pm Tr_S(p) = 0. \tag{3}$$

The general positive mass theorem says that the total energy-momentum vector (E, P_1, P_2, P_3) is a forward pointing time-like vector, i.e.

$$E \geq \sqrt{\sum_{i=1}^{3} P_i^2},$$

where E is the total energy and $P = (P_1, P_2, P_3)$ is the total linear momentum (defined precisely below). We outline an analogous proof as for the Riemannian case.

Step 1. Without loss of generality we may assume the following asymptotic condition for g and p:

$$g_{ij} = (1 + \frac{E}{2|x|})^4 \delta_{ij} + O(|x|^{-2})$$
$$p_{ij} = \mathcal{L}(2\|P\| \cdot |x|^{-1} \frac{\partial}{\partial x_3}) + O(|x|^{-3}),$$

where P is the total linear momentum vector given by

$$P_i = \frac{1}{8\pi} \oint_{S_\infty} [(P_{ij}\nu^j - (TrP)\nu^i] dA,$$

and E is the total energy given by

$$E = \frac{1}{16\pi} \oint_{S_\infty} (g_{ij,j} - g_{jj,i})\nu_i dA,$$

and $\mathcal{L}$ is an operator defined by

$$\mathcal{L}X := L_X g - 2^{-1} div(X).$$

Step 2. Assume for the sake of contradiction that $E < |P|$. Then, by a calculation based on the asymptotics above, we have $H + Tr(p) < 0$ on the boundary of the region $|x_3| < C$ for C large. Therefore this region is trapped, and there exists an apparent horizon Σ which is asymptotic to a plane $x_3 = a$ and with $H_\Sigma = -Tr(p)$. (The existence here is complicated by the fact that the horizon equation is not a variational equation.)

Step 3. The solution which is constructed in the previous step satisfies the condition that its linearized operator has nonnegative first eigenvalue on compact sets, and an analagous stability-type argument produces a contradiction.

8 The Penrose Inequality(Riemannian Case)

We will call an outermost minimal sphere in an asymptotically flat manifold M^3 with nonnegatvie scalar curvature a *horizon*. We see that the Schwarzschild manifold has one horizon and an explicit calculation shows that the surface area A of the horizon satisfies:

$$m = \sqrt{\frac{A}{16\pi}}.$$

In general, the existence theory for minimal surfaces implies that there exists a finite collection of outermost minimal spheres so that if we remove the regions interior to them, then the manifold becomes diffeomorphis to $\mathbb{R}^3$ with a finite number of disjoint balls removed. Thus we may assume that our initial data set M is topologically the exterior of balls, and that the boundary spheres are minimal. We may furthermore assume that there are no compact minimal surfaces in the interior of M (the existence of one would produce another outermost minimal sphere). Now the Penrose inequality asserts that the total area of these outermost minimal spheres is bounded above by the horizon area for a Schwarzschild metric with the same total mass.

Theorem 8.1 (Penrose Inequality)**.** *Let (M^3, g) be a complete, smooth asymptotically flat manifold with nonnegative scalar curvature and total mass m whose outermost minimal spheres have total surface area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}},$$

with equality if and only if (M^3, g) is isometric to the Schwarzschild metric $(\mathbb{R}^3 \setminus \{0\}, s)$ of mass m outside their respective horizons.

The Penrose Inequality was first conjectured by Penrose [Pen] in 1973 and was proven in a slight weaker form by Huisken and Ilmanen [HI] in 1997 using inverse mean curvature flow and in full by Bray [B] in 1998 using a totally different method. (Gibbons, Tod, Bartnik, Herzlich and Bray had obtained earlier partial results). We note that the general case of the Penrose Inequality (for arbitrary initial data sets) is still open.

8.1 The Geroch and Jang/Wald Approach

We first describe the Hawking mass which for certain surfaces in M gives a reasonable definition of the gravitational mass enclosed by that surface.

Definition 8.2 (Hawking Mass). *Let* $\Sigma^2 \subset M^3$. *The* Hawking mass of Σ , $m_H(\Sigma)$ *is defined to be*

$$m_H(\Sigma) := \sqrt{\frac{Area(\Sigma)}{16\pi}}(1 - \frac{1}{16\pi}\int_\Sigma H^2 da).$$

In the Schwarzschild manifold, it is easily seen that the Hawking mass of any sphere S_r is equal to the mass m, and for general asymptotically flat M

$$\lim_{r\to\infty} m_H(S_r) = m$$

where S_r denotes a coordinate sphere of large radius. A calculation (see [BS]) shows that if g is rotationally symmetric and the scalar curvature $R_M \geq 0$, then $m_H(S_r)$ is an increasing function of r.

Geroch [G] showed that if M^3 has nonnegative scalar curvature, then the Hawking mass of Σ is nondecreasing when the surface Σ flows outward at a speed equal to the inverse of the mean curvature. More precisely assume Σ_t is a family of *connected* surfaces evolving by the equation

$$\frac{\partial x}{\partial t} = \frac{1}{H}\nu(x), \tag{4}$$

where H is the mean curvature of Σ_t and ν is the unit vector which is opposite to the mean curvature direction. Geroch then derived the important monotonicity property

$$\frac{d}{dt}m_H(\Sigma_t) \geq 0.$$

Using this monotonicity result, Jang and Wald [JW] gave a formal proof of the Penrose inequality in case there is a single outermost minimal sphere. Their formal argument supposes that Σ_0 is an outermost minimal sphere. Assume that the inverse mean curvature flow equation (4) with initial data Σ_0 has a family of smooth solution Σ_t for $0 \leq t < \infty$ such that for large t, the surface Σ_t is asymptotic to a coordinate sphere with large radius. It then follows that

$$\lim_{t\to\infty} m_H(\Sigma_t) = m.$$

On the other hand, Geroch monotonicity implies that

$$m_H(\Sigma_0) = \sqrt{\frac{Area(\Sigma_0)}{16\pi}} \leq m_H(\Sigma_t).$$

Hence we get

$$m \geq \sqrt{\frac{A}{16\pi}}$$

which gives the Penrose inequality in the case when M has only one outermost minimal sphere.

The main difficulty in making this formal argument rigorous is to prove the existence and regularity of the inverse mean curvature flow. In general singularities occur in this flow, and these must be understood in order to have a hope of giving a rigorous proof. Huisken and Ilmanen succeeded in rigorizing this argument by constructing an appropriate weak solution of the inverse mean curvature flow. We now give an outline of their argument.

9 The Huisken/Ilmanen Approach

To understand the difficulties which may arise, we first analyze some special cases for the inverse mean curvature flow.

Example 1. The initial surface is a coordinate sphere in $\mathbb{R}^3$, i.e. $\Sigma_0 = S_{r_0}$ for some r_0. Then we have

$$\frac{d}{dt}(Area(\Sigma_t)) = Area(\Sigma_t).$$

So we have

$$\Sigma_t = S_{e^{t/2}r_0}.$$

Thus in this example the flow exists and has the desired behavior. We next consider two examples for which the flow becomes singular.

Example 2. The initial surface Σ_0 is a "thin" torus of revolution. In this case, the mean curvature vector points into the solid torus and Σ_t exists for a short time, but at some finite time we will have $\min_{\Sigma_t} H \to 0$ and $1/H$ goes to infinity so the flow does not make sense after that. Indeed, there must be a topology change if the large time flow is to resemble large spheres.

Example 3. The initial surface is the disjoint union of two spheres, i.e. $\Sigma_0 = S_{r_1}(P) \cup S_{r_2}(Q)$. In this case, the classical flow Σ_t must develop self intersections in a finite time.

From the examples above, we see that in order for a flow to exist for all t, we must allow jumps and changes in topology.

9.1 The Level Set Approach

To construct a solution of the inverse mean curvature flow, Huisken and Ilmanen rewrite the flow as an equation for the level sets of a function. Let $u(x)$ be a function such that $\Sigma_t := \{x : u(x) = t\}$ is smooth for all t. By direct calculation we see that if Σ_t is a solution to the inverse mean curvature flow, then

$$div(\frac{\nabla u}{|\nabla u|}) = |\nabla u| \tag{5}$$

Remark 9.1. *The level set formulation allows jumps in a natural way since if u is constant on an open set Ω, then the level sets "jump" across Ω.*

Definition 9.2. *$\Sigma^2 \subset M^3$ is called outer minimizing if $Area(\Sigma) \leq Area(\Sigma_1)$ for any surface Σ_1 enclosing Σ. Σ is called strictly outer minimizing if equality holds if and only if $\Sigma = \Sigma_1$.*

The following illustrates the connection between the inverse mean curvature flow and the outer minimizing property.

Lemma 9.3. *If $u(x)$ is a solution to the inverse mean curvature flow equation (5), then for any $t > 0$, the level set Σ_t is outer minimizing.*

Proof. Let Σ be any surface enclosing Σ_t and Ω is the region between Σ and Σ_t. Integration by parts gives

$$\int_\Omega div(\frac{\nabla u}{|\nabla u|})dv = \int_\Sigma \nu \cdot \frac{\nabla u}{|\nabla u|} da - Area(\Sigma_t).$$

On the other hand (5) implies that

$$div(\frac{\nabla u}{|\nabla u|}) = |\nabla u| \geq 0.$$

Thus we conclude that

$$Area(\Sigma_t) \leq Area(\Sigma).$$

(note: equality implies that $u \equiv t$ on Ω). □

The existence of outer minimizing sets follows from the Plateau theory.

Lemma 9.4. *For any $\Sigma \subset M$, there exists a unique smallest strictly outer minimizing surface $\hat{\Sigma}$. We will call $\hat{\Sigma}$ the strict minimizing hull of Σ.*

Remark 9.5. *Although Σ might be enclosed by more than one outer minimizing surface, the strictly minimizing hull is the maximal such surface.*

Assume $u(x)$ is a global solution of the inverse mean curvature flow equation (5). Let

$$\Omega_t := \{x : u(x) < t\}, \quad \Omega_t^+ := \{x : u(x) \leq t\}.$$

Then $\Sigma_t = \partial\Omega_t$ is outer minimizing and $\Sigma_t^+ = \partial\Omega_t^+$ is strictly outer minimizing. In fact Σ_t^+ is the strict minimizing hull of Σ_t. The surfaces Σ_t and Σ_t^+ differ precisely at jumps, i.e. when $u(x) = t$ on a set of positive measure.

9.2 Heuristic Description of the Huisken-Ilmanen Flow

A valid way to think of the Huisken/Ilmanen modified inverse mean curvature flow is as follows: If $\hat{\Sigma_t} = \Sigma_t$, then continue with the classical inverse mean curvature flow, but at any instant for which $\hat{\Sigma_t} \neq \Sigma_t$, jump to $\hat{\Sigma_t}$ and continue with the classical inverse mean curvature flow. The rigorous construction is different from this, but it captures the main idea, and explains the examples given above.

Example 2. The "thin" torus: After some time t_0 before the mean curvature goes to zero the surface Σ_{t_0} will cease to be outer minimizing. At this time the surface Σ_{t_0} will jump to its strict minimizing hull which will be a sphere, and from then on the flow will be smooth and asymptotic to large coordinate spheres.

Example 3. The disjoint union of spheres: A similar phenomenon will occur, and after time t_0, Σ_{t_0} becomes a single sphere enclosing the two.

Lemma 9.6 (Monotonicity of Hawking Mass). *The Hawking mass $m_H(\Sigma_t)$ is increasing for the Huisken-Ilmanen flow.*

Proof. This is a rough sketch. It suffices to show that the Hawking mass increases at the jumps. Assume $\hat{\Sigma_t} \neq \Sigma_t$, then we see $Area(\Sigma_t) = Area(\hat{\Sigma})$, $\hat{H} = H$ on $\Sigma_t \cap \hat{\Sigma_t}$ and $\hat{H} = 0$ on $\hat{\Sigma_t} \setminus \Sigma_t$. Therefore

$$\int_{\hat{\Sigma}} \hat{H}^2 da \leq \int_{\Sigma} H^2 da.$$

This implies

$$m_H(\hat{\Sigma_t}) \geq m_H(\Sigma_t).$$

$\square$

A major problem with the Huisken-Ilmanen flow is how to rigorize the construction, (e.g. the jump times are not known to be discrete). We give a hint as to how that is done.

9.3 Elliptic Regularization

Consider the following perturbed version of the inverse mean curvature flow equation (5)

$$div(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}}) = \sqrt{|\nabla u|^2 + \varepsilon^2}. \tag{6}$$

We notice that the perturbed equation (6) is a non-degenerate elliptic equation and it has the following geometric meaning. If $u(x)$ is a solution to the perturbed equation (6), then

$$G_t(x) := graph(\frac{u(x)}{\varepsilon} - \frac{t}{\varepsilon}) \subset M \times \mathbb{R}$$

is a solution to the unperturbed equation (5).

We now sketch Huisken-Ilmanen's proof of a version of the Penrose Inequality.

Step 1. There is a smooth solution u_ε of the perturbed equation (6) satisfying $u_\varepsilon = 0$ on the initial outermost minimal sphere Σ_0 and u_ε has suitable behavior near infinity.

Step 2. The $\{u_\varepsilon\}$ above satisfy local uniform Lipschitz bounds and hence a subsequence converges to some function u uniformly.

Step 3. The limit function u is a Lipschitz weak solution of the unperturbed equation (5) in the sense that

$$\int_\Omega (|\nabla u| + u|\nabla u|) d\mu \leq \int_\Omega (|\nabla v| + v|\nabla u|) d\mu,$$

for any v such that $v - u$ has compact support in Ω for any compact set Ω in M. Moreover there can be no other Lipschitz weak solution whose zero set is Σ_0.

Step 4. The surface Σ_t is connected and is $C^{1,\alpha}$ for any $t > 0$, and Σ_t becomes asymptotic to a coordinate sphere for t large. Therefore by Lemma 9.6, $m_H(\Sigma_t)$ can be defined and is monotone increasing in t. Using a similar argument as for the formal prrof of Jang/Wald, we can show that the Riemannian Penrose Inequality holds assuming there is one outermost minimal sphere. More generally the statement is as follows.

Theorem 9.7 (Huisken-Ilmanen)**.** *If M^3 is a complete, asymptotically flat Riemannian manifold with nonnegative scalar curvature, with total mass m and with outermost minimal spheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$. Then*

$$m \geq \sqrt{\frac{A_{max}}{16\pi}},$$

where

$$A_{max} := \max\{Area(\Sigma_1), \ldots, Area(\Sigma_k)\}.$$

Moreover, equality holds if and only if the region of M outside $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ is isometric to the exterior Schwarzschild metric(in particular, $k = 1$).

10 Bray's Approach

After the Huisken/Ilmanen was written, H Bray [B] found a clever argument to use the Positive Mass Theorem [SY4] to prove the full version of the Penrose Inequality(Riemannian case). In this section we will discuss his approach.

We first may simplify the assumptions. By Schoen-Yau's argument in the proof of the Positive Mass Theorem [SY7] we can assume $R_g \equiv 0$ and

$$g_{ij} = u^4 \delta_{ij}$$

outside a compact set, where u is a function satisfies:

$$\begin{cases} \triangle_g u = 0, \\ u(x) = 1 + \frac{m}{2|x|} + O(|x|^{-2}). \end{cases}$$

Bray defines a continuous family of conformal metrics $\{g_t\}$ on M^3, where

$$g_t = u_t^4 g_0,$$

for some suitable function $u_t(x)$(described later) and $u_0(x) \equiv 1$. Given the metric g_t, define

$$\Sigma(t) := \text{the outer minimal area enclosure of } \Sigma_0 \text{ in } (M^3, g),$$

where Σ_0 is the union of the original outer-minimizing spheres in (M^3, g_0). The time rate of change of u_t is given by v_t where v_t satisfies

$$\begin{cases} \triangle_{g_0} v_t = 0, & \text{outside } \Sigma_t, \\ v_t = 0, & \text{on } \Sigma(t), \\ \lim_{x\to\infty} v_t(x) = -e^{-t}, & \end{cases} \tag{7}$$

and set $v_t \equiv 0$ inside $\Sigma(t)$. Then we set $\frac{d}{dt}(u_t) = v_t$, or,

$$u_t(x) = 1 + \int_0^t v_s(x)ds.$$

Theorem 10.1. *With the notation above, there exists a solution $u_t(x)$ and Σ_t for any $t \geq 0$ such that $u_t(x)$ is Lipschitz in t, C^1 in x globally and C^∞ in x outside $\Sigma(t)$. Moreover $\Sigma(t_1) \cap \Sigma(t_2) = \varnothing$ for $t_1 \neq t_2$*

The monotonicity associated with this flow is given as follows.

Proposition 10.2. *The mass $m(t)$ of (M^3, g_t) is nonincreasing.*

Proof. Since the flow has an autonomous character, it suffices to show $m'(0) \leq 0$. Assume

$$u_t(x) = a(t) + \frac{b(t)}{2|x|} + O(|x|^{-2}). \tag{8}$$

Note that $g_t = u_t^4 g_0$ is asymptotically flat with the expansion

$$g_t = [(a(t) + \frac{b(t)}{2|x|})^4 (1 + \frac{m(0)}{2|x|})^4]\delta_{ij} + O(|x|^{-2}).$$

Computing, we get

$$m(t) = a(t)(b(t) + m(0)b(t)).$$

From the definition of v_t in (7) we find

$$v_0 = -1 + \frac{C_0}{2|x|} + O(|x|^{-2}), \tag{9}$$

where C_0 is the Newtonian capacity defined by

$$C_0 = \inf\{\frac{1}{2\pi} \int_{M \setminus \Sigma(0)} |\nabla\varphi|^2 : \varphi = 0 \text{ on } \Sigma(0) \text{ and } \varphi = 1 \text{ at } \infty\}.$$

Now since $u_0 \equiv 1$ and $\frac{d}{dt}(u_t) = v_t$, comparing (8) and (9), we see

$$\begin{aligned} a(0) = 1, &\quad \dot{a}(0) = -1, \\ b(0) = 0, &\quad \dot{b}(0) = C_0. \end{aligned}$$

Thus

$$m'(0) = C_0 - 2m(0).$$

We now prove the following inequality which will complete the proof of monotonicity. $\square$

Proposition 10.3. $m(0) \geq 2^{-1} C_0$.

Proof. This will come by applying the positive mass theorem. We double M by reflection along Σ_0 and extend v_0 to the doubled manifold by odd reflection. Define

$$\tilde{g}_0 = (2^{-1}(1 - v_0))^4 g_0.$$

We can then show that the mass $\tilde{m}$ of $\tilde{g}$ is in fact $m(0) - 2^{-1}C_0$. The positive mass theorem therefore gives

$$\tilde{m} = m(0) - 2^{-1}C_0 \geq 0$$

which is the desired conclusion. $\square$

We may now complete a sketch of Bray's proof of the Penrose Inequality: By Theorem 10.1, we see that Σ_{t_2} strictly encloses Σ_{t_1} for $t_2 > t_1 \geq 0$. Also we can prove that $A(t) :=$ total area of Σ_t is constant, i.e. $A(t) \equiv A_0$ for any $t \geq 0$, and we have shown that the mass $m(t)$ is nonincreasing for $t \geq 0$. It is shown by Bray that $(M \setminus \Sigma(t), g_t)$ converges to a Schwarzschild metric. This, together with Proposition 10.2, implies that

$$m = m(0) \geq \lim_{t \to \infty} m(t) = m_{sch} = \sqrt{\frac{A}{16\pi}}.$$

which gives the Penrose Inequality.

11 Higher Codimensions

The minimal submanifold theory used up to now has been all in codimension one. We now consider some applications of minimal submanifolds with higher codimension. One of the main motivations for many of the phenomena in the subject is the Bernstein Theorem:

Theorem 11.1 (Bernstein Theorem)**.** *Any entire minimal graph in $\mathbb{R}^3$ is a plane.*

While the original proof involved a PDE type argument the theorem has been generalized in more geometric ways. Osserman [O] gave a generalization using the fact that Gauss map is conformal, and replaced the graphical assumption with the assumption that the Gauss image omit a sufficiently large set on the two sphere. This theory was improved dramatically by Fujimoto [Fu]. A great achievement for the codimension one theory was to prove the Bernstein theorem for entire minimal graphs of dimension seven or less, and to find counterexamples in higher dimensions (see [F] for an account of this). Schoen, Simon and Yau [SSY] proved this result in dimensions up to 6 for complete stable minimal hypersurfaces with a volume bound using curvature estimate.

It is natural to ask if there is an analogue of the Bernstein theorem for higher codimension minimal submanifolds. The theory of Osserman concerning the size of the omitted set for complete minimal surfaces was generalized to higher codimension (see [CO]). As a first guess, one might expect that entire graphs could be holomorphic with respect to an orthogonal complex structure on $\mathbb{R}^n$. That this is not the case was shown by Osserman [O]. One may then ask what is the suitable global hypothesis for a minimal surface in higher codimensions to be holomophic. Only partial results of this type are known.

It turns out that in various higher dimensions and codimensions there are classes of submanifolds which satisfy first order reductions of the minimal submanifold equation and are automatically volume minimizing like the holomorphic submanifolds. We now give a general discussion of these calibrated submanifolds.

12 Calibrations

To construct volume minimizing submanifolds, Harvey and Lawson in their paper [HL] defined the concept of a calibration.

Definition 12.1. *A k-form α on a Riemanian manifold (M^n, g) is called a calibration if*

1. $d\alpha = 0$;
2. $|\alpha(\pi_x)| \leq 1$, *for any k-dimensional subspace π_x in T_xM and any $x \in M$.*

Definition 12.2. *Let α be a calibrating k-form on M. A k-dimensional submanifold Σ^k is said to be calibrated by α if $\alpha(T_x\Sigma) = 1$ for any $x \in \Sigma$. In other words, Σ is calibrated by α if the restriction of α to Σ is the volume form on Σ.*

The most important result about calibrated submanifolds is the following minimizing property for calibrated submanifolds.

Proposition 12.3. *Let α be a calibrating k-form on M and let Σ^k be a submanifold calibrated by α. Then Σ is volume minimizing in its (relative) integral homology class, i.e.*

$$|\Sigma| = \inf\{|\Sigma_1| : \Sigma_1 \textit{ is homologous to} \Sigma,\ \partial\Sigma_1 = \partial\Sigma\}$$

where $|\cdot|$ denotes the volume.

Proof. Since Σ and Σ_1 are homologous (with $\partial\Sigma = \partial\Sigma_1$), we assume $\Sigma - \Sigma_1 = \partial C$, where C is a $(k+1)-$ dimensional chain. Then Stokes' Theorem gives

$$\int_{\Sigma-\Sigma_1} \alpha = \int_{\partial C} \alpha = \int_C d\alpha = 0.$$

On the other hand since Σ is calibrated by α, we see that

$$|\Sigma| = \int_\Sigma \alpha = \int_{\Sigma_1} \alpha \leq |\Sigma_1|,$$

where, in the last inequality, we used property (2) of the definition of calibrating form. □

Complex submanifolds in R^{2m} can be viewed as calibrated submanifolds. A complex structure on R^{2m} is a linear isomorphism $J : R^{2m} \to R^{2m}$ such that $J^2 = -I$, where I is the identity map. A complex structure J is said to be compatible with the Euclidean metric if J is also an isometry, i.e. $J(v) \cdot J(w) = v \cdot w$, where $\cdot$ is the Euclidean inner product of R^{2m}. For a compatible complex structure J, we define the *Kähler form(or symplectic form)* ω by $\omega(v, w) := J(v) \cdot w$. A $2k$-dimensional subspace V^{2k} in R^{2m} is called *complex with respect to* J if it is invariant under J, i.e. $J(V) = V$. A $2k$-dimensional submanifold Σ^{2k} is called *complex(or holomorphic) with respect to* J if $T_x\Sigma$ is complex for every $x \in \Sigma$.

The following Wirtinger Inequality(for proof, see [L]) implies that $\omega^k/k!$ is a calibrating form.

Lemma 12.4 (Wirtinger Inequality)**.** *Let ω be the Kähler form for a compatible complex structure J in R^{2m}. Define $\alpha := \omega^k/k!$, where $\omega^k = \omega \wedge \ldots \wedge \omega$. Then $|\alpha(V)| \leq 1$ and $\alpha(V) = 1$ if and only if V is complex.*

An immediate consequence of the Wirtinger Inequality is

Corollary 12.5. *The form α defined in the above lemma is a calibrating form which calibrates complex submanifolds. Therefore any complex submanifold is volume minimizing in its relative homology class (i.e. for its boundary).*

We now give the standard complex structure on R^{2m}. Let $x_1, \ldots x_m, y_1, \ldots y_m$ be the standard coordinates in R^{2m}. Define J by

$$J(\frac{\partial}{\partial x_j}) := \frac{\partial}{\partial y_j}, \quad J(\frac{\partial}{\partial y_j}) := -\frac{\partial}{\partial x_j}.$$

Therefore the Kähler form $\omega = \sum_{j=1}^{m} dx_j \wedge dy_j$. Let $z_j = x_j + \sqrt{-1}y_j$ be the standard complex coordinates.

Now consider a graph over the $z_1, \ldots z_k$ plane given by

$$z_\alpha = f_\alpha(z_1, \ldots z_k), \quad \alpha = k+1, \ldots, m.$$

It is easy to check that the graph is holomorphic if $\frac{\partial f_\alpha}{\partial \bar{z}_j} = 0$ for $j = 1, \ldots k$. From the above discussion we see that the graph of f is a volume minimizing submanifold.

Corollary 12.5 says that every complex submanifold is volume minimizing; however, there has been very little success is showing that volume minimizing submanifolds are complex even in situations where one may expect them to be. There are a few results in dimension two.

Theorem 12.6 (Siu-Yau [SiY])**.** *Let M^{2m} be a Kähler manifold with positive bisectional curvature. Any stable minimal 2-sphere is either holomorphic or anti-holomorphic.*

Theorem 12.7 (Micallef [M])**.** *In $\mathbb{R}^4$, any 2-dimensional complete stable minimal surfaces with quadratic area growth is holomorphic for some compatible complex structure J. Also any 2-dimensional entire stable minimal graph in $\mathbb{R}^4$ is holomorphic for some compatible complex structure.*

Theorem 12.8 (Micallef [M])**.** *In R^{2m}, any 2-dimensional stable minimal surface with genus zero and with finite total curvature is holomorphic.*

Remark 12.9. *Recently, Arezzo and Micallef [AM] gave examples for n large(near 20) of stable minimal Σ with genus one and finite total curvature which are not holomorphic.*

There is also the following existence result for holomorphic disks which is important in symplectic geometry. The proof uses the $\bar{\partial}$ operator directly, and up to this time there is no proof which uses area minimization.

Definition 12.10. *A submanifold Σ^k in R^{2k} is called Lagrangian if $\omega|_\Sigma = 0$, where ω is the Kähler form.*

Theorem 12.11 (Gromov [Gr1])**.** *Let Σ^k be a compact embedded Lagrangian submanifold in R^{2k}. Then there exists a holomorphic 2-disk D such that $\partial D \subset \Sigma$.*

12.1 The Special Lagrangian Calibration

Another important calibrating form is the special lagrangian form, which was originally defined in Harvey and Lawson's paper [HL]. This form and the associated class of special lagrangian submanifolds can be defined generally on a Calabi-Yau manifold. For simplicity we discuss here the flat case. In $\mathbb{R}^{2m}$ let $z_1, \ldots, z_m$ be the standard complex coordinates. Define a real m-form $\alpha := Re(dz_1 \wedge \ldots \wedge dz_m)$. Standard linear algebra shows that $|\alpha(V)| \leq 1$ for any m-dimensional subspace V(see [HL] for a detailed proof). Thus we have the following result.

Lemma 12.12. *$\alpha := Re(dz_1 \wedge \ldots \wedge dz_m)$ is a calibrating m-form in $\mathbb{R}^{2m}$.*

Definition 12.13. *An m-subspace V in $\mathbb{R}^{2m}$ is called special Lagrangian if it is calibrated by α.*

We have the following characterization of special Lagrangian subspace. We refer the reader to [HL] for a proof.

Theorem 12.14. *Let $\alpha = Re(dz^1 \wedge dz^2 \wedge \ldots \wedge dz^m)$ and $\mu = Im(dz^1 \wedge dz^2 \wedge \ldots \wedge dz^m)$ in $\mathbb{C}^m$. Then α is a calibrating m-form. Moreover, if P is an oriented m-plane in T_xC^m, the following statements are equivalent:*

(a) P is special Lagrangian;

(b) $\mu(P) = 0$ and P is Lagrangian;

(c) There is a linear map $A \in SU(m)$ such that A maps the x-plane (the m-plane in $\mathbb{C}^m$ spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$) onto P.

Remark 12.15. *If P is any lagrangian subspace, then we see that $|dz(P)| = 1$. Thus we can write $dz(P) = e^{\sqrt{-1}\beta}$ for some angle β(called lagrangian angle) which is well-defined mod 2π. Therefore we see that P is special Lagrangian if and only if P is lagragian and $\beta = 0$.*

This leads us to the definition of special lagrangian submanifolds. These bear certain formal analogies to the class of holomorphic submanifolds.

Definition 12.16. *A submanifold $\Sigma^m \subset \mathbb{R}^{2m}$ is called special lagragian if each tangent plane of Σ is special lagrangian.*

Note that there is an S^1 family of calibrating forms in $\mathbb{C}^n$ associated with the form dz. Let

$$\alpha_\theta = Re(e^{\sqrt{-1}\theta} dz^1 \wedge dz^2 \wedge \ldots \wedge dz^m), \tag{10}$$

and

$$\mu_\theta = Im(e^{\sqrt{-1}\theta} dz^1 \wedge dz^2 \wedge \ldots \wedge dz^m). \tag{11}$$

As above, the form α_θ is a calibrating form and there is an associated class of submanifolds calibrated by α_θ. We will refer to these as *special lagrangian with respect to α_θ*. One of the reasons we are interested in this S^1 family of special Lagrangian geometry is the following theorem (see [HL] or the discussion in the next section).

Theorem 12.17. *Let Σ be a smooth submanifold of real dimension m in $\mathbb{R}^{2m}$. Then Σ is both minimal and lagrangian if and only if Σ is special lagrangian with respect to α_θ for some choice of θ.*

13 Existence Theory for Special Lagrangian Submanifolds

The existence of special lagrangian submanifolds is of great interest in both geometry and string theory. Theorem 12.17 reduces this problem to the existence of minimal lagrangian submanifolds. To construct such submanifolds, one idea is to minimize volume among lagrangian competitors. The auther's joint work with Jon Wolfson [SW2] explores this variational approach in detail and gives an approach to the construction of a smooth minimal lagrangian submanifold in each lagrangian homology class in a 4-dimension Kähler-Einstein manifold. Here we provide some background and elementary properties related to this approach.

13.1 Hamiltonian Stationary Submanifolds

To solve this lagrangian variational problem, we need to find suitable deformation which preserve the lagrangian condition. Ambient symplectic deformations (i.e. deformations of $\mathbb{R}^{2m}$ which preserve the symplectic form $\omega = \sum_{j=1}^m dx_j \wedge dy_j$) are good candidates. One family of such deformations consists of the hamiltonian deformations which arise from a smooth ambient function.

Definition 13.1. *Let $h(x_1, \dots x_m, y_1, \dots y_m)$ be a smooth function on R^{2m} with compact support. The hamiltonian vector field associated to h is defined to be*

$$X_h := J\nabla h = \sum_{j=1}^{m} \{ \frac{\partial h}{\partial x_j} \frac{\partial}{\partial y_j} - \frac{\partial h}{\partial y_j} \frac{\partial}{\partial x_j} \}$$

It is not difficult to check that the flow generated by any hamiltonian vector field preserves the symplectic form. Therefore the image of any lagrangian submanifold under a hamiltonian deformation is lagrangian.

Definition 13.2. *A lagrangian submanifold Σ is called hamiltonian stationary if the first variation of Σ under any compactly supported hamiltonian deformation is zero, i.e*

$$\delta|\Sigma|(X_h) = 0$$

for any smooth function h on R^{2m} whose restriction to Σ has compact support.

The following proposition says that the mean curvature vector of any lagrangian submanifold is a (multi-valued) hamiltonion vector field (see [HL] or [SW1] for a proof).

Proposition 13.3. *Let $\Sigma^m \subset R^{2m}$ be a Lagrangian submanifold. We then have $dz(T_x\Sigma) = e^{\sqrt{-1}\beta}$ for some angle β (the lagrangian angle) defined mod 2π. Moreover, $H = J\nabla^{\Sigma}\beta$ where H is the mean curvature and ∇^{Σ} is the induced connection on Σ.*

The above proposition implies that if we minimize volume among a class for which the mean curvature is an allowable variation, then the solution will be minimal lagrangian (hence special Lagrangian with respect to some α_θ).

13.2 The Euler-Lagrange Equations

In this sections , we give two versions of the Euler-Lagrange equations for hamiltonian stationary submanifolds. The first is the following geometric version.

Proposition 13.4. *Let $\Sigma^m \subset \mathbb{R}^{2m}$ be a lagrangian submanifold which is hamiltonian stationary. Let H be the mean curvature vector. Define a one form σ_H on Σ by $\sigma = H \rfloor \omega$, where $\rfloor$ denotes the interior product and ω is the standard symplectic form on $\mathbb{R}^{2m}$. Then we have the following Euler-Lagrange equations*

$$d\sigma_H = 0, \qquad \delta\sigma_H = 0.$$

Proof. From above we have that locally $\sigma_H = d\beta$ (this is equivalent to saying $H = J\nabla\beta$). Thus we have $d\sigma_H = 0$ for any lagrangian submanifold. The first variation formula and the hamiltonian stationarity then give that

$$\begin{aligned} 0 &= \delta\Sigma(X_h) \\ &= -\int_\Sigma X_h \cdot H d\mu \\ &= -\int_\Sigma J\nabla h \cdot J\nabla\beta d\mu \\ &= -\int_\Sigma \nabla h \cdot \nabla\beta d\mu \end{aligned}$$

Hence we see that $\triangle\beta = 0$, i.e. β is a harmonic function on Σ. This is equivalent to the condition $\delta\sigma_H = 0$. □

Now we give the analytical version of the Euler-Lagrange equations. Standard lagrangian geometry implies that any lagrangian submanifold which is graphical over a lagrangian plane, can be written as the graph of the gradient of a potential function defined on the plane. More precisely, if Σ is a lagrangian submanifold which is graphical over the x-plane, then there is a function $u(x)$ such that the graph may be defined by $y = \nabla u$, i.e.

$$y_i = \frac{\partial u}{\partial x_i},$$

for $i = 1, \ldots, m$.

If we in addition assume that Σ is also hamiltonian stationary, a standard first variation computation gives the following Euler-Lagrange equation

$$\sum_{j=1}^{m} \frac{\partial}{\partial x_j}(\triangle_g \frac{\partial u}{\partial x_j}) = 0. \tag{12}$$

Here the induced metric is given by $g_{ij} = \delta_{ij} + \sum_k u_{ik}u_{jk}$ with subscripts denoting partial derivatives of u. This is a fourth order quasilinear scalar equation for u which is of bi-harmonic type.

13.3 Examples

In this section we give some examples of hamiltonian stationary submanifolds.

Example 1. If $m = 1$, hamiltonian stationarity implies that the lagrangian angle β is a linear function of s (the arclength parameter). Thus the hamiltonian stationary curves in R^2 are the lines and circles.

Example 2(Clifford Tori). Consider $\mathbb{R}^{2m}$ as $\mathbb{C}^m$, and define $\Sigma = S^1(r_1) \times \ldots \times S^m(r_m)$, where $S^i(r_i)$ is the circle with radius r_i in the i-th copy of $\mathbb{C}$. These are called the *Clifford tori*, and they are hamiltonian stationary. It is unknown whether for $m > 1$ these minimize volume in their hamiltonian isotopy class. For $m = 1$ this is true and is equivalent to the isoperimetric inequality for plane regions.

Example 3(Helein/Romon). If $m = 2$, then Helein and Romon [HR] showed that there are infinite many distinct hamiltonian stationary tori in R^4. Their proof uses explicit representation formulae for hamiltonian stationary surfaces in $\mathbb{R}^4$ which arise from the theory of integrable systems.

Example 4(Hamiltonian stationary cones). Let

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

be the unit 3-sphere in $\mathbb{C}^2$. Let $\pi : S^3 \to S^2$ be the standard Hopf map defined by

$$\pi((z_1, z_2)) = \frac{z_1}{z_2} \in \mathbb{C} \cup \{\infty\} = S^2.$$

The fiber of this projection is a great circle, i.e. for $p \in S^2$, $\pi^{-1}(p) = \{e^{\sqrt{-1}\theta}(z_1, z_2) : \theta \in [0, 2\pi)\}$ for any point (z_1, z_2) in $\pi^{-1}(p)$. For a point $q \in S^3$ we will call the great circle $\{e^{\sqrt{-1}\theta}q : \theta \in [0, 2\pi]\}$ the *Hopf fiber through* q and simply denote it as $e^{\sqrt{-1}\theta}q$. Now let $\gamma(t) \subset S^3$ be a curve satisfying:

1. γ is horizontal in the sense that $\gamma'(t) \perp T_{\gamma(t)}(e^{\sqrt{-1}\theta}\gamma(t))$ (i.e. γ is perpendicular to the Hopf fibers);
2. $\pi(\gamma)$ is a circle in S^2.

It is shown in [SW1] that there are infinitely many closed curves in S^3 satisfying these two properties. These are precisely the curves γ in S^3 with the property that the corresponding cones in $\mathbb{R}^4$ over γ are hamiltonian stationary.

14 The Lagrangian Plateau Problem in $\mathbb{R}^4$

Let Γ be a smooth Jordan curve in R^4. We would like to find a Lagrangian disk bounded by γ which has the least area among all such disks.

First we need to ask whether γ bounds any lagrangian disk. One can easily derive a necessary condition using Stokes' Theorem. Let $\eta = \sum_{i=1}^{2}(x_i dy_i - y_i dx_i)$. Clearly η is a primitive of the standard symplectic form ω. If Γ bounds an oriented lagrangian surface Σ, then Stokes' Theorem implies that

$$0 = \int_\Sigma \omega = \int_\Gamma \eta.$$

Therefore, a necessary condition for a closed curve Γ to bound a lagrangian disk is $\int_\Gamma \eta = 0$.

It turns out that this is also a sufficient condition. Quantitative results obtained by Gromov[Gr2] and Allcock[A] show that if Γ is a closed curve in $\mathbb{R}^4$ such that $\int_\Gamma \eta = 0$, then Γ bounds a (singular) lagrangian disk D with area bounded in terms of the lenrth of Γ, i.e. $Area(D) \leq cLength(\Gamma)^2$, where c is an absolute constant. (The situation is completely different if we allow nonorientable surfaces. Qiu [Q1] proved that any closed curve in R^4 bounds a lagrangian Möbius band with similarly bounded area).

We state the following result (whose proof is contained in [SW2]) concerning the existence and regularity of the Lagrangian Plateau Problem:

Theorem 14.1. *Let Γ be a Jordan curve in R^4 such that $\int_\Gamma \eta = 0$. Then there exists a map $F : D \to R^4$, where D is the unit disk in R^2 such that $F|_{\partial D}$ is a $1-1$ parametrization of Γ. Also F satisfies:*

1. *F is Lagrangian, i.e. $F^*\omega = 0$ and F has the least area among all Lagrangian disks bounded by Γ;*
2. *F is weakly conformal;*
3. *F is Lipschitz in D and continuous in $\bar{D}$;*
4. *F is a smooth immersion except at a discrete set of points. Those singularities are either branch points or points at which $F(D)$ has a non-flat tangent cone described in Example 4 above.*

Remark 14.2. *For the Plateau boundary condition, $\Sigma = F(D)$ is hamiltonian stationary, but will not be minimal since the mean curvature H does not vanish along the boundary Γ and hence is not an allowable variation for the problem. The condition that a curve in $\mathbb{R}^4$ bound a minimal lagrangian surface is much more restrictive than the condition that it bound a lagrangian disk.*

15 The Free Boundary Problem

As remarked above, the solution of Lagrangian Plateau Problem need not be minimal. In order to produce minimal lagrangian surfaces, we need to consider more flexible boundary conditions. We describe one of these here.

Let S be a complex (real two dimensional) surface in R^4 and Γ be a nontrivial (in the relative homotopy sense) curve on S. We call a surface Σ with boundary $\partial\Sigma \subset S$ *a solution to the free boundary problem with respect to* (S, Γ) if

$$\begin{aligned} Area(\Sigma) &= \inf\{Area(\Sigma_1) : \Sigma_1 \text{ lagrangian} \\ &\quad\ \text{such that } \partial\Sigma_1 \subset S \text{ and is homotopy to } \Gamma\}. \end{aligned}$$

Lemma 15.1. *Let S be a complex surface in R^4 and Γ be a nontrivial (in the relative homotopy sense) curve on S. A smooth solution Σ for the free boundary problem with respect to (S, Γ) is minimal lagrangian.*

Proof. Clearly since Σ is a solution to the free boundary problem with respect to (S, Γ), the first variation gives that

$$\int_\Sigma \langle X, H\rangle d\mu = 0,$$

for any vector field X along Σ such that X is tangent to S along $\partial\Sigma$. Let X be a hamiltonian vector field X_h. Since $H = J\nabla\beta$, where β is the lagrangian angle, we see that

$$\int_\Sigma \langle \nabla h, \nabla\beta\rangle d\mu = 0.$$

Integrating by parts and using $\triangle\beta = 0$, we have

$$\int_{\partial\Sigma} h\frac{\partial\beta}{\partial\nu}ds = 0,$$

where ν is the conormal vector along $\partial\Sigma$. Since S is complex, h can be arbitray on $\partial\Sigma$ (i.e., any h defined on $\partial\Sigma$ can be extended to an ambient function on R^4 such that $J\nabla h$ is tangent to S along $\partial\Sigma$). We see that

$$\frac{\partial\beta}{\partial\nu} = 0.$$

Hence $\nabla\beta$ is tangent to $\partial\Sigma$, and in particular tangent to S. Since S is complex, $H = J\nabla\beta$ is tangent to S so H is an allowable variation. Therefore the first variation gives

$$\int_{\Sigma}\langle H, H\rangle d\mu = 0,$$

and Σ is minimal. □

Remark 15.2. *The preceeding argument shows more generally that if Σ is a compact lagrangian stationary submanifold, then we can conclude $H = 0$ using the mean curvature as the variational vector field. in other words, it does not use the minimizing property of Σ.*

Remark 15.3. *The boundary regularity for this free boundary problem has been studied by Weiyang Qiu [Q2] in his Stanford PhD dissertation.*

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